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Absract The moments of multivariate Wishart distribution is known up to the fourth order. Product moments of the elements of Wishart matrix are not available in general. In this paper some product moments of the elements of bivariate Wishart matrix are derived.

1. Introduction

The moments of multivariate Wishart distribution is known up to the fourth order. Some researchers have derived moments of some special type of functions namely, sample trace and determinant of Wishart Matrix which have got applications in estimation theory. Only few special cases of product moments are published in the literature which are mostly considered for applications in correlation analysis. This paper aims at deriving some moments product moments of bivariate Wishart distribution. This will lead to generalization of some of the published results in the area.

Let $X_1, X_2, \cdots X_N \, (N > p)$ be a p-dimensional independent normal random vector with mean vector $\bar{X}$ so that the sums of squares and cross product matrix is given by $\sum_{j=1}^{N}(X_j - \bar{X})(X_j - \bar{X})' = A$. The random symmetric positive definite matrix A is said to have a Wishart distribution with parameters p, $m = N - 1 > p$ and $\Sigma(p \times p) > 0$, written as $A \sim W_p(m, \Sigma)$ if its probability density function is given by

$$f(A) = \frac{|A|^{(m-p-1)/2} \exp\left(-\frac{1}{2} tr \Sigma^{-1} A\right)}{2^{mp/2} \pi^{p(p-1)/4} |\Sigma|^{m/2} \prod_{i=1}^{p} \Gamma\left(\frac{1}{2}(m+1-i)\right)}, \quad A > 0, m \geq p$$

(See e.g. Anderson, 2003, 252).

In this paper we consider deriving product moments of the distribution of the symmetrix matrix A for the bivariate case i.e. for $p = 2$ in which case $A = (a_{ik}), i = 1,2; k = 1,2$ where

$a_{ii} = mS_i^2 = \sum_{j=1}^{N}(X_{ij} - \bar{X}_i)^2, \quad m = N - 1, (i = 1,2)$ and

$a_{12} = \sum_{j=1}^{N}(X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) = mRS_1S_2$.

We want to derive the moments $E\left(S_1^{2l_1}S_2^{2l_2}R^l\right)$ for finite l_1, l_2, l . Some of them may not have closed forms. This paper may help estimate some parametric functions of the elements of $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_1 & \sigma_2^2 \end{pmatrix}$ where $\sigma_1 > 0, \sigma_2 > 0$, and ρ, $(-1 < \rho < 1)$, is the product moment correlation coefficient between X_1 and X_2.

Fisher (1915) derived the distribution of A for $p = 2$ in order to study the distribution of correlation coefficient from a normal sample. Wishart (1928) obtained the distribution for arbitrary p as the joint distribution of sample variances and covariances from multivariate normal population. Because of its important role in multivariate statistical analysis, various authors have derived it from different perspectives. See the references in Gupta and Nagar (200, 87-88).

The first moment of A, trA, $\det(A)$, A^{-1} and similar beautiful quantities are known (Muirhead, 1982). A nice update of moments of Wishart distribution is given in Gupta and Nagar (2000). But very few product moments can be deduced from the works published so far. Hence we derive some product moments of Wishart distribution when the parent variables have a bivariate normal distribution.

2. Description of the Problem and Proposed Method for Solution

The moments of sample variances $S_i^2 (i = 1,2)$ are well known. The moments of product moment correlation coefficient R are also known though they do not have closed forms. Ghosh (1966) expressed the moments of R in terms of hypergeometric functions. The first moment is the simplest one and is given by

$$E(R) = \frac{2}{m}\left(\frac{\Gamma((m+1)/2)}{\Gamma(m/2)}\right)^2 \rho \;\; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{m+2}{2}; \rho^2\right), \; -1 < \rho < 1, m > 1.$$

There are a number of representations of moments of the correlation coefficient in Johnson, Kotz and Balakrishnan (1995, 553). Understandably the derivation of the joint moments of $S_1^{2l_1}, S_2^{2l_2}$ and R^l will be formidably difficult. By differentiating the moment generating function of $A \sim W_p(m, \Sigma)$, $m > p$,de Waal and Nel (1973) derived the following results:

$$(a) E(A^2) = m\left((m+1)\Sigma + (tr\Sigma)I_p\right)\Sigma$$
$$(b) E(A^3) = m\left((m^2 + 3m + 4)\Sigma^2 + 2(m+1)(tr\Sigma)\Sigma + (m+1)(tr\Sigma^2)I_p + (tr\Sigma)^2 I_p\right)\Sigma$$

Some product moments e.g. $E(S_1 S_2^3 R), E(S_1^3 S_2 R)$ and $E(S_1^2 S_2^2 R^2)$ can be deduced from the above identity in (a).

Theorem 2.1 For finite l_1, l_2 and l , the product moments $E\left(S^{2l_1} S^{2l_2} R^l\right)$ denoted by $\mu(l_1, l_2, l)$ are given by

$$\mu(l_1,l_2,l)=\frac{2^{l_1+l_2}}{m^{l_1+l_2}L(m,\rho)}\left(1-\rho^2\right)^{l_1+l_2}\sigma_1^{2l_1}\sigma_2^{2l_2}$$
$$\times\sum_{k=0}^{\infty}\rho^k\frac{2^k}{k!}\Gamma\left(\frac{k+m}{2}+l_1\right)\Gamma\left(\frac{k+m}{2}+l_2\right)\frac{\Gamma\left(\frac{k+1+l}{2}\right)}{\Gamma\left(\frac{k+m+l}{2}\right)},\tag{2.1}$$

where $m>0,\sigma_1>0,\sigma_2>0,-1<\rho<1$ and $L(m,\rho)=\dfrac{4\pi\,\Gamma(m-1)}{2^m\Gamma\left(\frac{m-1}{2}\right)\left(1-\rho^2\right)^{m/2}}$.

Proof. The pdf of the elements of A can be written as

$$f_1(a_{11},a_{22},a_{12})=\frac{\left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}}{2^m\sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)}\left(a_{11}a_{22}-a_{12}^2\right)^{(m-3)/2}$$
$$\times\exp\left(-\frac{1}{2}\frac{a_{11}}{(1-\rho^2)\sigma_1^2}\right)\exp\left(-\frac{1}{2}\frac{a_{22}}{(1-\rho^2)\sigma_2^2}\right)\exp\left(\frac{a_{12}}{(1-\rho^2)\sigma_1\sigma_2}\right)$$

where $a_{11}>0,a_{22}>0,-\infty<a_{12}<\infty,-1<\rho<1$, $m>0,\sigma_1>0,\sigma_2>0$. Under the transformation $a_{11}=ms_1^2,a_{22}=ms_2^2,a_{12}=mrs_1s_2$ with Jacobian $m^3s_1s_2$, the pdf of S_1^2,S_2^2 and R is given by

$$f_2(s_1^2,s_2^2,r)=\frac{m^m\left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}}{4\pi\Gamma(m-1)}s_1^{m-2}\exp\left(-\frac{1}{2}\frac{ms_1^2}{(1-\rho^2)\sigma_1^2}\right)$$
$$\times s_2^{m-2}\exp\left(-\frac{1}{2}\frac{ms_2^2}{(1-\rho^2)\sigma_2^2}\right)\left(1-r^2\right)^{(m-3)/2}\exp\left(\frac{m\rho rs_1s_2}{(1-\rho^2)\sigma_1\sigma_2}\right)\tag{2.2}$$

where we have also used the duplication formula of gamma function
$\Gamma(z)=\frac{2^{z-1}}{\sqrt{\pi}}\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{z}{2}\right)$ with $z=m-1$. By expanding the last term, we have

$$f_3(s_1^2,s_2^2,r)=\frac{m^m\left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}}{4\pi\,\Gamma(m-1)}$$
$$\times\sum_{k=0}^{\infty}\frac{(m\rho r)^k}{(1-\rho^2)^k(\sigma_1\sigma_2)^k\,k!}s_1^{m-2+k}\exp\left(-\frac{1}{2}\frac{ms_1^2}{(1-\rho^2)\sigma_1^2}\right)$$
$$\times s_2^{m-2+k}\exp\left(-\frac{1}{2}\frac{ms_2^2}{(1-\rho^2)\sigma_2^2}\right)r^k\left(1-r^2\right)^{(m-3)/2}$$

Then the product moments $E\left(S_1^{2l_1}S_2^{2l_2}R^l\right)$ are given by

$$E\left(S_1^{2l_1}S_2^{2l_2}R^{l}\right)=\frac{m^m\left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}}{4\pi\,\Gamma(m-1)}$$

$$\times\sum_{k=0}^{\infty}\frac{(m\rho)^k}{(1-\rho^2)^k(\sigma_1\sigma_2)^k\,k\,!}\int_0^{\infty}s_1^{m+k+2l_1-2}\exp\left(-\frac{1}{2}\frac{ms_1^2}{(1-\rho^2)\sigma_1^2}\right)ds_1^2$$

$$\times\int_0^{\infty}s_2^{m+k+2l_2-2}\exp\left(-\frac{1}{2}\frac{ms_2^2}{(1-\rho^2)\sigma_2^2}\right)ds_2^2\int_{-1}^{1}r^{k+l}\left(1-r^2\right)^{(m-3)/2}dr \quad ,$$

$$=\frac{m^m\left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}}{4\pi\,\Gamma(m-1)}$$

$$\times\sum_{k=0}^{\infty}\frac{(m\rho)^k}{(1-\rho^2)^k(\sigma_1\sigma_2)^k\,k\,!}\int_0^{\infty}\left(s_1^2\right)^{(m+k)/2+l_1-1}\exp\left(-\frac{1}{2}\frac{ms_1^2}{(1-\rho^2)\sigma_1^2}\right)ds_1^2$$

$$\times\int_0^{\infty}\left(s_2^2\right)^{(m+k)/2+l_2-1}\exp\left(-\frac{1}{2}\frac{ms_2^2}{(1-\rho^2)\sigma_2^2}\right)ds_2^2\int_0^{1}u^{(k+l-1)/2}\left(1-u\right)^{(m-3)/2}du$$

which can further be evaluated as

$$\mu(l_1,l_2,l)=\frac{2^{l_1+l_2+m}}{4\pi m^{l_1+l_2}}\frac{\Gamma\left(\dfrac{m-1}{2}\right)}{\Gamma\left(m-1\right)}\left(1-\rho^2\right)^{l_1+l_2+m/2}\sigma_1^{2l_1}\sigma_2^{2l_2}$$

$$\times\sum_{k=0}^{\infty}\frac{(2\rho)^k}{k\,!}\Gamma\left(\frac{k+m}{2}+l_1\right)\Gamma\left(\frac{k+m}{2}+l_2\right)\frac{\Gamma\left(\dfrac{k+1+l}{2}\right)}{\Gamma\left(\dfrac{k+m+l}{2}\right)}.$$

Theorem 2.2 The probability density function of the correlation coefficient R is given by

$$h(r)=\frac{2^{m-2}\,\Gamma^2(m/2)\left(1-\rho^2\right)^{-m/2}}{\pi\,\Gamma(m-1)}\left(1-r^2\right)^{(m-3)/2}E\left(e^{\rho r\sqrt{U_1U_2}}\right)$$

where $m>0,-1<\rho<1$ and $U_i\sim\chi_m^2\ (i=1,2)$**.**

Proof. By making the transformation

$$\frac{ms_1^2}{\sigma_1^2(1-\rho^2)}=u_1,\ \frac{ms_2^2}{\sigma_2^2(1-\rho^2)}=u_2$$

with Jacobian $J(s_1^2,s_2^2\to u_1,u_2)=\left(\sigma_1\sigma_2/m\right)^2\left(1-\rho^2\right)^2$ in (2.2) we have

$$f_4(u_1,u_2,r)=\frac{\left(1-\rho^2\right)^{-m/2}}{4\pi\ \Gamma(m-1)}\ \left(1-r^2\right)^{(m-3)/2} e^{\rho r\sqrt{uv}}\left(u_1u_2\right)^{m/2-1} e^{-(u_1+u_2)/2}.$$

Then the theorem is obvious by the integration over u_1 and u_2 as the following:

$$h(r)=\frac{\left(1-\rho^2\right)^{-m/2}}{4\pi\ \Gamma(m-1)}\ \left(1-r^2\right)^{(m-3)/2}\int_0^\infty\int_0^\infty e^{\rho r\sqrt{u_1u_2}}\left(u_1u_2\right)^{m/2-1} e^{-(u_1+u_2)/2}du_1du_2.$$

Note that by expanding the term $e^{\rho r\sqrt{u_1u_2}}$ and completing the integration over u_1 and u_2 we have the well known probability density function of R :

$$h(r)=\frac{2^{m-2}\left(1-\rho^2\right)^{m/2}}{\pi\ \Gamma(m-1)}\ \left(1-r^2\right)^{(m-3)/2}\sum_{k=0}^{\infty}\frac{(2\rho r)^k}{k\,!}\ \Gamma^2\left(\frac{m+k}{2}\right),\ -1<r<1$$

Muirhead (1982, 154).

3. Some Special Cases of Product Moments $\mu(2l_1,2l_2,l)$

To evaluate the product moments given by (2.1) for some special cases we need the following lemma.

Lemma 3.1 Let $b_{k,m}=\frac{2^k}{k!}\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k+m}{2}\right)$, $m>0,k>0$. Then

(i) $\sum_{k=0}^{\infty}\rho^k b_{k,m}=\dfrac{4\pi\ \Gamma(m-1)}{2^m\Gamma\left(\dfrac{m-1}{2}\right)}\left(1-\rho^2\right)^{-m/2}=L(m,\rho)$

(ii) $\sum_{k=0}^{\infty}k\rho^k b_{k,m}=m\rho^2\left(1-\rho^2\right)^{-1}L(m,\rho)=w_1(m,\rho)\ L(m,\rho)$

(iii) $\sum_{k=0}^{\infty}k^{(2)}\rho^k b_{k,m}=\left(m(m+1)\rho^4+m\rho^2\right)\left(1-\rho^2\right)^{-2}L(m,\rho)=w_{(2)}(m,\rho)L(m,\rho)$

(iv) $\sum_{k=0}^{\infty}k^{(3)}\rho^k b_{k,m}=w_{(3)}(m,\rho)L(m,\rho),$ where

$$\begin{aligned}w_{(3)}(m,\rho)&=\left((m^3+3m^2+2m)\rho^6+(3m^2+6m)\rho^4\right)\left(1-\rho^2\right)^{-3}\\&=\left(m(m+1)(m+2)\rho^6+3m(m+2)\rho^4\right)\left(1-\rho^2\right)^{-3}\end{aligned}$$

(v) $\sum_{k=0}^{\infty} k^{(4)}\rho^k \ b_{k,m} = w_{(4)}(m,\rho)L(m,\rho)$ where

$$w_{(4)}(m,\rho) = m\left[(m^3+6m^2+11m+6)\rho^8+(6m^2+30m+36)\rho^6+(3m+6)\rho^4\right]\left(1-\rho^2\right)^{-4}$$
$$=\left[m(m+1)(m+2)(m+3)\rho^8+6m(m+2)(m+3)\rho^6+3m(m+2)\rho^4\right]\left(1-\rho^2\right)^{-4}$$

(vi) $\sum_{k=0}^{\infty} k^2\rho^k b_{k,m} = \left(m^2\rho^4+2m\rho^2\right)\left(1-\rho^2\right)^{-2} L(m,\rho) = w_2(m,\rho)\ L(m,\rho)$

(vii) $\sum_{k=0}^{\infty} k^3\rho^k b_{k,m} = \left(4m\rho^2+(6m^2+4m)\rho^4+m^3\rho^6\right)\left(1-\rho^2\right)^{-3} L(m,\rho) = w_3(m,\rho)L(m,\rho)$

(viii) $\sum_{k=0}^{\infty} k^4\rho^k \ b_{k,m} = w_4(m,\rho)L(m,\rho)$ where

$$w_4(m,\rho) = \left[(m^4+18m^2+12m)\rho^8+(12m^3-20m^2+8m)\rho^6\right.$$
$$\left.+(46m^2-4m)\rho^4+32m\rho^2\right]\left(1-\rho^2\right)^{-4}$$

Proof.

The proof of (i) is obvious by virtue of $\mu(0,0,0)=1$ where $\mu(l_1,l_2,l)$ is defined in (2.1). The identity in (i) can be rewritten as

$$\sum_{k=0}^{\infty}\rho^k b_{k,m} = L(m,0)\left(1-\rho^2\right)^{-m/2}. \tag{3.1}$$

Differentiating both sides of the identity in Lemma 3.1 (i) with respect to ρ we have

$$\sum_{k=0}^{\infty} k\rho^{k-1}b_{k,m} = mL(m,0)\rho\left(1-\rho^2\right)^{-m/2-1} \tag{3.2}$$

which yields the identity in (ii) above. Differentiating the identity in (3.2) again we have

$$\sum_{k=0}^{\infty} k\,(k-1)\rho^{k-2}\ b_{k,m} = mL(m,0)\left[\rho\left\{(-m/2-1)\left(1-\rho^2\right)^{-m/2-2}(-2\rho)\right\}+\left(1-\rho^2\right)^{-m/2-1}\right]$$
$$= mL(m,0)\left[1+(m+1)\rho^2\right]\left(1-\rho^2\right)^{-m/2-2}.$$

(3.3)

which yields the identity in (iii) above. Differentiating the identity in (3.3) we have

$$\sum_{k=0}^{\infty} k(k-1)(k-2)\rho^{k-3}\ b_{k,m}$$
$$= mL(m,0)\left[\left(1+(m+1)\rho^2\right)\left\{(-m/2-2)\left(1-\rho^2\right)^{-m/2-3}(-2\rho)\right\} + \{(m+1)2\rho\}\left(1-\rho^2\right)^{-m/2-2}\right]$$
$$= mL(m,0)\left[(m^2+3m+2)\rho^3 + (3m+6)\rho\right]\left(1-\rho^2\right)^{-m/2-3}$$
(3.4)

which yields the identity in (iv). Differentiating the identity in (3.4) we have

$$\sum_{k=0}^{\infty} k(k-1)(k-2)(k-3)\rho^{k-4}\ b_{k,m}$$
$$= mL(m,0)\Big[\left((m^2+3m+2)\rho^3 + (3m+6)\rho\right)\left\{(-m/2-3)\left(1-\rho^2\right)^{-m/2-4}(-2\rho)\right\}$$
$$+\left\{(m^2+3m+2)3\rho^2 + (3m+6)\right\}\left(1-\rho^2\right)^{-m/2-3}\Big]$$
$$= mL(m,0)\left[(m^3+6m^2+11m+6)\rho^4 + (6m^2+30m+36)\rho^2 + (3m+6)\right]\left(1-\rho^2\right)^{-m/2-4}$$

which yields the identity in (v).

The identity in (vi) can be proved as follows:

$$\sum_{k=0}^{\infty} k^2\rho^k b_{k,m} = \sum_{k=0}^{\infty}\left(k^{(2)}+k\right)\rho^k b_{k,m}$$
$$= \left(w_{(2)}(m,\rho) + w_{(1)}(m,\rho)\right)L(m,\rho)$$
$$= \left[w_{(2)}(m,\rho)\left(1-\rho^2\right)^2 + w_{(1)}(m,\rho)\left(1-\rho^2\right)^2\right]L(m,\rho)\left(1-\rho^2\right)^{-2}$$
$$= \left[\left((m^2+m)\rho^4 + m\rho^2\right) + m\rho^2\left(1-\rho^2\right)^2\right]L(m,\rho)\left(1-\rho^2\right)^{-2}$$

The identity in (vii) can be proved as follows:

$$\sum_{k=0}^{\infty} k^3\rho^k b_{k,m} = \sum_{k=0}^{\infty}\left(k^{(3)}+3k^{(2)}+k\right)\rho^k b_{k,m}$$
$$= \left(w_{(3)}(m,\rho) + 3w_{(2)}(m,\rho) + w_{(1)}(m,\rho)\right)L(m,\rho)$$
$$= \left[w_{(3)}\left(1-\rho^2\right)^3 + 3w_{(2)}(m,\rho)\left(1-\rho^2\right)^3 + w_{(1)}(m,\rho)\left(1-\rho^2\right)^3\right]L(m,\rho)\left(1-\rho^2\right)^{-2}$$
$$= \Big[\left((m^3+3m^2+2m)\rho^6 + (3m^2+6m)\rho^4\right) + 3\left((m^2+m)\rho^4 + m\rho^2\right)\left(1-\rho^2\right)$$
$$+m\rho^2\left(1-\rho^2\right)^2\Big]L(m,\rho)\left(1-\rho^2\right)^{-3}$$

The identity in (vii) can be proved as follows:

$$\sum_{k=0}^{\infty} k^4 \rho^k b_{k,m} = \sum_{k=0}^{\infty} \left(k^{(4)} + 6k^{(3)} 7k^{(2)} + k \right) \rho^k b_{k,m}$$

$$= \left(w_{(4)}(m,\rho) + 6w_{(3)}(m,\rho) + 7w_{(2)}(m,\rho) + w_{(1)}(m,\rho) \right) L(m,\rho)$$

$$= \Big[w_{(4)}(m,\rho)\left(1-\rho^2\right)^4 + 6w_{(3)}(m,\rho)\left(1-\rho^2\right)^4 + 7w_{(2)}(m,\rho)\left(1-\rho^2\right)^4$$

$$+ w_{(1)}(m,\rho)\left(1-\rho^2\right)^4 \Big] L(m,\rho)\left(1-\rho^2\right)^{-4}$$

$$= \Big[\left((m^4 + 6m^3 + 11m^2 + 6m)\rho^8 + (6m^3 + 30m^2 + 36m)\rho^6 + (3m^2 + 6m)\rho^4 \right)$$

$$+ 6\left(m^3 + 3m^2 + 2m)\rho^6 + (3m^2 + 6m)\rho^4 \right)\left(1-\rho^2\right)$$

$$+ 7\left((m^2 + m)\rho^4 + m\rho^2 \right)\left(1-\rho^2\right)^2 + m\rho^2\left(1-\rho^2\right)^3 \Big] L(m,\rho)\left(1-\rho^2\right)^{-4}$$

$$= w_4(m,\rho) L(m,\rho)$$

where $w_{(i)}(m,\rho)$ and $w_i(m,\rho)$, $(i = 1,2,3,4)$ are defined in the lemma.

Since, $\dfrac{L(m+c,\rho)}{L(m,\rho)} = \dfrac{\Gamma\left((m+c)/2\right)}{\Gamma\left(m/2\right)} \; (1-\rho^2)^{-c/2}$, we have the following special cases:

$$\frac{L(m-4,\rho)}{L(m,\rho)} = \frac{4}{(m-2)(m-4)} \; (1-\rho^2)^2$$

$$\frac{L(m-2,\rho)}{L(m,\rho)} = \frac{2}{m-2} \; (1-\rho^2)$$

$$\frac{L(m+2,\rho)}{L(m,\rho)} = \frac{m}{2} \; (1-\rho^2)^{-1}$$

$$\frac{L(m+4,\rho)}{L(m,\rho)} = \frac{m(m+2)}{4} \; (1-\rho^2)^{-2}$$

By Direct application of Lemma 3.1 to Theorem 2.1, special cases of moments $\mu(2l_1, 2l_2, l)$ of Wishart distribution having probability density function given by (2.2) can be calculated. Some are tabulated below where in general $m > 0, \sigma_1 > 0, \sigma_2 > 0, -1 < \rho < 1$:

$$\mu(-2,0,0) = \frac{m^2 \sigma_1^{-4}}{(m-2)(m-4)}, \; m > 4$$

$$\mu(-2,1,0) = \frac{m}{(m-2)(m-4)} (-4\rho^2 + m) \sigma_1^{-4} \sigma_2^2$$

$$= m\left(\frac{\rho^2}{(m-2)\rho^2} + \frac{m}{(m-2)(m-4)} \right) \sigma_1^{-4} \sigma_2^2, \; m > 4$$

$$\mu(-2,1,2) = m\ \frac{1+(m-5)\rho^2}{(m-2)(m-4)}\sigma_1^{-4}\sigma_2^2,\ \ m>4$$

$$\mu(-2,2,0) = \left(24\rho^4 - 8(m+2)\rho^2 + m(m+2)\right)\frac{(\sigma_2/\sigma_1)^4}{(m-2)(m-4)}$$

$$= \left(\frac{24(1-\rho^2)^2}{(m-2)(m-4)} + \frac{8(1-\rho^2)}{m-2} + 1\right)(\sigma_2/\sigma_1)^4,\ m>4$$

$$\mu(-2,2,2) = \left(6(5-m)\rho^4 + (m^2-m-24)\rho^2 + (m+2)\right)\frac{(\sigma_2/\sigma_1)^4}{(m-2)(m-4)}$$

$$= \left(\frac{6(1-\rho^2)^2}{(m-2)(m-4)} + \frac{(1+6\rho^2)(1-\rho^2)}{m-2} + \rho^2\right)(\sigma_2/\sigma_1)^4,\ \ m>4$$

$$\mu(-2,2,4) = \left((m^2-12m+35)\rho^4 + 6(m-5)\rho^2 + 3\right)\frac{(\sigma_2/\sigma_1)^4}{(m-2)(m-4)}$$

$$= \left(\frac{3(1-\rho^2)^2}{(m-2)(m-4)} + \frac{6\rho^2(1-\rho^2)}{m-2} + \rho^4\right)(\sigma_2/\sigma_1)^4,\ \ m>4$$

$$\mu(-1,0,0) = \frac{m\sigma_1^{-2}}{(m-2)},\ m>2$$

$$\mu(-1,1,0) = \frac{m-2\rho^2}{m-2}\left(\frac{\sigma_2}{\sigma_1}\right)^2 = \left(\frac{2(1-\rho^2)}{m-2}+1\right)\left(\frac{\sigma_2}{\sigma_1}\right)^2,\ m>2$$

$$\mu(-1,1,2) = \frac{(m-3)\rho^2+1}{m-2}\left(\frac{\sigma_2}{\sigma_1}\right)^2 = \left(\rho^2 + \frac{1-\rho^2}{m-2}\right)\left(\frac{\sigma_2}{\sigma_1}\right)^2,\ m>2$$

$$\mu(-1,2,0) = \left(8\rho^4 - 2(m+3)\rho^2 + m(m+2)\right)\frac{\sigma_1^{-2}\sigma_2^4}{m(m-2)},\ m>2$$

$$\mu(-1,2,2) = \left((-2m+6)\rho^4 + (m^2+m-14)\rho^2 + (m+2)\right)\frac{\sigma_1^{-2}\sigma_2^4}{m(m-2)},\ m>2$$

$$\mu(0,-2,0) = \frac{m^2}{(m-2)(m-4)}\ \sigma_2^{-4},\ m>4$$

$$\mu(0,-1,0) = \frac{m\sigma_2^{-2}}{m-2}\ ,\ m>2$$

$$\mu(0,1,0) = \sigma_2^2$$

$$\mu(0,1,2) = \left((m-1)\rho^2+1\right)\frac{\sigma_2^2}{m}$$

$$\mu(0,2,2) = \left(-2(m-1)\rho^4 + (m-1)(m+4)\rho^2 + (m+2)\right)\frac{\sigma_2^2}{m^2}$$

$$\mu(0,2,4) = \left((m-1)(m-3)\rho^4 + 6(m-1)\rho^2 + 3\right)\frac{\sigma_2^4}{m^2}$$

$$\mu(1,1,0) = \left(m+2\rho^2\right)\frac{\sigma_1^2\sigma_2^2}{m}$$

$$\mu(1,-2,2)=\frac{m\left((m-5)\rho^2+1\right)}{(m-2)(m-4)}\sigma_1^2\sigma_2^{-4}\text{, } m>4$$

$$\mu(1,0,0)=\sigma_1^2$$

$$\mu(1,1,2)=\left((m+1)\rho^2+1\right)\frac{\sigma_1^2\sigma_2^2}{m}$$

$$\mu(2,-1,0)=\left(8\rho^4-4(m+2)\rho^2+m(m+2)\right)\frac{1}{m(m-2)}\sigma_1^4\sigma_2^{-2}\text{, } m>2$$

$$\mu(2,-1,2)=\left(-4(m-3)\rho^4+(m^2+m-14)\rho^2+(m+2)\right)\frac{1}{m(m-2)}\sigma_1^4\sigma_2^{-2}\text{, } m>2$$

$$\mu(2,0,0)=\frac{m+2}{m}\ \sigma_1^4$$

$$\mu(2,0,2)=\left(-2(m-1)\rho^4+(m^2+3m-4)\rho^2+(m+2)\right)\frac{\sigma_1^4}{m^2}$$

$$\mu(2,1,0)=\frac{m+2}{m^2}\left((m+3)\rho^2+1\right)\sigma_1^4\sigma_2^2$$

$$\mu(2,1,2)=\frac{m+2}{m^2}\left((m+3)\rho^2+1\right)\sigma_1^4\sigma_2^2$$

$$\mu(2,2,0)=\frac{m+2}{m^3}\left(8\rho^4+8(m+2)\rho^2+m(m+2)\right)\sigma_1^4\sigma_2^4$$

$$\mu(2,2,2)=\frac{m+2}{m^3}\left(4(m+3)\rho^4-(m^2+7m+4)\rho^2+m\right)\sigma_1^4\sigma_2^4$$

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