Characterizations of Mappings in $\gamma$-Open Sets

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Abstract. The notion of semi-convergence of filters was introduced by Latif (1999) who investigated some characterizations related to semi-open continuous functions. In the spirit of Latif (1999), Min (2002) used the idea of semi-convergence of filters to introduce a new class of sets, called $\gamma$–open sets, and the notions of $\gamma$–closure, $\gamma$–interior and $\gamma$–continuity and investigated some properties. In this paper we introduce $\gamma$–continuous, $\gamma$–irresolute, $\gamma$–open, $\gamma$–closed, pre–$\gamma$–open and pre–$\gamma$–closed mappings and investigate properties and characterizations of these new types of mappings.

1. Introduction

The notion of $\gamma$–open set (originally called $\gamma$–sets) in topological spaces was introduced by Min [Min, 2002]. We continue to explore further properties and characterizations of $\gamma$–continuous, $\gamma$–irresolute and $\gamma$–open mappings. We also introduce and study properties and characterizations of $\gamma$–closed, pre–$\gamma$–open and pre–$\gamma$–closed mappings.

Throughout this paper, $(X, \tau)$ (simply $X$) always mean topological space on which no separation axioms are assumed unless explicitly stated. Let $S$ be a subset of $X$. The closure (resp., interior) of $S$ will be denoted by $\text{Cl}(S)$ (resp., $\text{Int}(S)$). A subset $S$ of $X$ is called a semi-open set [Levine, 1963] (resp., $\alpha$–open set [Njastad, 1965]) if $S \subseteq \text{Cl}(\text{Int}(S))$ (resp., $S \subseteq \text{Int}([\text{Cl}(\text{Int}(S))]$)). The complement of a semi-open set (resp., $\alpha$–open set) is called semi-closed set (resp., $\alpha$–closed set). The family of all semi-open sets (resp., $\alpha$–open sets) in a topological space $(X, \tau)$ will be denoted by $\text{SO}(X)$ (resp., $\tau^\circ$). A subset $M(x)$ of a space $X$ is called a semi-neighbourhood of a point $x \in X$ if there exists a semi-open set $S$ such that $x \in S \subseteq M(x)$. In [Latif, 1999] Latif introduced the notion of semi-convergence of filters and investigated some characterizations related to semi-open continuous function. Now, we recall the concept of semi-convergence of filters. Let $S(x) = \{A \in \text{SO}(X) : x \in A\}$ and let $S_x = \{A \subseteq X :$ there exists $\mu \subseteq S(x)$ such that $\mu$ is finite and $\cap \mu \subseteq A\}$. Then, $S_x$ is called the semi-neighbourhood filter at $x$. For any filter $\Gamma$ on $X$, we say that $\Gamma$ semi-converges to $x$ if and only if $\Gamma$ is finer than the semi-neighbourhood filter at $x$.

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A subset $U$ of $X$ is called a $\gamma - open$ set if whenever a filter $\Gamma$ semi-converges to $x$ and $x \in U$, $U \in \Gamma$. The complement of a $\gamma - open$ set is called a $\gamma - closed$ set. The intersection of all $\gamma - closed$ sets containing $A$ is called the $\gamma - closure$ of $A$, denoted by $Cl_\gamma(A)$. A subset $A$ is also $\gamma - closed$ if and only if $A = Cl_\gamma(A)$. We denote the family of all $\gamma - open$ sets of $(X, \tau)$ by $\tau^\gamma$. It is shown in [Min, 2002] that $\tau^\gamma$ is a topology on $X$. In a topological space $(X, \tau)$, it is always true that $\tau \subseteq \tau^\alpha \subseteq S(X) \subseteq \tau^\gamma$.

2. Characterizations of Mappings

The purpose of this section is to explore properties and characterizations of $\gamma - continuous$, $\gamma - irresolute$, $\gamma - open$, $\gamma - closed$, $pre - \gamma - open$ and $pre - \gamma - closed$ functions.

A. $\gamma - Continuous$ Functions

In [Min, 2002] Min introduced the notion of $\gamma - continuous$ mapping and gave certain characterizations of it. The purpose of this section is to investigate further properties and characterizations of $\gamma - continuous$ functions.

Definition 2.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\gamma - continuous$ if $f^{-1}(V) \in \tau^\gamma$ for every $V \in \sigma$.

Theorem 2.2. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then the following are equivalent:

1. $f$ is $\gamma - continuous$;
2. The inverse image of each closed set in $Y$ is a $\gamma - closed$ set in $X$;
3. $Cl_\gamma(f^{-1}(V)) \subseteq f^{-1}[Cl(V)]$, for every $V \subseteq Y$;
4. $f[Cl_\gamma(U)] \subseteq Cl[f(U)]$, for every $U \subseteq X$;
5. For any point $x \in X$ and any open set $V$ of $Y$ containing $f(x)$, there exists $U \in \tau^\gamma$ such that $x \in U$ and $f(U) \subseteq V$;
6. $Bd_\gamma(f^{-1}(V)) \subseteq f^{-1}[Bd(V)]$, for every $V \subseteq Y$;
7. $f[D_\gamma(U)] \subseteq Cl[f(U)]$, for every $U \subseteq X$;
8. $f^{-1}[Int(V)] \subseteq Int_\gamma[f^{-1}(V)]$, for every $V \subseteq Y$.

Proof (1) $\Rightarrow$ (2) : Let $F \subseteq Y$ be closed. Since $f$ is $\gamma - continuous$, $f^{-1}(Y - F) = X - f^{-1}(F)$ is $\gamma - open$. Therefore, $f^{-1}(F)$ is $\gamma - closed$ in $X$.

(2) $\Rightarrow$ (3) : Since $Cl(V)$ is closed for every $V \subseteq Y$, then $f^{-1}[Cl(V)]$ is $\gamma - closed$. Therefore $f^{-1}[Cl(V)] = Cl_\gamma(f^{-1}(Cl(V))) \supseteq Cl_\gamma(f^{-1}(V))$.

(3) $\Rightarrow$ (4) : Let $U \subseteq X$ and $f(U) = V$. Then $f^{-1}[Cl(V)] \supseteq Cl_\gamma(f^{-1}(V))$. Thus $f^{-1}[Cl(f(U))] \supseteq Cl_\gamma(f^{-1}(f(U))) \supseteq Cl_\gamma(U)$ and $Cl[f(U)] \supseteq f[Cl_\gamma(U)]$. 
Remark 2.4. Let \( W \subseteq Y \) be a closed set, and \( U = f^{-1}(W) \), then
\[
\{ f(Cl_f(U)) \subseteq Cl([f(U)] = Cl([f(f^{-1}(W))]) \subseteq Cl(W) = W. \]
Thus
\[
Cl_f(U) \subseteq f^{-1}(Cl_f(U)) \subseteq f^{-1}(W) = U. \]
Therefore \( U \) is \( \gamma \)-closed.

\begin{enumerate}
\item (2) \( \Rightarrow \) (1): Let \( V \subseteq Y \) be an open set, then \( Y - V \) is closed. Then \( f^{-1}(Y - V) = X - f^{-1}(V) \) is \( \gamma \)-closed in \( X \) and hence \( f^{-1}(V) \) is \( \gamma \)-open in \( X \).
\item (1) \( \Rightarrow \) (5): Let \( f : X \to Y \) be \( \gamma \)-continuous. For any \( x \in X \) and any open set \( V \) of \( Y \) containing \( f(x) \), \( U = f^{-1}(V) \in \tau' \), and \( f(U) = f([f^{-1}(V)]) \subseteq V \).
\item (5) \( \Rightarrow \) (1): Let \( V \subseteq \sigma \). We prove \( f^{-1}(V) \in \tau' \). Let \( x \in f^{-1}(V) \). Then \( f(x) \in V \) and there exists \( U \in \tau' \) such that \( x \in U \) and \( f(x) \in f(U) \subseteq V \). Hence \( x \in U \subseteq f^{-1}[f(U)] \subseteq f^{-1}(V) \). It shows that \( f^{-1}(V) \) is a \( \gamma \)-neighborhood of each of its points. Therefore \( f^{-1}(V) \in \tau' \).
\item (6) \( \Rightarrow \) (8): Let \( V \subseteq Y \). Then by hypothesis, \( f^{-1}(V) \subseteq \gamma^{-1}(U) \subseteq f^{-1}[Bd(V)] \)
\[
\Rightarrow f^{-1}(V) - Int_x[f^{-1}(V)] \subseteq f^{-1}[V - Int(V)] = f^{-1}(V) - f^{-1}[Int(V)]
\]
\[
\Rightarrow f^{-1}[Int(V)] \subseteq f^{-1}[Bd(V)].
\]
\item (8) \( \Rightarrow \) (6): Let \( V \subseteq Y \). Then by hypothesis, \( f^{-1}[Int(V)] \subseteq Int_x[f^{-1}(V)] \)
\[
\Rightarrow f^{-1}(V) - Int_x[f^{-1}(V)] \subseteq f^{-1}(V) - f^{-1}[Int(V)] = f^{-1}[V - Int(V)]
\]
\[
\Rightarrow Bd_x[f^{-1}(V)] \subseteq f^{-1}[Bd(V)]
\]
\item (1) \( \Rightarrow \) (7): It is obvious, since \( f \) is \( \gamma \)-continuous and by (4) \( f(Cl_f(U)) \subseteq Cl(f(U)) \) for each \( U \subseteq X \). So \( f(D_f(U)) \subseteq Cl(f(U)) \).
\item (7) \( \Rightarrow \) (1): Let \( U \subseteq Y \) be an open set, \( V = Y - U \) and \( f^{-1}(V) = W \). Then by hypothesis
\[
f(D_f(W)) \subseteq Cl([f(W)] = Cl([f(f^{-1}(V))]) \subseteq Cl(V) = W.
\]
Thus \( D_f(f^{-1}(V)) \subseteq f^{-1}(V) \) and \( f^{-1}(V) \) is \( \gamma \)-closed. Therefore, \( f \) is \( \gamma \)-continuous.
\item (1) \( \Rightarrow \) (8): Let \( V \subseteq Y \). Then \( f^{-1}[Int(V)] \) is \( \gamma \)-open in \( X \). Thus
\[
f^{-1}[Int(V)] = Int_x[f^{-1}(Int(V))]) \subseteq Int_x[f^{-1}(V)].
\]
\item (8) \( \Rightarrow \) (1): Let \( V \subseteq Y \) be an open set. Then \( f^{-1}(V) = f^{-1}[Int(V)] \subseteq Int_x[f^{-1}(V)]. \) Therefore, \( f^{-1}(V) \) is \( \gamma \)-open. Hence \( f \) is \( \gamma \)-continuous.
\end{enumerate}

In the next Theorem, \( \#_f - c. \) denotes the set of points \( x \) of \( X \) for which a function \( f : (X, \tau) \to (Y, \sigma) \) is not \( \gamma \)-continuous.

**Theorem 2.3.** \( \#_f - c. \) is identical with the union of the \( \gamma \)-frontiers of the inverse images of \( \gamma \)-open sets containing \( f(x) \).

**Proof** Suppose that \( f \) is not \( \gamma \)-continuous at point \( x \) of \( X \). Then there exists an open set \( V \subseteq Y \) containing \( f(x) \) such that \( f(U) \) is not a subset of \( V \) for every \( U \in \tau' \) containing \( x \). Hence, we have \( U \cap [X - f^{-1}(V)] \neq \emptyset \) for every \( U \in \tau' \) containing \( x \). It follows that \( x \in Cl_f[X - f^{-1}(V)] \). We also have \( x \in f^{-1}(V) \subseteq Cl_f[f^{-1}(V)] \). This means that \( x \in Fr_f(f^{-1}(V)) \).

Now, let \( f \) be \( \gamma \)-continuous at \( x \) in \( X \) and let \( V \in \gamma \) be an open set containing \( f(x) \). Then, \( x \in f^{-1}(V) \) is a \( \gamma \)-open set of \( X \). Thus, \( x \in Int_x[f^{-1}(V)] \) and therefore \( x \notin Fr_f[f^{-1}(V)] \) for every open set \( V \) containing \( f(x) \).

**Remark 2.4.** (1) Every continuous function is \( \gamma \)-continuous but the converse may not be true.

(2) If a function \( f : (X, \tau) \to (Y, \sigma) \) is \( \gamma \)-continuous and a function \( g : (Y, \sigma) \to (Z, \theta) \) is \( \gamma \)-continuous, then \( g \circ f : (X, \tau) \to (Z, \theta) \) may not be \( \gamma \)-continuous.

(3) If a function \( f : (X, \tau) \to (Y, \sigma) \) is \( \gamma \)-continuous and a function \( g : (Y, \sigma) \to (Z, \theta) \)
is continuous, then \( g \circ f : (X, \tau) \rightarrow (Z, \mathcal{G}) \) is \( \gamma \)–continuous.

(4) Let \((X, \tau)\) and \((Y, \sigma)\) be topological spaces. If \( f : X \rightarrow Y \) is a function, and one of the following holds, then \( f \) is continuous.

(a) \( f^{-1}[\text{Int}_{\gamma}(B)] \subseteq \text{Int}[f^{-1}(B)] \) for each \( B \subseteq Y \),
(b) \( \text{Cl}[f^{-1}(B)] \subseteq f^{-1}[\text{Cl}_{\gamma}(B)] \) for each \( B \subseteq Y \),
(c) \( f[\text{Cl}(A)] \subseteq \text{Cl}_{\gamma}[f(A)] \) for each \( A \subseteq X \).

Lemma 2.5. Let \( A \subseteq Y \subseteq X \), \( Y \) is \( \gamma \)–open in \( X \) and \( A \) is \( \gamma \)–open in \( Y \). Then \( A \) is \( \gamma \)–open in \( X \).

Proof Since \( A \) is \( \gamma \)–open in \( Y \), there exists an open set \( U \subseteq X \) such that \( A = Y \cap U \). Thus \( A \) being the intersection of two \( \gamma \)–open sets in \( X \), is \( \gamma \)–open in \( X \).

Theorem 2.6. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a mapping and \( \{U_i : i \in I\} \) be a cover of \( X \) such that \( U_i \in \tau^I \) for each \( i \in I \). Suppose that \( f|U_i : U_i \rightarrow Y \) is \( \gamma \)–continuous for each \( i \in I \). Then prove that \( f \) is \( \gamma \)–continuous.

Proof Let \( V \subseteq Y \) be an open set, then \( (f|U_i)^{-1}(V) \) is \( \gamma \)–open in \( U_i \) for each \( i \in I \). Since \( U_i \) is \( \gamma \)–open in \( X \) for each \( i \in I \). So by Lemma 2.5, \( (f|U_i)^{-1}(V) \) is \( \gamma \)–open in \( X \) for each \( i \in I \). But, \( f^{-1}(V) = \bigcup \{ (f|U_i)^{-1}(V) : i \in I \} \), then \( f^{-1}(V) \in \tau^I \) because \( \tau^I \) is a topology on \( X \). This implies that \( f \) is \( \gamma \)–continuous.

\section{B. \( \gamma \)–Irresolute Functions}

In this section, the functions to be considered are those for which inverses of \( \gamma \)–open sets are \( \gamma \)–open. We investigate some new properties and characterizations of such functions.

Definition 2.7. [Min, 2002]. Let \((X, \tau)\) and \((Y, \sigma)\) be topological spaces. A function \( f : X \rightarrow Y \) is called \( \gamma \)–irresolute if the inverse image of each \( \gamma \)–open set of \( Y \) is a \( \gamma \)–open set in \( X \).

Theorem 2.8. [Min, 2002]. Let \((X, \tau) \rightarrow (Y, \sigma)\) be a function between topological spaces. Then the following are equivalent:

(1) \( f \) is \( \gamma \)–irresolute;
(2) the inverse image of each \( \gamma \)–closed set in \( Y \) is a \( \gamma \)–closed set;
(3) \( \text{Cl}_{\gamma}[f^{-1}(V)] \subseteq f^{-1}[\text{Cl}_{\gamma}(V)] \) for every \( V \subseteq Y \);
(4) \( f[\text{Cl}_{\gamma}(U)] \subseteq \text{Cl}_{\gamma}[f(U)] \) for every \( U \subseteq X \);
(5) \( f^{-1}[\text{Int}_{\gamma}(B)] \subseteq \text{Int}_{\gamma}[f^{-1}(B)] \) for every \( B \subseteq Y \).

Theorem 2.9. Prove that a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \gamma \)–irresolute if and only if for each
point \( p \) in \( X \) and each \( \gamma \)-open set \( B \) in \( Y \) with \( f(p) \in B \), there is a \( \gamma \)-open set \( A \) in \( X \) such that \( p \in A, f(A) \subseteq B \).

**Proof**  
Necessity. Let \( p \in X \) and \( B \in \sigma^\gamma \) such that \( f(p) \in B \). Let \( A = f^{-1}(B) \). Since \( f \) is \( \gamma \)-irresolute, \( A \) is \( \gamma \)-open in \( X \). Also \( p \in f^{-1}(B) = A \) as \( f(p) \in B \). Thus we have 
\[ f(A) = f[f^{-1}(B)] \subseteq B. \]

Sufficiency. Let \( B \in \sigma^\gamma \), let \( A = f^{-1}(B) \). We show that \( A \) is \( \gamma \)-open in \( X \). For this let \( x \in A \). It implies that \( f(x) \in B \). Then by hypothesis, there exists \( A_x \in \tau^\gamma \) such that \( x \in A_x \) and \( f(A_x) \subseteq B \). Then \( A_x \subseteq f^{-1}[f(A_x)] \subseteq f^{-1}(B) = A \). Thus \( A = \bigcup \{ A_x : x \in A \} \). It follows that \( A \) is \( \gamma \)-open in \( X \). Hence \( f \) is \( \gamma \)-irresolute.

**Definition** 2.10. Let \( (X, \tau) \) be a topological space. Let \( x \in X \) and \( N \subseteq X \). We say that \( N \) is a \( \gamma \)-neighbourhood of \( x \) if there exists a \( \gamma \)-open set \( M \) of \( X \) such that \( x \in M \subseteq N \).

**Theorem** 2.11. Prove that a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \gamma \)-irresolute if and only if for each \( x \in X \), the inverse image of each \( \gamma \)-neighbourhood of \( f(x) \), is a \( \gamma \)-neighbourhood of \( x \).

**Proof**  
Necessity. Let \( x \in X \) and let \( B \) be a \( \gamma \)-neighbourhood of \( f(x) \). Then there exists \( U \in \sigma^\gamma \) such that \( f(x) \in U \subseteq B \). This implies that \( x \in f^{-1}(U) \subseteq f^{-1}(B) \). Since \( f \) is \( \gamma \)-irresolute, so \( f^{-1}(U) \in \tau^\gamma \). Hence \( f^{-1}(B) \) is a \( \gamma \)-neighbourhood of \( x \).

Sufficiency. Let \( B \in \sigma^\gamma \). Put \( A = f^{-1}(B) \). Let \( x \in A \). Then \( f(x) \in B \). But then, \( B \) being \( \gamma \)-open set, is a \( \gamma \)-neighborhood of \( f(x) \). So by hypothesis, \( A = f^{-1}(B) \) is a \( \gamma \)-neighborhood of \( x \). Hence by definition, there exists \( A_x \in \tau^\gamma \) such that \( x \in A_x \subseteq A \). Thus \( A = \bigcup \{ A_x : x \in A \} \). It follows that \( A \) is a \( \gamma \)-open set in \( X \). Therefore \( f \) is \( \gamma \)-irresolute.

**Theorem** 2.12. Prove that a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \gamma \)-irresolute if and only if for each \( x \in X \), and each \( \gamma \)-neighborhood \( U \) of \( f(x) \), there is a \( \gamma \)-neighborhood \( V \) of \( x \) such that \( f(V) \subseteq U \).

**Proof**  
Necessity. Let \( x \in X \) and let \( U \) be a \( \gamma \)-neighborhood of \( f(x) \). Then there exists \( O_{f(x)} \in \sigma^\gamma \) such that \( f(x) \in O_{f(x)} \subseteq U \). It follows that \( x \in f^{-1}[O_{f(x)}] \subseteq f^{-1}(U) \). By hypothesis, \( f^{-1}[O_{f(x)}] \in \tau^\gamma \). Let \( V = f^{-1}(U) \). Then it follows that \( V \) is a \( \gamma \)-neighborhood of \( x \) and \( f(V) \subseteq f[f^{-1}(U)] \subseteq U \).

Sufficiency. Let \( B \in \sigma^\gamma \). Put \( O = f^{-1}(B) \). Let \( x \in O \). Then \( f(x) \in B \). Thus \( B \) is a \( \gamma \)-neighborhood of \( f(x) \). So by hypothesis, there exists a \( \gamma \)-neighborhood \( V_x \) of \( x \) such that \( f(V_x) \subseteq B \). Thus it follows that \( x \in V_x \subseteq f^{-1}[f(V_x)] \subseteq f^{-1}(B) = O \). Since \( V_x \) is a \( \gamma \)-neighborhood of \( x \), so there exists an \( O_x \in \tau^\gamma \) such that \( x \in O_x \subseteq V_x \). Hence \( x \in O_x \subseteq O \). \( O_x \in \tau^\gamma \). Thus \( O = \bigcup \{ O_x : x \in O \} \). It follows that \( O \) is \( \gamma \)-open in \( X \). Therefore, \( f \) is \( \gamma \)-irresolute.

**Theorem** 2.13. Prove that a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \gamma \)-irresolute if and only if 
\[ f[D_\gamma(A)] \subseteq f(A) \cup D_\gamma[f(A)], \text{ for all } A \subseteq X. \]

**Proof**  
Necessity. Let \( f : X \rightarrow Y \) be \( \gamma \)-irresolute. Let \( A \subseteq X \), and \( a_0 \in D_\gamma(A) \). Assume that \( f(a_0) \not\in f(A) \) and let \( V \) denote a \( \gamma \)-neighborhood of \( f(a_0) \). Since \( f \) is \( \gamma \)-irresolute, so by Theorem 2.12, there exists a \( \gamma \)-neighborhood \( U \) of \( a_0 \) such that \( f(U) \subseteq V \). From \( a_0 \in D_\gamma(A) \), it follows that \( U \cap A \neq \emptyset \); there exists, therefore, at least one element \( a \in U \cap A \) such that \( f(a) \in f(A) \) and \( f(a) \in V \). Since \( f(a_0) \not\in f(A) \), we have \( f(a) \neq f(a_0) \). Thus every \( \gamma \)-neighborhood of \( f(a_0) \) contains an element of \( f(A) \) different from \( f(a_0) \), consequently, \( f(a_0) \in D_\gamma[f(A)] \). This proves necessity of the condition.

Sufficiency. Assume that \( f \) is not \( \gamma \)-irresolute. Then by Theorem 2.12, there exists \( a_0 \in X \)
and a $\gamma$–neighbourhood $V$ of $f(a_0)$ such that every $\gamma$–neighbourhood $U$ of $a_0$ contains at least one element $a \in U$ for which $f(a) \not\in V$. Put $A = \{ a \in X : f(a) \not\in V \}$. Then $a_0 \not\in A$ since $f(a_0) \in V$, and therefore $f(a_0) \not\in f(A)$; also $f(a_0) \not\in D_\gamma[f(A)]$ since $f(A) \cap (V - \{ f(a_0) \}) = \emptyset$. It follows that $f(a_0) \not\in f[D_\gamma(A)] - [f(A) \cup D_\gamma(f(A))] \neq \emptyset$, which is a contradiction to the given condition. The condition of the Theorem is therefore sufficient and the theorem is proved.

**Theorem 2.14.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $1 - 1$ function. Then $f$ is $\gamma$–irresolute if and only if $f[D_\gamma(A)] \subseteq D_\gamma[f(A)]$, for all $A \subseteq X$.

**Proof** Necessity. Let $f$ be $\gamma$–irresolute. Let $A \subseteq X$, $a_0 \in D_\gamma(A)$ and $V$ be a $\gamma$–neighbourhood of $f(a_0)$. Since $f$ is $\gamma$–irresolute, so by Theorem 2.12, there exists a $\gamma$–neighbourhood $U$ of $a_0$ such that $f(U) \subseteq V$. But $a_0 \in D_\gamma(A)$; hence there exists an element $a \in U \cap A$ such that $a \neq a_0$; then $f(a) \in f(A)$ and, since $f$ is $1 - 1$, $f(a) \neq f(a_0)$. Thus every $\gamma$–neighbourhood $V$ of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$; consequently $f(a_0) \in D_\gamma[f(A)]$. We have therefore $f[D_\gamma(A)] \subseteq D_\gamma[f(A)]$.

**Sufficiency.** Follows from Theorem 2.13.

### C. $\gamma$–Open Functions

In [Min, 2002] Min defined $\gamma$–open mappings as a generalization of open mappings and investigated some properties of such mappings. The purpose of this section is to add some more characterizations of $\gamma$–open mappings.

**Definition 2.15.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. A function $f : X \rightarrow Y$ is called $\gamma$–open if for every open set $G$ in $X$, $f(G)$ is a $\gamma$–open set in $Y$.

**Theorem 2.16.** Prove that a mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\gamma$–open if and only if for each $x \in X$, and $U \in \tau$ such that $x \in U$, there exists a $\gamma$–open set $W \subseteq Y$ containing $f(x)$ such that $W \subseteq f(U)$.

**Proof** Follows immediately from Definition 2.15.

**Theorem 2.17.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be $\gamma$–open. If $W \subseteq Y$ and $F \subseteq X$ is a closed set containing $f^{-1}(W)$, then there exists a $\gamma$–closed $H \subseteq Y$ containing $W$ such that $f^{-1}(H) \subseteq F$.

**Proof** Let $H = Y - f(X - F)$. Since $f^{-1}(W) \subseteq F$, we have $f(X - F) \subseteq (Y - W)$. Since $f$ is $\gamma$–open, then $H$ is $\gamma$–closed and $f^{-1}(H) = X - f^{-1}(f(X - F)) \subseteq X - (X - F) = F$.

**Theorem 2.18.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be $\gamma$–open and let $B \subseteq Y$. Then $f^{-1}[\text{Cl}_\gamma(\text{Int}_\gamma(\text{Cl}_\gamma(B)))] \subseteq \text{Cl}[f^{-1}(B)]$.

**Proof** $\text{Cl}[f^{-1}(B)]$ is closed in $X$ containing $f^{-1}(B)$. By Theorem 2.17, there exists a $\gamma$–closed set $B \subseteq H \subseteq Y$, such that $f^{-1}(H) \subseteq \text{Cl}[f^{-1}(B)]$. Thus, $f^{-1}[\text{Cl}_\gamma(\text{Int}_\gamma(\text{Cl}_\gamma(B)))] \subseteq f^{-1}[\text{Cl}_\gamma(\text{Int}_\gamma(\text{Cl}_\gamma(H)))] \subseteq f^{-1}(H) \subseteq \text{Cl}[f^{-1}(B)]$.

**Theorem 2.19.** Prove that a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\gamma$–open if and only if $f(\text{Int}(A)) \subseteq \text{Int}_\gamma(f(A))$, for all $A \subseteq X$.

**Proof** Necessity. Let $A \subseteq X$. Let $x \in \text{Int}(A)$. Then there exists $U_x \in \tau$ such that $x \in U_x \subseteq A$. So $f(x) \in f(U_x) \subseteq f(A)$ and by hypothesis, $f(U_x) \in \sigma^\gamma$. Hence $f(x) \in \text{Int}_\gamma(f(A))$. Thus $f(\text{Int}(A)) \subseteq \text{Int}_\gamma(f(A))$. 


Sufficiency. Let \( U \in \tau \). Then by hypothesis, \( f(\text{Int}(U)) \subseteq \text{Int}_y(f(U)) \). Since \( \text{Int}(U) = U \) as \( U \) is open. Also \( \text{Int}_y(f(U)) \subseteq f(U) \). Hence \( f(U) = \text{Int}_y(f(U)) \). Thus \( f(U) \) is \( \gamma \)-open in \( Y \). So \( f \) is \( \gamma \)-open.

We remark that the equality does not hold in the preceding Theorem as the following example shows.

**Example 2.20.** Let \( X = Y = \{1, 2\} \). Suppose \( \tau \) be antidiscrete topology on \( X \) and \( \sigma \) be discrete topology on \( Y \). Then \( \tau \) = \( \tau \) and \( \sigma \) = \( \sigma \). Let \( f = \text{Id.}, A = \{1\} \). Then \( \phi = f(\text{Int}(A)) \neq \text{Int}_y[f(A)] = \{1\} \).

**Theorem 2.21.** Prove that a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \gamma \)-open if and only if \( \text{Int}(f^{-1}(B)) \subseteq f^{-1}[\text{Int}_y(B)] \), for all \( B \subseteq Y \).

**Proof** Necessity. Let \( B \subseteq Y \). Since \( \text{Int}(f^{-1}(B)) \) is open in \( X \) and \( f \) is \( \gamma \)-open, \( f(\text{Int}(f^{-1}(B))) \) is \( \gamma \)-open in \( Y \). Also we have \( f(\text{Int}(f^{-1}(B))) \subseteq f(f^{-1}(B)) \subseteq B \). Hence, \( f(\text{Int}(f^{-1}(B))) \subseteq \text{Int}_y(B) \).

Therefore \( \text{Int}(f^{-1}(B)) \subseteq f^{-1}[\text{Int}_y(B)] \).

Sufficiency. Let \( A \subseteq X \). Then \( f(A) \subseteq Y \). Hence by hypothesis, we obtain \( \text{Int}(A) \subseteq \text{Int}(f^{-1}(A)) \subseteq f^{-1}[\text{Int}_y(f(A))] \). Thus \( f(\text{Int}(A)) \subseteq \text{Int}_y[f(A)] \), for all \( A \subseteq X \). Hence, by Theorem 2.19, \( f \) is \( \gamma \)-open.

**Theorem 2.22.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a mapping. Then a necessary and sufficient condition for \( f \) to be \( \gamma \)-open is that \( f^{-1}(\text{Cl}_y(B)) \subseteq \text{Cl}[f^{-1}(B)] \), for every subset \( B \) of \( Y \).

**Proof** Necessity. Assume \( f \) is \( \gamma \)-open. Let \( B \subseteq Y \). Let \( x \in f^{-1}(\text{Cl}_y(B)) \). Then \( f(x) \in \text{Cl}_y(B) \). Let \( U \in \tau \) such that \( x \in U \). Since \( f \) is \( \gamma \)-open, then \( f(U) \) is a \( \gamma \)-open set in \( Y \). Therefore, \( B \cap f(U) \neq \phi \). Then \( B \cap f^{-1}(B) \neq \phi \). Hence \( x \in \text{Cl}[f^{-1}(B)] \). We conclude that \( f^{-1}[\text{Cl}_y(B)] \subseteq \text{Cl}[f^{-1}(B)] \).

Sufficiency. Let \( B \subseteq Y \). Then \( (Y - B) \subseteq Y \). By hypothesis, \( f^{-1}[\text{Cl}_y(Y - B)] \subseteq \text{Cl}[f^{-1}(Y - B)] \). This implies \( X - \text{Cl}[f^{-1}(Y - B)] \subseteq X - f^{-1}[\text{Cl}_y(Y - B)] \). Hence \( X - \text{Cl}[X - f^{-1}(B)] \subseteq f^{-1}[Y - \text{Cl}_y(Y - B)] \). By applying Theorem 10 [Latif, 1993], \( \text{Int}(f^{-1}(B)) \subseteq f^{-1}[\text{Int}_y(B)] \). Now form Theorem 2.21, it follows that \( f \) is \( \gamma \)-open.

**D. \( \gamma \)-Closed Functions**

In this section we introduce \( \gamma \)-closed functions and study certain properties and characterizations of this type of functions.

**Definition 2.23.** A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called \( \gamma \)-closed if the image of each closed set in \( X \) is a \( \gamma \)-closed set.

**Theorem 2.24.** Prove that a mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \gamma \)-closed if and only \( \text{Cl}_y[f(A)] \subseteq f(\text{Cl}(A)) \) for each \( A \subseteq X \).

**Proof** Necessity. Let \( f \) be \( \gamma \)-closed and let \( A \subseteq X \). Then \( f(A) \subseteq f(\text{Cl}(A)) \) and \( f(\text{Cl}(A)) \) is a \( \gamma \)-closed set in \( Y \). Thus \( \text{Cl}_y[f(A)] \subseteq f(\text{Cl}(A)) \).

Conversely, suppose that \( \text{Cl}_y[f(A)] \subseteq f(\text{Cl}(A)) \), for each \( A \subseteq X \). Let \( A \subseteq X \) be a closed set.
Then \( Cl_\gamma[f(A)] \subseteq f(Cl(A)) = f(A) \). This shows that \( f(A) \) is a \( \gamma \) - closed set. Hence \( f \) is \( \gamma \) - closed.

**Theorem 2.25.** Let \( f : (X, \tau) \to (Y, \sigma) \) be \( \gamma \) - closed. If \( V \subseteq Y \) and \( E \subseteq X \) is an open set containing \( f^{-1}(V) \), then there exists a \( \gamma \) - open set \( G \subseteq Y \) containing \( V \) such that \( f^{-1}(G) \subseteq E \).

**Proof** Let \( G = Y - f(X - E) \). Since \( f^{-1}(V) \subseteq E \), we have \( f(X - E) \subseteq Y - V \). Since \( f \) is \( \gamma \) - closed, then \( G \) is a \( \gamma \) - open set and \( f^{-1}(G) = X - f^{-1}[f(X - E)] \subseteq X - (X - E) = E \).

**Theorem 2.26.** Suppose that \( f : (X, \tau) \to (Y, \sigma) \) is a \( \gamma \) - closed mapping. Then \( Int_\gamma[Cl_\gamma(f(A))] \subseteq f(Cl(A)) \) for every subset \( A \) of \( X \).

**Proof** Suppose \( f \) is a \( \gamma \) - closed mapping and \( A \) is an arbitrary subset of \( X \). Then \( f(Cl(A)) \) is \( \gamma \) - closed in \( Y \). Then \( Int_\gamma[Cl_\gamma(f(Cl(A)))] \subseteq f(Cl(A)) \). But \( Int_\gamma[Cl_\gamma(f(Cl(A)))] \subseteq Int_\gamma[Cl_\gamma(f(A))] \). Hence \( Int_\gamma[Cl_\gamma(f(A))] \subseteq f(Cl(A)) \).

**Theorem 2.27.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a \( \gamma \) - closed function, and \( B, C \subseteq Y \).

1. If \( U \) is an open neighbourhood of \( f^{-1}(B) \), then there exists a \( \gamma \) - open neighbourhood \( V \) of \( B \) such that \( f^{-1}(B) \subseteq f^{-1}(V) \subseteq U \).

2. If \( f \) is also onto, then if \( f^{-1}(B) \) and \( f^{-1}(C) \) have disjoint open neighbourhoods, so have \( B \) and \( C \).

**Proof**

1. Let \( V = Y - f(X - U) \). Then \( V^c = Y - V = f(U^c) \). Since \( f \) is \( \gamma \) - closed, \( V \) is a \( \gamma \) - open set. Since \( f^{-1}(B) \subseteq U \), we have \( V^c = f(U^c) \subseteq f[f^{-1}(B^c)] \subseteq B^c \). Hence, \( B \subseteq V \), and thus \( V \) is a \( \gamma \) - open neighbourhood of \( B \). Further \( U^c \subseteq f^{-1}(f(U^c)) = f^{-1}(V^c) = f^{-1}([f^{-1}(V)]^c) \). This proves that \( f^{-1}(V) \subseteq U \).

2. If \( f^{-1}(B) \) and \( f^{-1}(C) \) have disjoint open neighbourhoods \( M \) and \( N \), then by (1), we have \( \gamma \) - open neighbourhoods \( U \) and \( V \) of \( B \) and \( C \) respectively such that \( f^{-1}(B) \subseteq f^{-1}(U) \subseteq Int_\gamma(M) \) and \( f^{-1}(C) \subseteq f^{-1}(V) \subseteq Int_\gamma(N) \). Since \( M \) and \( N \) are disjoint, so are \( Int_\gamma(M) \) and \( Int_\gamma(N) \), and hence \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint as well. It follows that \( U \) and \( V \) are disjoint too as \( f \) is onto.

**Theorem 2.28.** Prove that a surjective mapping \( f : (X, \tau) \to (Y, \sigma) \) is \( \gamma \) - closed if and only if for each subset \( B \) of \( Y \) and each open set \( U \) in \( X \) containing \( f^{-1}(B) \), there exists a \( \gamma \) - open set \( V \) in \( Y \) containing \( B \) such that \( f^{-1}(V) \subseteq U \).

**Proof** Necessity. This follows from (1) of Theorem 2.27.

Sufficiency. Suppose \( F \) is an arbitrary closed set in \( X \). Let \( y \) be an arbitrary point in \( Y - f(F) \). Then \( f^{-1}(y) \subseteq X - f^{-1}[f(F)] \subseteq (X - F) \) and \( (X - F) \) is open in \( X \). Hence by hypothesis, there exists a \( \gamma \) - open set \( V_y \) containing \( y \) such that \( f^{-1}(V_y) \subseteq (X - F) \). This implies that \( y \in V_y \subseteq [Y - f(F)] \). Thus \( Y - f(F) = \cup \{ V_y : y \in Y - f(F) \} \). Hence \( Y - f(F) \), being a union of \( \gamma \) - open sets is \( \gamma \) - open. Thus its complement \( f(F) \) is \( \gamma \) - closed. This shows that \( f \) is \( \gamma \) - closed.

**Theorem 2.29.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a bijection. Then the following are equivalent:

1. \( f \) is \( \gamma \) - closed.
2. \( f \) is \( \gamma \) - open.
3. \( f^{-1} \) is \( \gamma \) - continuous.

**Proof** (a) \( \Rightarrow \) (b) : Let \( U \in \tau \). Then \( X - U \) is closed in \( X \). By (a), \( f(X - U) \) is \( \gamma \) - closed in \( Y \). But \( f(X - U) = f(X) - f(U) = Y - f(U) \). Thus \( f(U) \) is \( \gamma \) - open in \( Y \). This shows that \( f \) is \( \gamma \) - open.
(b) ⇒ (c) : Let \( U \subseteq X \) be an open set. Since \( f \) is \( \gamma \)-open. So \( f(U) = (f^{-1})^{-1}(U) \) is \( \gamma \)-open in \( Y \). Hence \( f^{-1} \) is \( \gamma \)-continuous.

(c) ⇒ (a) : Let \( A \) be an arbitrary closed set in \( X \). Then \( X - A \) is open in \( X \). Since \( f^{-1} \) is \( \gamma \)-continuous, \( (f^{-1})^{-1}(X - A) \) is \( \gamma \)-open in \( Y \). But \( (f^{-1})^{-1}(X - A) = f(X - A) = Y - f(A) \). Thus \( f(A) \) is \( \gamma \)-closed in \( Y \). This shows that \( f \) is \( \gamma \)-closed.

**Example 2.30.** Let \( X = Y = \{x, y, z, \} \) and \( \tau = \sigma = \{X, \emptyset, \{x, y\}, \{x\} \} \). Then, a mapping \( f : X \rightarrow Y \) which is defined by \( f(x) = x, f(y) = z \) and \( f(z) = y \), is \( \gamma \)-open and \( \gamma \)-closed but neither open nor closed.

### E. Pre \( \gamma \) – Open Functions

The purpose of this section is to introduce and discuss certain properties and characterizations of \( \pre \gamma \) – open functions.

**Definition 2.31.** Let \( (X, \tau) \) and \( (Y, \sigma) \) be topological spaces. Then a function \( f : X \rightarrow Y \) is said to be \( \pre \gamma \) – open if and only if for each \( A \in \tau \), \( f(A) \in \sigma \).

**Theorem 2.32.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \mu) \) be any two \( \pre \gamma \) – open functions. Then the composition function \( g \circ f : X \rightarrow Z \) is a \( \pre \gamma \) – open function.

**Proof** Let \( U \in \tau \). Then \( f(U) \in \sigma \) since \( f \) is \( \pre \gamma \) – open. But then \( g(f(U)) \in \mu \) as \( g \) is \( \pre \gamma \) – open. Hence, \( g \circ f \) is \( \pre \gamma \) – open.

**Theorem 2.33.** Prove that a mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \pre \gamma \) – open if and only if for each \( x \in X \) and for any \( \gamma \) – neighbourhood \( U \) of \( x \) in \( X \), there exists \( V \in \sigma \) such that \( f(x) \in V \) and \( V \subseteq f(U) \).

**Proof** Routine.

**Theorem 2.34.** Prove that a mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \pre \gamma \) – open if and only if for each \( x \in X \) and for any \( \gamma \) – neighbourhood \( U \) of \( x \) in \( X \), there exists a \( \gamma \) – neighbourhood \( V \) of \( f(x) \) in \( Y \) such that \( V \subseteq f(U) \).

**Proof** Necessity. Let \( x \in X \) and let \( U \) be a \( \gamma \) – neighbourhood of \( x \). Then there exists \( W \in \tau \) such that \( x \in W \subseteq U \). Then \( f(x) \in f(W) \subseteq f(U) \). But \( f(W) \in \sigma \) as \( f \) is \( \pre \gamma \) – open. Hence \( V = f(W) \) is a \( \gamma \) – neighbourhood of \( f(x) \) and \( V \subseteq f(U) \).

Sufficiency. Let \( U \in \tau \). Let \( x \in U \). Then \( U \) is a \( \gamma \) – neighbourhood of \( x \). So by hypothesis, there exists a \( \gamma \) – neighbourhood \( V \) of \( f(x) \) such that \( f(x) \in V \subseteq f(U) \). It follows at once that \( f(U) \) is a \( \gamma \) – neighbourhood of each of its points. Therefore \( f(U) \) is \( \gamma \) – open. Hence \( f \) is \( \pre \gamma \) – open.

**Theorem 2.35.** Prove that a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \pre \gamma \) – open if and only if \( f(\text{Int}_\tau(A)) \subseteq \text{Int}_\sigma(f(A)) \), for all \( A \subseteq X \).

**Proof** Necessity. Let \( A \subseteq X \). Let \( x \in \text{Int}_\tau(A) \). Then there exists \( U_x \in \tau \) such that \( x \in U_x \subseteq A \). So \( f(x) \in f(U_x) \subseteq f(A) \) and by hypothesis, \( f(U_x) \in \sigma \). Hence \( f(x) \in \text{Int}_\sigma(f(A)) \). Thus \( f(\text{Int}_\tau(A)) \subseteq \text{Int}_\sigma(f(A)) \).

Sufficiency. Let \( U \in \tau \). Then by hypothesis, \( f(\text{Int}_\tau(U)) \subseteq \text{Int}_\sigma(f(U)) \). Since \( \text{Int}_\tau(U) = U \) as \( U \) is \( \gamma \) – open. Also \( \text{Int}_\sigma(f(U)) \subseteq f(U) \). Hence \( f(U) = \text{Int}_\sigma(f(U)) \). Thus \( f(U) \) is \( \gamma \) – open in \( Y \). So \( f \) is
\[ pre - \gamma - open. \]

We remark that the equality does not hold in Theorem 2.35 as the following example shows.

**Example 2.36.** Let \( X = Y = \{1, 2\} \), suppose \( X \) is antidiscrete and \( Y \) is discrete. Let \( f = \text{Id} \), \( A = \{1\} \). Then \( \phi = f[\text{Int}_Y(A)] \neq \text{Int}_X(f(A)) = \{1\} \).

**Theorem 2.37.** Prove that a function \( f : (X, \tau) \to (Y, \sigma) \) is \( pre - \gamma - open \) if and only if \( \text{Int}_Y(f^{-1}(B)) \subseteq f^{-1}[\text{Int}_X(B)] \), for all \( B \subseteq Y \).

**Proof** Necessity. Let \( B \subseteq Y \). Since \( \text{Int}_Y(f^{-1}(B)) \) is \( \gamma - open \) in \( X \) and \( f \) is \( pre - \gamma - open \), \( f[\text{Int}_Y(f^{-1}(B))] \) is \( \gamma - open \) in \( Y \). Also we have \( f[\text{Int}_Y(f^{-1}(B))] \subseteq f^{-1}(B) \). Hence, \( f[\text{Int}_Y(f^{-1}(B))] \subseteq \text{Int}_X(f^{-1}(B)) \). Therefore \( \text{Int}_Y(f^{-1}(B)) \subseteq f^{-1}[\text{Int}_X(B)] \).

Sufficiency. Let \( A \subseteq X \). Then \( f(A) \subseteq Y \). Hence by hypothesis, we obtain \( \text{Int}_X(A) \subseteq \text{Int}_Y(f^{-1}(f(A))) \subseteq f^{-1}[\text{Int}_X(f(A))] \). This implies that \( f[\text{Int}_X(A)] \subseteq f[\text{Int}_X(f(A))] \subseteq f^{-1}[\text{Int}_X(f(A))] \). Thus \( f[\text{Int}_X(A)] \subseteq f^{-1}[\text{Int}_X(f(A))] \), for all \( A \subseteq X \). Hence, by Theorem 2.35, \( f \) is \( pre - \gamma - open \).

**Theorem 2.38.** Prove that a mapping \( f : (X, \tau) \to (Y, \sigma) \) is \( pre - \gamma - open \) if and only if \( \text{Int}_Y(f^{-1}(B)) \subseteq f^{-1}[\text{Cl}_X(f^{-1}(B))] \), for every subset \( B \) of \( Y \).

**Proof** Necessity. Let \( B \subseteq Y \). Now by Theorem 2.39, \( f^{-1}[\text{Cl}_X(f^{-1}(B))] \subseteq \text{Cl}_Y(f^{-1}(B)) \).

Sufficiency. Let \( B \subseteq Y \). Then \( (Y - B) \subseteq Y \). By hypothesis, \( \text{Int}_Y(f^{-1}(X - \text{Cl}_X(Y - B)) \subseteq f^{-1}[\text{Cl}_X(Y - B)] \). Hence \( X - \text{Cl}_X(f^{-1}(Y - B)) \subseteq X - f^{-1}[\text{Cl}_X(Y - B)] \). By Theorem 3.7 (Latif, 2005), \( \text{Int}_Y(f^{-1}(B)) \subseteq f^{-1}[\text{Int}_X(f^{-1}(B))] \). Now by Theorem 2.37, it follows that \( f \) is \( pre - \gamma - open \).

**Theorem 2.39.** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \mu) \) be two mappings such that \( g \circ f : (X, \tau) \to (Z, \mu) \) is \( \gamma - irreolute \). Then

\begin{enumerate}
  \item If \( g \) is a \( pre - \gamma - open \) injection, then \( f \) is \( \gamma - irreolute \).
  \item If \( f \) is a \( pre - \gamma - open \) surjection, then \( g \) is \( \gamma - irreolute \).
\end{enumerate}

**Proof** (1) Let \( U \in \sigma^Y \). Then \( g(U) \in \mu^X \) since \( g \) is \( pre - \gamma - open \). Also \( g \circ f \) is \( \gamma - irreolute \). Therefore, we have \( (g \circ f)^{-1}(g(U)) \) \( \in \tau^X \). Since \( g \) is an injection, so we have :

\[ (g \circ f)^{-1}(g(U)) = (f^{-1} \circ g^{-1})(g(U)) = f^{-1}[g^{-1}(g(U))] = f^{-1}(U). \]

Consequently \( f^{-1}(U) \) is \( \gamma - open \) in \( X \). This proves that \( f \) is \( \gamma - irreolute \).

(2) Let \( V \in \mu^X \). Then \( (g \circ f)^{-1}(V) \in \tau^X \) since \( g \circ f \) is \( \gamma - irreolute \). Also \( f \) is \( pre - \gamma - open \), \( f^{-1}[g^{-1}(g(U))] = g^{-1}(f^{-1}(U)) \). Hence \( g \) is \( \gamma - irreolute \).

**F. Pre – \gamma – Closed Functions**

In this last section, we introduce and explore several properties and characterizations of \( pre - \gamma - closed \) functions.

**Definition 2.40.** A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( pre - \gamma - closed \) if and only if the
image set \( f(A) \) is \( \gamma \)-closed for each \( \gamma \)-closed subset \( A \) of \( X \).

**Theorem 2.41.** The composition of two \( \gamma \)-closed mappings is a \( \gamma \)-closed mapping.

**Proof** The straightforward proof is omitted.

**Theorem 2.42.** Prove that a mapping \( f : (X, \tau) \to (Y, \sigma) \) is \( \gamma \)-closed if and only if \( Cl_\gamma[f(A)] \subseteq f[Cl_\gamma(A)] \) for every subset \( A \) of \( X \).

**Proof** Necessity. Suppose \( f \) is a \( \gamma \)-closed mapping and \( A \) is an arbitrary subset of \( X \). Then \( f(Cl_\gamma(A)) \) is \( \gamma \)-closed in \( Y \). Since \( f(A) \subseteq f(Cl_\gamma(A)) \), we obtain \( Cl_\gamma[f(A)] \subseteq f(Cl_\gamma(A)) \).

Sufficiency. Suppose \( F \) is an arbitrary \( \gamma \)-closed set in \( X \). By hypothesis, we obtain \( f(F) \subseteq Cl_\gamma[f(F)] \subseteq f(Cl_\gamma(F)) = f(F) \). Hence \( f(F) = Cl_\gamma[f(F)] \). Thus \( f(F) \) is \( \gamma \)-closed in \( Y \). It follows that \( f \) is \( \gamma \)-closed.

**Theorem 2.43.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a mapping such that \( \text{Int}[Cl(f(A))] \subseteq f[Cl_\gamma(A)] \) for every subset \( A \) of \( X \). Then \( f \) is \( \gamma \)-closed.

**Proof** Suppose \( A \) is an arbitrary \( \gamma \)-closed set in \( X \). Then by hypothesis, we have \( \text{Int}[Cl(f(A))] \subseteq f[Cl_\gamma(A)] = f(A) \). Take \( B = Cl(f(A)) \). Then \( B \) is closed in \( Y \). Also it implies that \( \text{Int}(B) \subseteq f(A) \subseteq B \). Hence \( f(A) \) is semi-closed in \( Y \). Since \( \sigma \subseteq \sigma^\gamma \), thus \( f(A) \) is \( \gamma \)-closed in \( Y \). This implies that \( f \) is \( \gamma \)-closed.

**Theorem 2.44.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a mapping such that \( \text{Int}[Cl(f(A))] \subseteq f[Cl_\gamma(A)] \) for every subset \( A \) of \( X \).

1. If \( U \) is a \( \gamma \)-open neighbourhood of \( f^{-1}(B) \), then there exists a \( \gamma \)-open neighbourhood \( V \) of \( B \) such that \( f^{-1}(B) \subseteq f^{-1}(V) \subseteq U \).

2. If \( f \) is also onto, then if \( f^{-1}(B) \) and \( f^{-1}(C) \) have disjoint \( \gamma \)-open neighbourhoods, so have \( B \) and \( C \).

**Proof** (1) Let \( V = Y - f(X - U) \). Then \( V^c = Y - V = f(U^c) \). Since \( f \) is \( \gamma \)-closed, so \( V \) is \( \gamma \)-open. Since \( f^{-1}(B) \subseteq U \), we have \( V^c = f(U^c) \subseteq f[f^{-1}(B^c)] \subseteq B^c \). Hence, \( B \subseteq V \), and thus \( V \) is a \( \gamma \)-open neighbourhood of \( B \). Further \( U^c \subseteq f^{-1}[f(U^c)] = f^{-1}(V^c) = [f^{-1}(V)]^c \). This proves that \( f^{-1}(V) \subseteq U \).

(2) If \( f^{-1}(B) \) and \( f^{-1}(C) \) have disjoint \( \gamma \)-open neighbourhoods \( M \) and \( N \), then by (1), we have \( \gamma \)-open neighbourhoods \( U \) and \( V \) of \( B \) and \( C \) respectively such that \( f^{-1}(B) \subseteq f^{-1}(U) \subseteq \text{Int}_\gamma(M) \) and \( f^{-1}(C) \subseteq f^{-1}(V) \subseteq \text{Int}_\gamma(N) \). Since \( M \) and \( N \) are disjoint, so are \( \text{Int}_\gamma(M) \) and \( \text{Int}_\gamma(N) \), and hence \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint as well. It follows that \( U \) and \( V \) are disjoint too as \( f \) is onto.

**Theorem 2.45.** Prove that a surjective mapping \( f : (X, \tau) \to (Y, \sigma) \) is \( \gamma \)-closed if and only if for each subset \( B \) of \( Y \) and each \( \gamma \)-open set \( U \) in \( X \) containing \( f^{-1}(B) \), there exists a \( \gamma \)-open set \( V \) in \( Y \) containing \( B \), such that \( f^{-1}(V) \subseteq U \).

**Proof** Necessity. This follows from (1) of Theorem 2.44.

Sufficiency. Suppose \( F \) is an arbitrary \( \gamma \)-closed set in \( X \). Let \( y \) be an arbitrary point in \( Y - f(F) \). Then \( f^{-1}(y) \subseteq X - f^{-1}[f(F)] \subseteq (X - F) \) and \( (X - F) \) is \( \gamma \)-open in \( X \). Hence by hypothesis, there exists a \( \gamma \)-open set \( V_y \) containing \( y \) such that \( f^{-1}(V_y) \subseteq (X - F) \). This implies that \( y \in V_y \subseteq [Y - f(F)] \). Thus \( Y - f(F) = \bigcup \{ V_y : y \in Y - f(F) \} \). Hence \( Y - f(F) \), being a union of \( \gamma \)-open sets is \( \gamma \)-open. Thus its complement \( f(F) \) is \( \gamma \)-closed. This shows that \( f \) is \( \gamma \)-closed.
Theorem 2.46. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection. Then the following are equivalent:

1. $f$ is pre-$\gamma$-closed.
2. $f$ is pre-$\gamma$-open.
3. $f^{-1}$ is $\gamma$-irresolute.

Proof (1) $\Rightarrow$ (2): Let $U \in \tau^\gamma$. Then $X - U$ is $\gamma$-closed in $X$. By (1), $f(X - U)$ is $\gamma$-closed in $Y$. But $f(X - U) = f(X) - f(U) = Y - f(U)$. Thus $f(U)$ is $\gamma$-open in $Y$. This shows that $f$ is pre-$\gamma$-open.

(2) $\Rightarrow$ (3): Let $A \subseteq X$. Since $f$ is pre-$\gamma$-open, so by Theorem 2.38, $f^{-1}[\text{Cl}_{\gamma}(f(A))] \subseteq \text{Cl}_{\gamma}[f^{-1}(f(A))]$. It implies that $\text{Cl}_{\gamma}[f(A)] \subseteq f[\text{Cl}_{\gamma}(A)]$. Thus $\text{Cl}_{\gamma}[(f^{-1})^{-1}(A)] \subseteq (f^{-1})^{-1}[\text{Cl}_{\gamma}(A)]$, for all $A \subseteq X$. Then by Theorem 2.8, it follows that $f^{-1}$ is $\gamma$-irresolute.

(3) $\Rightarrow$ (1): Let $A$ be an arbitrary $\gamma$-closed set in $X$. Then $X - A$ is $\gamma$-open in $X$. Since $f^{-1}$ is a $\gamma$-irresolute, $(f^{-1})^{-1}(X - A)$ is $\gamma$-open in $Y$. But $(f^{-1})^{-1}(X - A) = f(X - A) = Y - f(A)$. Thus $f(A)$ is $\gamma$-closed in $Y$. This shows that $f$ is pre-$\gamma$-closed.

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