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Characterizations and Applications of γ -Open Sets

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Characterizations and Applications of γ –Open Sets

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Abstract. The notion of semi-convergence of filters was introduced by Latif (1999) who investigated some characterizations related to semi-open continuous functions. In the spirit of Latif (1999), Min (2002) used the idea of semi-convergence of filters to introduce a new class of sets, called $\gamma -$ *open* sets, and the notions of $\gamma -$ *closure*, $\gamma -$ *interior* and $\gamma -$ *continuity* and investigated some properties. In this paper we continue to explore further properties of these notions as well as characterizations of $\gamma -$ *open* sets. We introduce and study topological properties of $\gamma -$ *derived*, $\gamma -$ *border*, $\gamma -$ *frontier*, and $\gamma -$ *exterior* of a set using the concept of $\gamma -$ *open* sets.

1. Introduction

The notion of γ –open set (originally called $\gamma -$ sets) in topological spaces was introduced by Min [*Min*, 2002]. For these sets, we introduce the notions of γ –derived, $\gamma -$ *border*, $\gamma -$ *frontier*, and $\gamma -$ *exterior* of a set and show that some of their properties are analogous to those for open sets. Also, we give some additional properties of $\gamma -$ *closure* and $\gamma -$ *interior* of a set.

Throughout this paper, (X, τ) (simply X) always mean topological space on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure (resp., *interior*) of S will be denoted by $Cl(S)$ (resp., $Int(S)$). A subset S of X is called a semi-open set [*Levine*, 1963] (resp., $\alpha -$ *open set* [Njåstad, 1965]) if $S \subseteq Cl[Int(S)]$ (*resp*., $S \subseteq Int[Cl(Int(S))]$). The complement of a semi-open set (resp., $\alpha -$ *open set*) is called semi-closed set (resp., $\alpha -$ *closed set*). The family of all semi-open sets (resp., $\alpha -$ *open sets*) in a topological space (X, τ) will be denoted by $SO(X)$ (resp., τ^{α}). A subset $M(x)$ of a space X is called a semi-neighbourhood of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subseteq M(x)$. In [*Latif*, 1999] Latif introduced the notion of semi-convergence of filters and investigated some characterizations related to semi-open continuous function. Now, we recall the concept of semi-convergence of filters. Let $S(x) = \{A \in SO(X) : x \in A\}$ and let $S_x = \{A \subseteq X : \textit{there exists } \mu \subseteq S(x) \textit{ such that } \mu \textit{ is finite and } \cap\mu \subseteq A\}$. Then, S_x is called the semi-neighbourhood filter at x. For any filter Γ on X, we say that Γ semi-converges to x if and only if Γ is finer than the semi-neighbourhood filter at x.

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Key Words and Phrases: Topological Space, Semi-open set, Interior, Open set, Derived Set, Border, Frontier, Exterior, $\gamma -$ *interior*, $\gamma -$ *closure*, $\gamma -$ *open* set, $\gamma -$ *closed* set, $\gamma -$ *derived*, $\gamma -$ *border*, $\gamma -$ *frontier*, $\gamma -$ *exterior*.

A subset U of X is called a $\gamma-$ *open* set if whenever a filter Γ semi-converges to x and $x \in U$, $U \in \Gamma$. The complement of a $\gamma-$ *open* set is called a $\gamma-$ *closed* set. The intersection of all $\gamma-$ *closed* sets containing A is called the $\gamma-$ *closure* of A, denoted by $Cl_\gamma(A)$. A subset A is also $\gamma-$ closed if and only if $A = Cl_\gamma(A)$. We denote the family of all $\gamma-$ open sets of (X,τ) by τ^γ. It is shown in $[Min, 2002]$ that τ^γ is a topology on X. In a topological space (X, τ), it is always true that $\tau \subseteq \tau^\alpha \subseteq S(X) \subseteq \tau^\gamma$.

2. Characterizations of γ −Open Sets

Definition 2.1. $[Latif, 1999]$. *Let (X,τ) be a topological space. Let $(x_i : i \in I)$ be a net in X, and let $x \in X$. Then $(x_i : i \in I)$ semi-converges to x if and only if $(x_i : i \in I)$ is eventually in every semi-open set containing x.*

Definition 2.2. *If Γ is a filter on X, and $\Lambda_\Gamma = \{(x,F) : x \in F \in \Gamma\}$. Then Λ_Γ is directed by the relation $(x_1,F_1) \leq (x_2,F_2)$ if and only if $F_2 \subseteq F_1$, so the map $P : \Lambda_\Gamma \to X$ defined by $P(x,F) = x$ is a net in X. It is called the net based on Γ.*

Theorem 2.3. $[Latif, 1999]$. *A net $(x_i : i \in I)$ semi-converges to x in X if and only if the filter generated by $(x_i : i \in I)$ semi-converges to x.*

Theorem 2.4. $[Latif, 1999]$. *Let X be a topological space. Then a filter Γ semi-converges to x in X if and only if the net based on Γ semi-converges to x.*

The next theorem gives an equivalent formulation of a $\gamma-$ open set.

Theorem 2.5. *Let (X,τ) be a topological space. Then a subset U of X is a $\gamma-$ open set if and only if any net $(x_i : i \in I)$ semi-converging to any point $x \in U$, implies that $(x_i : i \in I)$ is eventually in U.*

Proof *Necessity. Let U be a $\gamma-$ open set and $x \in U$. Let $(x_i : i \in I)$ be a net in X such that it semi-converges to x. Then by Theorem 2.3, the filter generated by $(x_i : i \in I)$ semi-converges to x. By hypothesis, U is a $\gamma-$ open set and contains x. Thus $U \in \Gamma$. Since the filter Γ is generated by the filter base C consisting of the sets $B_{i_0} = \{x_i : i \geq i_0\}$, $i_0 \in I$. Hence there exists $i_* \in I$ such that $B_{i_*} \subseteq U$. It shows that $(x_i : i \in I)$ is eventually in U.*

Sufficiency. Suppose that $U \subseteq X$ such that every net $(x_i : i \in I)$ semi-converging to a point $x \in U$, implies that the net $(x_i : i \in I)$ is eventually in U. Let Γ be a filter such that Γ semi-converges to x and $x \in U$. Then by Theorem 2.4, the net based on Γ semi-converges to x. Hence by hypothesis, the net based on Γ is eventually in U. Then for some $(p^*,F^*) \in \Lambda_\Gamma$, we have $(p,F) \geq (p^*,F^*)$ implies $p \in F$. But then $F^* \subseteq U$; otherwise, there is some $q \in F^* - U$, and that $(q,F^*) \geq (p^*,F^*)$, but $q \notin U$. Hence $U \in \Gamma$, so U is a $\gamma-$ open set.

Theorem 2.6. *Let (X,τ) be a topological space, $x \in X$, $S \subseteq X$. Then $x \in Cl_\gamma(S)$ if and only if there is a filterbase in S which semi-converges to x.*

Proof Suppose first that such a filter base Γ exists, and let N be a semi-neighborhood of x. Then there exists $A \in \Gamma$ with $A \subseteq N$. Since $A \subseteq S$, this means that N must meet S.

Conversely, let $x \in Cl_\gamma(S)$ and let $\Gamma = \{N \cap S : N \in S_x\}$. We check the definition of filter base. $\Gamma \neq \phi$ since, for every $N \in S_x$, $N \cap S \neq \phi$ as $x \in Cl_\gamma(S)$; and, if $A = N_1 \cap S$, $B = N_2 \cap S$, then $A \cap B = N_1 \cap N_2 \cap S \in \Gamma$. Finally to verify that Γ semi-converges to x, let $N \in S_x$. Then $N \cap S \in \Gamma$ and $N \cap S \subseteq N$.

Definition 2.7. *A topological space (X,τ) is called semi-compact if and only if for each semi-open covering Γ of X, there exists a finite subcovering Λ of X.*

The next two results follow by the fact that every semi-open set in X is a $\gamma-$open set in X.

Theorem 2.8. *Let (X,τ) be a topological space. Then (X,τ) is semi-compact if (X,τ^γ) is compact.*

Proposition 2.9. *Suppose that (X,τ) is a topological space such that (X,τ^γ) is compact. Then (X,τ) is compact.*

The next example shows that the converse of the above proposition may not be always true.

Example 2.10. *Let $I = [0,1]$ be a subspace of R with the usual topology τ_N on R. Then I is compact. We note that every subset of I is a $\gamma-$open set. Hence I with the topology of all $\gamma-$open sets is not compact.*

We observe that every open set is a $\gamma-$open set . Hence the next three propositions follow.

Proposition 2.11. *Let (X,τ) be a topological space such that (X,τ^γ) is separable. Then (X,τ) is separable.*

Proposition 2.12. *Suppose that (X,τ) is a topological space such that (X,τ^γ) is connected. Then (X,τ) is connected.*

Proposition 2.13. *Suppose that (X,τ) is a topological space such that (X,τ^γ) is Lindelöf. Then (X,τ) is Lindelöf.*

The converse of the last three propositions is not true follows from the following example.

Example 2.14. *Consider R with the usual topology τ_N. Then (R,τ_N) is separable, connected, and Lindelöf but (R,τ_N^γ) being a discrete space is not separable, connected and Lindelöf.*

Definition 2.15. *A topological space (X,τ) is irreducible if every pair of nonempty open subsets of the space X has a nonempty intersection.*

Theorem 2.16. *Prove that a topological space (X,τ) is an irreducible space if and only if each nonempty $W \in SO(X,\tau)$ is dense in (X,τ^γ).*

Proof Suppose that (X,τ) is irreducible. Let $W \in SO(X,\tau) - \{\phi\}$. Then there exists $U \in \tau$ such that $U \subseteq W \subseteq Cl(U)$. Let $x \in X$ and $W^* \in \tau^\gamma$ such that $x \in W^*$. Then W^* is in the semi-neighbourhood filter S_x. Thus there exist $U_1, U_2, \ldots, U_n \in S_{(x)}$ such that $U = U_1 \cap U_2 \cap \ldots \cap U_n \subseteq W^*$. Then there exist $V_1, V_2, \ldots, V_n \in \tau - \{\phi\}$ such that $V = V_1 \cap V_2 \cap \ldots \cap V_n \subseteq U = U_1 \cap U_2 \cap \ldots \cap U_n \subseteq W^*$. Then $U \cap V_1 \cap V_2 \cap \ldots \cap V_n \neq \phi$ for (X,τ) is an irreducible space. It gives $W \cap W^* \neq \phi$. We conclude that W is dense in (X, τ^γ).

Conversely suppose that U and V is a pair of nonempty open subsets of X. Then U is a nonempty semi-open set in X and V is a nonempty $\gamma - open$ set. Thus by hypothesis $U \cap V \neq \phi$.

Theorem 2.17. *Prove that a topological space* (X, τ) *is an irreducible space if and only if each nonempty* $\gamma - open$ *set W is dense in* (X, τ^γ).

Proof Suppose that (X, τ) is an irreducible space. Let $W^* \in \tau^\gamma - \{\phi\}$. Let $x \in X$ and $W \in \tau^\gamma$ such that $x \in W$. Fix $x^* \in W^*$. Then there exist $W_1, W_2, \ldots, W_m \in S_{(x)}$ and $W_1^*, W_2^*, \ldots, W_n^* \in S_{(x^*)}$ such that $W_1 \cap W_2 \cap \ldots \cap W_m \subseteq W$ and $W_1^* \cap W_2^* \cap \ldots \cap W_n^* \subseteq W^*$. For each $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$, there exist $U_i \in \tau$ and $U_j^* \in \tau$ such that $U_i \subseteq W_i \subseteq Cl(U_i)$ and $U_j^* \subseteq W_j^* \subseteq Cl(U_j^*)$. By hypothesis (X, τ) being irreducible, we infer that $U_1 \cap U_2 \cap \ldots \cap U_m \cap U_1^* \cap U_2^* \cap \ldots \cap U_n^* \neq \phi$. It follows that $W \cap W^* \neq \phi$.We conclude that W^* is dense in (X, τ^γ).

Conversely suppose that each nonempty $\gamma - open$ set is dense in (X, τ^γ). Notice that $\tau \subset \tau^\gamma$. *Hence* it follows immediately that (X, τ) is an irreducible space.

Theorem 2.18. (X, τ) *is irreducible if and only if* (X, τ^γ) *is irreducible*.

Proof *Suppose that* (X, τ) *is irreducible. Let U and V be two nonempty* $\gamma - open$ *sets in X. Let* $x \in U$ *and* $y \in V$. *Then* $U \in S_x$ *and* $V \in S_y$. *Then there exist* $U_1, U_2, \ldots, U_m \in S_{(x)}$ *and* $V_1, V_2, \ldots, V_n \in S_{(y)}$ *such that* $U_1 \cap U_2 \cap \ldots \cap U_m \subseteq U$ *and* $V_1 \cap V_2 \cap \ldots \cap V_n \subseteq V$. *So there exists* $G_i \in \tau$ *such that* $G_i \subseteq U_i \subseteq Cl(G_i)$, $\forall\, i = 1, 2, \ldots, m$, *and there exists* $H_j \in \tau$ *such that* $H_j \subseteq V_j \subseteq Cl(H_j)$, $\forall\, j = 1, 2, \ldots, n$. *Now since* (X, τ) *is irreducible, so* $(G_1 \cap G_2 \cap \ldots \cap G_m) \cap (H_1 \cap H_2 \cap \ldots \cap H_n) \neq \phi$. *It follows that* $U \cap V \neq \phi$. *We conclude that* (X, τ^γ) *is irreducible*.

The converse follows immediately from the fact that $\tau \subseteq \tau^\gamma$.

Theorem 2.19. *Let* (X, τ) *be a topological space such that* τ *is finite. Then* (X, τ) *is an irreducible space if and only if the intersection of all nonempty* $\gamma - open$ *sets in X is nonempty*.

Proof Suppose that (X, τ) is an irreducible space. Assume that the intersection of all nonempty $\gamma - open$ sets in X is empty. Then for any $p \in X$, there exists a nonempty $\gamma - open$ set W_p in X such that $p \notin W_p$. Suppose that $q \in W_p$. Then W_p is in the semi-neighborhood filter S_q. Thus there exist $U_1, U_2, \ldots, U_n \in S_{(q)}$ such that $U_1 \cap U_2 \cap \ldots \cap U_n \subseteq W_p$. Then clearly $p \notin U_1 \cap U_2 \cap \ldots \cap U_n$ so there exists $k \in \{1, 2, \ldots, n\}$ such that $p \notin U_k$. Now $q \in U_k \in S_{(q)}$ implies that there exists a nonempty open set U in X such that $U \subseteq U_k \subseteq Cl(U)$. Then $p \notin U$ and hence the intersection of nonempty open sets in X is empty. This contradicts that (X, τ) is an irreducible space. It concludes that the intersection of all nonempty $\gamma - open$ sets in X is nonempty.

The converse follows by the fact that $\tau \subseteq \tau^\gamma$.

Theorem 2.20. (X, τ) *is an irreducible space if and only if any nonempty* $\gamma - open$ *set Y is connected in* (X, τ^γ).

Proof *Suppose that* (X, τ) *is an irreducible space. Let Y be a nonempty* $\gamma - open$ *set in X such that Y is disconnected in* (X, τ^γ). *Then there exist two nonempty* $\gamma - open$ *sets U and V in X such that* $Y \subseteq U \cup V$ *and* $U \cap V = \phi$. *It implies that* (X, τ^γ) *is not an irreducible space. Then by Theorem* 2.18, (X, τ) *is not an irreducible space. It contradicts the hypothesis. Thus every nonempty* $\gamma - open$ *set Y in X is connected in* (X, τ^γ).

Conversely suppose that every nonempty $\gamma-$*open set* Y *in* X *is connected in* (X,τ^{γ}). *Assume that* (X,τ) *is not an irreducible space. Then there exists a pair of nonempty open sets* G *and* H *in* (X,τ) *such that* $G\cap H=\phi$. *Let* $Y=G\cup H$. *Then* $Y\in\tau^{\gamma}$ *and* Y *is disconnected in* (X,τ^{γ}). *It contradicts the hypothesis. Hence we conclude that* (X,τ) *is irreducible.*

Theorem 2.21. *Let* (X,τ) *be a topological space and let* $Y\subseteq X$. *Then the following conditions are equivalent*:

(1) Y *is dense in* (X,τ^{γ}).

(2) *An improper* $\gamma-$*closed subset containing* Y *is* X.

(3) $Int_{\gamma}(Y)=X$.

Proof $(1)\Rightarrow(2)$: Let Z be a $\gamma-$*closed* subset of X with $Y\subseteq Z$. Then $X=Cl_{\gamma}(Y)\subseteq Z$.

$(2)\Rightarrow(3)$: Let $Z=Cl_{\gamma}(Y)$. Then Z is a $\gamma-$*closed* subset of X with $Y\subseteq Z$. Hence by hypothesis $Y=X$. Thus $Int_{\gamma}(Y)=X$.

$(3)\Rightarrow(1)$: Obvious.

3. Applications of γ −Open Sets

Definition 3.1. *Let* A *be a subset of a space* X. *A point* $x\in A$ *is said to be* $\gamma-$*limit point of* A *if for each* $\gamma-$*open set* U *containing* x, $U\cap(A-\{x\})\neq\phi$. *The set of all* $\gamma-$*limit points of* A *is called a* $\gamma-$*derived set of* A *and is denoted by* $D_{\gamma}(A)$.

Theorem 3.2. *For subsets* A,B *of a space* X, *the following statements hold*:

(1) $D_{\gamma}(A)\subseteq D(A)$, where $D(A)$ is the derived set of A;
(2) if $A\subseteq B$, then $D_{\gamma}(A)\subseteq D_{\gamma}(B)$;
(3) $D_{\gamma}(A)\cup D_{\gamma}(B)\subseteq D_{\gamma}(A\cup B)$ and $D_{\gamma}(A\cap B)\subseteq D_{\gamma}(A)\cap D_{\gamma}(B)$;
(4) $[D_{\gamma}(D_{\gamma}(A))-A]\subseteq D_{\gamma}(A)$;
(5) $D_{\gamma}[A\cup D_{\gamma}(A)]\subseteq A\cup D_{\gamma}(A)$.

Proof (1) It suffices to observe that every open set is $\gamma-$*open*.

(2) Obvious. (3) Follows by (2).

(4) If $x\in[D_{\gamma}(D_{\gamma}(A))-A]$ and U is a $\gamma-$*open* set containing x, then $U\cap[D_{\gamma}(A)-\{x\}]\neq\phi$. Let $y\in U\cap[D_{\gamma}(A)-\{x\}]$. Then, since $y\in D_{\gamma}(A)$ and $y\in U$, so $U\cap[A-\{y\}]\neq\phi$. Let $z\in U\cap[A-\{y\}]$. Then, $z\neq x$ for $z\in A$ and $x\notin A$. Hence, $U\cap[A-\{x\}]\neq\phi$. Therefore, $x\in D_{\gamma}(A)$.

(5) Let $x\in D_{\gamma}[A\cup D_{\gamma}(A)]$. If $x\in A$, the result is obvious. So, let $x\in[D_{\gamma}(A\cup D_{\gamma}(A))-A]$, then, for $\gamma-$*open* set U containing x, $U\cap[A\cup D_{\gamma}(A)-\{x\}]\neq\phi$. Thus, $U\cap(A-\{x\})\neq\phi$ or $U\cap[D_{\gamma}(A)-\{x\}]\neq\phi$. Now, it follows similarly from (4) that $U\cap[A-\{x\}]\neq\phi$. Hence, $x\in D_{\gamma}(A)$. Therefore, in any case, $D_{\gamma}[A\cup D_{\gamma}(A)]\subseteq[A\cup D_{\gamma}(A)]$.

In general, the converse of (1) may not be true and the equality does not hold in (3) of Theorem 3.2.

Example 3.3. *Let* $X=\{a,b,c\}$ *with topology* $T=\{\phi,\{a\},X\}$. *Thus* $\tau^{\gamma}=\{\phi,\{a\},\{a,b\},\{a,c\},X\}$.

Consider the following:

$(i)\ A = \{c\}$. *Then*, $D(A) = \{b\}$ *and* $D_\gamma(A) = \phi$. *Hence*, $D(A) \not\subseteq D_\gamma(A)$;

$(ii)\ C = \{a\}$ *and* $E = \{b,c\}$. *Then*, $D_\gamma(C) = \{b,c\}$ *and* $D_\gamma(E) = \phi$. *Hence*, $D_\gamma(C \cup E) \neq D_\gamma(C) \cup D_\gamma(E)$.

Theorem 3.4. *For any subset A of a space X*, $Cl_\gamma(A) = A \cup D_\gamma(A)$.

Proof *Since* $D_\gamma(A) \subseteq Cl_\gamma(A)$, $A \cup D_\gamma(A) \subseteq Cl_\gamma(A)$. *On the other hand*, *let* $x \in Cl_\gamma(A)$. *If* $x \in A$, *then the proof is complete*. *If* $x \notin A$, *each* $\gamma-$*open set U containing x intersects A at a point distinct from x*; *so* $x \in D_\gamma(A)$. *Thus*, $Cl_\gamma(A) \subseteq [A \cup D_\gamma(A)]$, *which completes the proof*.

Corollary 3.5. *A subset A is* $\gamma-$*closed if and only if it contains the set of its* $\gamma-$*Iimit points*.

Definition 3.6. $[p.178, Min, 2002]$. *A point* $x \in X$ *is said to be a* $\gamma-$*interior point of A if there exists a* $\gamma-$*open set U containing x such that* $U \subseteq A$. *The set of all* $\gamma-$*interior points of A is said to be* $\gamma-$*interior of A and denoted by* $Int_\gamma(A)$.

Theorem 3.7. *For subsets A,B of a space X*, *the following statements are true*:

(1) $Int_\gamma(A)$ is the largest $\gamma-$*open* set contained in A;
(2) A is $\gamma-$*open* if and only if $A = Int_\gamma(A)$;
(3) $Int_\gamma[Int_\gamma(A)] = Int_\gamma(A)$;
(4) $Int_\gamma(A) = [A - D_\gamma(X-A)]$;
(5) $[X - Int_\gamma(A)] = Cl_\gamma(X-A)$;
(6) $[X - Cl_\gamma(A)] = Int_\gamma(X-A)$;
(7) $A \subseteq B$, then $Int_\gamma(A) \subseteq Int_\gamma(B)$;
(8) $Int_\gamma(A) \cup Int_\gamma(B) \subseteq Int_\gamma(A \cup B)$;
(9) $Int_\gamma(A \cap B) \subseteq Int_\gamma(A) \cap Int_\gamma(B)$.

Proof (4) If $x \in [A - D_\gamma(X-A)]$, then $x \notin D_\gamma(X-A)$ and so there exists a $\gamma-$*open* set U containing x such that $U \cap (X-A) = \phi$. Then, $x \in U \subseteq A$ and hence $x \in Int_\gamma(A)$, that is, $[A - D_\gamma(X-A)] \subseteq Int_\gamma(A)$. On the other hand, if $x \in Int_\gamma(A)$, then $x \notin D_\gamma(X-A)$ since $Int_\gamma(A)$ is $\gamma-$*open* and $[Int_\gamma(A) \cap (X-A)] = \phi$. Hence, $Int_\gamma(A) = A - D_\gamma(X-A)$.

(5) $X - Int_\gamma(A) = X - [A - D_\gamma(X-A)] = (X-A) \cup D_\gamma(X-A) = Cl_\gamma(X-A)$.

In general, the converse of (8) may not be true.

Example 3.8. *Let* $X = \{a,b,c,d\}$ *with topology* $\tau = \{\phi, \{a\}, \{a,b\}, X\}$. *Thus*, $\tau^\gamma = \{\phi, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, X\}$.

Let $A = \{a,c\}$, $B = \{b,c\}$. *Then*, $Int_\gamma(A) = \{a,c\}$, $Int_\gamma(B) = \phi$ *and* $Int_\gamma(A \cup B) = \{a,b,c\}$. *Hence*, $Int_\gamma(A \cup B) \neq Int_\gamma(A) \cup Int_\gamma(B)$.

Definition 3.9. $Bd_\gamma(A) = A - Int_\gamma(A)$ *is said to be the* $\gamma-$*border of A*.

Theorem 3.10. *For a subset A of a space X*, *the following statements hold*:

(1) $Bd_\gamma(A) \subseteq Bd(A)$ *where* $Bd(A)$ *denotes the border of A*;

(2) $A = Int_\gamma(A) \cup Bd_\gamma(A)$;

(3) $Int_\gamma(A) \cap Bd_\gamma(A) = \phi$;

(4) A *is a* $\gamma-$*open set if and only if* $Bd_\gamma(A) = \phi$;

(5) $Bd_\gamma[Int_\gamma(A)] = \phi$;

(6) $Int_\gamma[Bd_\gamma(A)] = \phi$;

(7) $Bd_\gamma[Bd_\gamma(A)] = Bd_\gamma(A)$;

(8) $Bd_\gamma(A) = A \cap [Cl_\gamma(X - A)]$;

(9) $Bd_\gamma(A) = D_\gamma(X - A)$.

Proof (6) *If* $x \in Int_\gamma[Bd_\gamma(A)]$, *then* $x \in Bd_\gamma(A)$. *On the other hand, since* $Bd_\gamma(A) \subseteq A$, $x \in Int_\gamma[Bd_\gamma(A)] \subseteq Int_\gamma(A)$. *Hence*, $x \in Int_\gamma(A) \cap Bd_\gamma(A)$, *which contradicts* (3). *Thus*, $Int_\gamma[Bd_\gamma(A)] = \phi$.

(8) $Bd_\gamma(A) = A - Int_\gamma(A) = A - [X - Cl_\gamma(X - A)] = A \cap Cl_\gamma(X - A)$.

(9) $Bd_\gamma(A) = A - Int_\gamma(A) = A - [A - D_\gamma(X - A)] = D_\gamma(X - A)$.

In general, the converse of Theorem 3.10(1) may not be true.

Example 3.11. *Consider the topological space* (X, τ) *given in Example* 3.3. *If* $A = \{a, b\}$, *then* $Bd_\gamma(A) = \phi$ *and* $Bd(A) = \{b\}$. *Hence*, $Bd(A) \not\subseteq Bd_\gamma(A)$.

Definition 3.12. $Fr_\gamma(A) = Cl_\gamma(A) - Int_\gamma(A)$ *is said to be the* $\gamma-$*frontier of* A.

Theorem 3.13. *For a subset* A *of a space* X, *the following statements hold*:

(1) $Fr_\gamma(A) \subseteq Fr(A)$ *where* $Fr(A)$ *denotes the frontier of* A;

(2) $Cl_\gamma(A) = Int_\gamma(A) \cup Fr_\gamma(A)$;

(3) $Int_\gamma(A) \cap Fr_\gamma(A) = \phi$;

(4) $Bd_\gamma(A) \subseteq Fr_\gamma(A)$;

(5) $Fr_\gamma(A) = Bd_\gamma(A) \cup D_\gamma(A)$;

(6) A *is a* $\gamma-$*open set if and only if* $Fr_\gamma(A) = D_\gamma(A)$;

(7) $Fr_\gamma(A) = Cl_\gamma(A) \cap Cl_\gamma(X - A)$;

(8) $Fr_\gamma(A) = Fr_\gamma(X - A)$;

(9) $Fr_\gamma(A)$ *is* $\gamma-$*closed*;

(10) $Fr_\gamma[Fr_\gamma(A)] \subseteq Fr_\gamma(A)$;

(11) $Fr_\gamma[Int_\gamma(A)] \subseteq Fr_\gamma(A)$;

(12) $Fr_\gamma[Cl_\gamma(A)] \subseteq Fr_\gamma(A)$;

(13) $Int_\gamma(A) = A - Fr_\gamma(A)$.

Proof (2) $Int_\gamma(A) \cup Fr_\gamma(A) = Int_\gamma(A) \cup [Cl_\gamma(A) - Int_\gamma(A)] = Cl_\gamma(A)$.

(3) $Int_\gamma(A) \cap Fr_\gamma(A) = Int_\gamma(A) \cap [Cl_\gamma(A) - Int_\gamma(A)] = \phi$.

(5) *Since* $Int_\gamma(A) \cup Fr_\gamma(A) = Int_\gamma(A) \cup Bd_\gamma(A) \cup D_\gamma(A)$, $Fr_\gamma(A) = Bd_\gamma(A) \cup D_\gamma(A)$.

(7) $Fr_\gamma(A) = Cl_\gamma(A) - Int_\gamma(A) = Cl_\gamma(A) \cap Cl_\gamma(X - A)$

(9) $Cl_\gamma[Fr_\gamma(A)] = Cl_\gamma[Cl_\gamma(A) \cap Cl_\gamma(X - A)]$
$\subseteq Cl_\gamma[Cl_\gamma(A)] \cap Cl_\gamma[Cl_\gamma(X - A)] = Cl_\gamma(A) \cap Cl_\gamma(X - A) = Fr_\gamma(A)$.
Hence, $Fr_\gamma(A)$ *is* γ − *closed*.

(10) $Fr_\gamma[Fr_\gamma(A)] = Cl_\gamma[Fr_\gamma(A)] \cap Cl_\gamma[X - Fr_\gamma(A)] \subseteq Cl_\gamma[Fr_\gamma(A)] = Fr_\gamma(A)$.

(12) $Fr_\gamma(Cl_\gamma(A)) = Cl_\gamma[Cl_\gamma(A)] - Int_\gamma[Cl_\gamma(A)]$
$= Cl_\gamma(A) - Int_\gamma[Cl_\gamma(A)] \subseteq [Cl_\gamma(A) - Int_\gamma(A)] = Fr_\gamma(A)$.

(13) $A - Fr_\gamma(A) = A - [Cl_\gamma(A) - Int_\gamma(A)] = Int_\gamma(A)$.

The converse of (1) *and* (4) *of Theorem* 3.13 *are not true in general*, *as shown by Example* 3.14.

Example 3.14. *Consider the topological space* (X, τ) *given in Example* 3.3. *If* $A = \{c\}$, *then* $Fr(A) = \{b, c\} \not\subseteq \{c\} = Fr_\gamma(A)$, *and if* $B = \{a, b\}$, *then* $Fr_\gamma(B) = \{c\} \not\subseteq Bd_\gamma(B) = \phi$.

Definition 3.15. $Ext_\gamma(A) = Int_\gamma(X - A)$ *is said to be a* γ − *exterior of* A.

Theorem 3.16. *For a subset* A *of a space* X, *the following statements hold*:

(1) $Ext(A) \subseteq Ext_\gamma(A)$ *where* $Ext(A)$ *denotes the exterior of* A;

(2) $Ext_\gamma(A)$ *is* γ − *open*;

(3) $Ext_\gamma(A) = Int_\gamma(X - A) = X - Cl_\gamma(A)$;

(4) $Ext_\gamma[Ext_\gamma(A)] = Int_\gamma[Cl_\gamma(A)]$;

(5) *If* $A \subseteq B$, *then* $Ext_\gamma(A) \supseteq Ext_\gamma(B)$;

(6) $Ext_\gamma(A \cup B) \subseteq Ext_\gamma(A) \cup Ext_\gamma(B)$;

(7) $Ext_\gamma(A) \cap Ext_\gamma(B) \subseteq Ext_\gamma(A \cap B)$;

(8) $Ext_\gamma(X) = \phi$;

(9) $Ext_\gamma(\phi) = X$;

(10) $Ext_\gamma(A) = Ext_\gamma[X - Ext_\gamma(A)]$;

(11) $Int_\gamma(A) \subseteq Ext_\gamma[Ext_\gamma(A)]$;

(12) $X = Int_\gamma(A) \cup Ext_\gamma(A) \cup Fr_\gamma(A)$;

(13) $Ext_\gamma(A) \cup Ext_\gamma(B) \subseteq Ext_\gamma(A \cap B)$.

Proof (4) $Ext_\gamma[Ext_\gamma(A)] = Ext_\gamma[X - Cl_\gamma(A)] = Int_\gamma[X - (X - Cl_\gamma(A))] = Int_\gamma[Cl_\gamma(A)]$.

(10) $Ext_\gamma[X - Ext_\gamma(A)] = Ext_\gamma[X - Int_\gamma(X - A)] = Int_\gamma[X - (X - Int_\gamma(X - A))]$
$= Int_\gamma[Int_\gamma(X - A)] = Int_\gamma(X - A) = Ext_\gamma(A)$.
(11) $Int_\gamma(A) \subseteq Int_\gamma[Cl_\gamma(A)] = Int_\gamma[X - Int_\gamma(X - A)]$
$= Int_\gamma[X - Ext_\gamma(A)] = Ext_\gamma[Ext_\gamma(A)]$.
(13) $Ext_\gamma(A) \cup Ext_\gamma(B) = Int_\gamma(X - A) \cup Int_\gamma(X - B)$
$\subseteq Int_\gamma[(X - A) \cup (X - B)] = Int_\gamma[X - (A \cap B)] = Ext_\gamma(A \cap B)$.

Example 3.17. *Let* $X = \{1,2,3,4\}$ *with topology* $\tau = \{\phi, \{3,4\}, X\}$.

$\tau^\gamma = \{\phi, \{3,4\}, \{2,3,4\}, \{1,3,4\}, X\}$. *Let* $A = \{1\}$ *and* $B = \{2\}$. *Then* $Ext_\gamma(A) \neq Ext(A)$, $Ext_\gamma(A \cap B) \neq Ext_\gamma(A) \cap Ext_\gamma(B)$, $Ext_\gamma(A \cup B) \neq Ext_\gamma(A) \cup Ext_\gamma(B)$, *and* $Ext_\gamma(A) \cup Ext_\gamma(B) \neq Ext_\gamma(A \cap B)$.

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