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Characterizations of Mappings in γ-Open Sets

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Characterizations of Mappings in γ –Open Sets

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Abstract. The notion of semi-convergence of filters was introduced by Latif (1999) who investigated some characterizations related to semi-open continuous functions. In the spirit of Latif (1999), Min (2002) used the idea of semi-convergence of filters to introduce a new class of sets, called $\gamma-$ *open* sets, and the notions of $\gamma-$ *closure*, $\gamma-$ *interior* and $\gamma-$ *continuity* and investigated some properties. In this paper we introduce $\gamma-$ *continuous*, $\gamma-$ *irresolute*, $\gamma-$ *open*, $\gamma-$ *closed*, *pre* $-\gamma-$ *open* and *pre* $-\gamma-$ *closed* mappings and investigate properties and characterizations of these new types of mappings.

1. Introduction

The notion of γ –open set (originally called $\gamma-$ sets) in topological spaces was introduced by Min [*Min*, 2002]. We continue to explore further properties and characterizations of $\gamma-$ *continuous*, $\gamma-$ *irresolute* and $\gamma-$ *open* mappings. We also introduce and study properties and characterizations of $\gamma-$ *closed*, *pre* $-\gamma-$ *open* and *pre* $-\gamma-$ *closed* mappings.

Throughout this paper, (X,τ) (simply X) always mean topological space on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure (resp., *interior*) of S will be denoted by $Cl(S)$ (resp., $Int(S)$). A subset S of X is called a semi-open set [*Levine*, 1963] (resp., $\alpha-$ *open set* [Njåstad, 1965]) if $S \subseteq Cl[Int(S)]$ (*resp*., $S \subseteq Int[Cl(Int(S))]$). The complement of a semi-open set (resp., $\alpha-$ *open set*) is called semi-closed set (resp., $\alpha-$ *closed set*). The family of all semi-open sets (resp., $\alpha-$ *open sets*) in a topological space (X,τ) will be denoted by $SO(X)$ (resp., τ^{α}). A subset $M(x)$ of a space X is called a semi-neighbourhood of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subseteq M(x)$. In [*Latif*, 1999] Latif introduced the notion of semi-convergence of filters and investigated some characterizations related to semi-open continuous function. Now, we recall the concept of semi-convergence of filters. Let $S(x) = \{A \in SO(X) : x \in A\}$ and let $S_x = \{A \subseteq X : \textit{there exists } \mu \subseteq S(x) \textit{ such that } \mu \textit{ is finite and } \cap\mu \subseteq A\}$. Then, S_x is called the semi-neighbourhood filter at x. For any filter Γ on X, we say that Γ semi-converges to x if and only if Γ is finer than the semi-neighbourhood filter at x.

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A subset U of X is called a $\gamma-$*open* set if whenever a filter Γ semi-converges to x and $x \in U$, $U \in \Gamma$. The complement of a $\gamma-$*open* set is called a $\gamma-$*closed* set. The intersection of all $\gamma-$*closed* sets containing A is called the $\gamma-$*closure* of A, denoted by $Cl_\gamma(A)$. A subset A is also $\gamma-$closed if and only if $A = Cl_\gamma(A)$. We denote the family of all $\gamma-$open sets of (X,τ) by τ^γ. It is shown in [*Min*, 2002] that τ^γ is a topology on X. In a topological space (X, τ), it is always true that $\tau \subseteq \tau^\alpha \subseteq S(X) \subseteq \tau^\gamma$.

2. Characterizations of Mappings

The purpose of this section is to explore properties and characterizations of $\gamma-$*continuous*, $\gamma-$*irresolute*, $\gamma-$*open*, $\gamma-$*closed*, *pre*$-\gamma-$*open* and *pre*$-\gamma-$*closed* functions.

A. γ –Continuous Functions

In [*Min*, 2002] Min introduced the notion of $\gamma-$*continuous* mapping and gave certain characterizations of it. The purpose of this section is to investigate further properties and characterizations of $\gamma-$*continuous* functions.

Definition 2.1. *A function* $f : (X,\tau) \to (Y,\sigma)$ *is said to be* $\gamma-$*continuous if* $f^{-1}(V) \in \tau^\gamma$ *for every* $V \in \sigma$.

Theorem 2.2. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be a function. Then the following are equivalent*:

(1) f *is* $\gamma-$*continuous*;

(2) *The inverse image of each closed set in* Y *is a* $\gamma-$*closed set in* X;

(3) $Cl_\gamma[f^{-1}(V)] \subseteq f^{-1}[Cl(V)]$, *for every* $V \subseteq Y$;

(4) $f[Cl_\gamma(U)] \subseteq Cl[f(U)]$, *for every* $U \subseteq X$;

(5) *For any point* $x \in X$ *and any open set* V *of* Y *containing* $f(x)$, *there exists* $U \in \tau^\gamma$ *such that* $x \in U$ *and* $f(U) \subseteq V$;

(6) $Bd_\gamma[f^{-1}(V)] \subseteq f^{-1}[Bd(V)]$, *for every* $V \subseteq Y$;

(7) $f[D_\gamma(U)] \subseteq Cl[f(U)]$, *for every* $U \subseteq X$;

(8) $f^{-1}[Int(V)] \subseteq Int_\gamma[f^{-1}(V)]$, *for every* $V \subseteq Y$.

Proof $(1) \Rightarrow (2)$: Let $F \subseteq Y$ be closed. Since f is $\gamma-$*continuous*, $f^{-1}(Y-F) = X - f^{-1}(F)$ is $\gamma-$*open*. Therefore, $f^{-1}(F)$ is $\gamma-$*closed* in X.

$(2) \Rightarrow (3)$: Since $Cl(V)$ is closed for every $V \subseteq Y$, then $f^{-1}[Cl(V)]$ is $\gamma-$*closed*. Therefore $f^{-1}[Cl(V)] = Cl_\gamma[f^{-1}(Cl(V))] \supseteq Cl_\gamma[f^{-1}(V)]$.

$(3) \Rightarrow (4)$: Let $U \subseteq X$ and $f(U) = V$. Then $f^{-1}[Cl(V)] \supseteq Cl_\gamma[f^{-1}(V)]$. Thus $f^{-1}[Cl(f(U))] \supseteq Cl_\gamma[f^{-1}(f(U))] \supseteq Cl_\gamma(U)$ and $Cl[f(U)] \supseteq f[Cl_\gamma(U)]$.

(4) $\Rightarrow$ (2) : Let $W \subseteq Y$ be a closed set, and $U = f^{-1}(W)$, then $f[Cl_\gamma(U)] \subseteq Cl[f(U)] = Cl[f(f^{-1}(W))] \subseteq Cl(W) = W$. Thus $Cl_\gamma(U) \subseteq f^{-1}[f(Cl_\gamma(U))] \subseteq f^{-1}(W) = U$. So U is $\gamma -$ *closed*.

(2) $\Rightarrow$ (1) : Let $V \subseteq Y$ be an open set, then $Y - V$ is *closed*. Then $f^{-1}(Y - V) = X - f^{-1}(V)$ is $\gamma -$ *closed* in X and hence $f^{-1}(V)$ is $\gamma -$ *open* in X.

(1) $\Rightarrow$ (5) : Let $f : X \to Y$ be $\gamma -$ *continuous*. For any $x \in X$ and any open set V of Y containing $f(x)$, $U = f^{-1}(V) \in \tau^\gamma$, and $f(U) = f[f^{-1}(V)] \subseteq V$.

(5) $\Rightarrow$ (1) : Let $V \in \sigma$. We prove $f^{-1}(V) \in \tau^\gamma$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists $U \in \tau^\gamma$ such that $x \in U$ and $f(x) \in f(U) \subseteq V$. Hence $x \in U \subseteq f^{-1}[f(U)] \subseteq f^{-1}(V)$. It shows that $f^{-1}(V)$ is a $\gamma -$ *neighborhood* of each of its points. Therefore $f^{-1}(V) \in \tau^\gamma$.

(6) $\Rightarrow$ (8) : Let $V \subseteq Y$. Then by hypothesis, $Bd_\gamma[f^{-1}(V)] \subseteq f^{-1}[Bd(V)]$
$\Rightarrow f^{-1}(V) - Int_\gamma[f^{-1}(V)] \subseteq f^{-1}[V - Int(V)] = f^{-1}(V) - f^{-1}[Int(V)]$
$\Rightarrow f^{-1}[Int(V)] \subseteq Int_\gamma[f^{-1}(V)]$.

(8) $\Rightarrow$ (6) : Let $V \subseteq Y$. Then by hypothesis, $f^{-1}[Int(V)] \subseteq Int_\gamma[f^{-1}(V)]$
$\Rightarrow f^{-1}(V) - Int_\gamma[f^{-1}(V)] \subseteq f^{-1}(V) - f^{-1}[Int(V)] = f^{-1}[V - Int(V)]$
$\Rightarrow Bd_\gamma[f^{-1}(V)] \subseteq f^{-1}[Bd(V)]$.

(1) $\Rightarrow$ (7) : It is obvious, since f is $\gamma -$ *continuous* and by (4) $f[Cl_\gamma(U)] \subseteq Cl[f(U)]$ for each $U \subseteq X$. So $f[D_\gamma(U)] \subseteq Cl[f(U)]$.

(7) $\Rightarrow$ (1) : Let $U \subseteq Y$ be an open set, $V = Y - U$ and $f^{-1}(V) = W$. Then by hypothesis $f[D_\gamma(W)] \subseteq Cl[f(W)]$. Thus $f[D_\gamma(f^{-1}(V))] \subseteq Cl[f(f^{-1}(V))] \subseteq Cl(V) = V$. Then $D_\gamma[f^{-1}(V)] \subseteq f^{-1}(V)$ and $f^{-1}(V)$ is $\gamma -$ *closed*. Therefore, f is $\gamma -$ *continuous*.

(1) $\Rightarrow$ (8) : Let $V \subseteq Y$. Then $f^{-1}[Int(V)]$ is $\gamma -$ *open* in X. Thus $f^{-1}[Int(V)] = Int_\gamma[f^{-1}(Int(V))] \subseteq Int_\gamma[f^{-1}(V)]$. Therefore, $f^{-1}[Int(V)] \subseteq Int_\gamma[f^{-1}(V)]$.

(8) $\Rightarrow$ (1) : Let $V \subseteq Y$ be an open set. Then $f^{-1}(V) = f^{-1}[Int(V)] \subseteq Int_\gamma[f^{-1}(V)]$. Therefore, $f^{-1}(V)$ is $\gamma -$ *open*. Hence f is $\gamma -$ *continuous*.

In the next Theorem, $\#\gamma - c.$ denotes the set of points x of X for which a function $f : (X,\tau) \to (Y,\sigma)$ is not $\gamma -$ *continuous*.

Theorem 2.3. *$\#\gamma - c.$ is identical with the union of the $\gamma -$ frontiers of the inverse images of $\gamma -$ open sets containing $f(x)$.*

Proof *Suppose that f is not $\gamma -$ continuous at a point x of X. Then there exists an open set $V \subseteq Y$ containing $f(x)$ such that $f(U)$ is not a subset of V for every $U \in \tau^\gamma$ containing x. Hence, we have $U \cap [X - f^{-1}(V)] \neq \phi$ for every $U \in \tau^\gamma$ containing x. It follows that $x \in Cl_\gamma[X - f^{-1}(V)]$. We also have $x \in f^{-1}(V) \subseteq Cl_\gamma[f^{-1}(V)]$. This means that $x \in Fr_\gamma(f^{-1}(V))$.*

Now, let f be $\gamma -$ continuous at $x \in X$ and $V \subseteq Y$ any open set containing $f(x)$. Then, $x \in f^{-1}(V)$ is a $\gamma -$ open set of X. Thus, $x \in Int_\gamma[f^{-1}(V)]$ and therefore $x \notin Fr_\gamma[f^{-1}(V)]$ for every open set V containing $f(x)$.

Remark 2.4. (1) *Every continuous function is $\gamma -$ continuous but the converse may not be true.*

(2) *If a function $f : (X,\tau) \to (Y,\sigma)$ is $\gamma -$ continuous and a function $g : (Y,\sigma) \to (Z,\vartheta)$ is $\gamma -$ continuous, then $g \circ f : (X,\tau) \to (Z,\vartheta)$ may not be $\gamma -$ continuous.*

(3) *If a function $f : (X,\tau) \to (Y,\sigma)$ is $\gamma -$ continuous and a function $g : (Y,\sigma) \to (Z,\vartheta)$*

is continuous, *then* $g \circ f : (X, \tau) \to (Z, \vartheta)$ *is* $\gamma-$*continuous*.

(4) *Let* (X, τ) *and* (Y, σ) *be topological spaces*. *If* $f : X \to Y$ *is a function*, *and one of the following*

(*a*) $f^{-1}[Int_\gamma(B)] \subseteq Int[f^{-1}(B)]$ *for each* $B \subseteq Y$,

(*b*) $Cl[f^{-1}(B)] \subseteq f^{-1}[Cl_\gamma(B)]$ *for each* $B \subseteq Y$,

(*c*) $f[Cl(A)] \subseteq Cl_\gamma[f(A)]$ *for each* $A \subseteq X$.

holds, *then* f *is continuous*.

Lemma 2.5. *Let* $A \subseteq Y \subseteq X$, Y *is* $\gamma-$*open in* X *and* A *is* $\gamma-$*open in* Y. *Then* A *is* $\gamma-$*open in* X.

Proof Since A is $\gamma-$*open* in Y, there exists an open set $U \subseteq X$ such that $A = Y \cap U$. Thus A being the intersection of two $\gamma-$*open* sets in X, is $\gamma-$*open* in X.

Theorem 2.6. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a mapping and* $\{U_i : i \in I\}$ *be a cover of* X *such that* $U_i \in \tau^\gamma$ *for each* $i \in I$. *Suppose that* $f|U_i : U_i \to Y$ *is* $\gamma-$*continuous for each* $i \in I$. *Then prove that* f *is* $\gamma-$*continuous*.

Proof *Let* $V \subseteq Y$ *be an open set*, *then* $\left(f|U_i\right)^{-1}(V)$ *is* $\gamma-$*open in* U_i *for each* $i \in I$. *Since* U_i *is* $\gamma-$*open in* X *for each* $i \in I$. *So by Lemma* 2.5, $\left(f|U_i\right)^{-1}(V)$ *is* $\gamma-$*open in* X *for each* $i \in I$. *But*, $f^{-1}(V) = \cup\left\{\left(f|U_i\right)^{-1}(V) : i \in I\right\}$, *then* $f^{-1}(V) \in \tau^\gamma$ *because* τ^γ *is a topology on* X. *This implies that* f *is* $\gamma-$*continuous*.

B. γ –Irresolute Functions

In this section, the functions to be considered are those for which inverses of $\gamma-$*open* sets are $\gamma-$*open*. We investigate some new properties and characterizations of such functions.

Definition 2.7. [*Min*, 2002]. *Let* (X, τ) *and* (Y, σ) *be topological spaces*. *A function* $f : X \to Y$ *is called* $\gamma-$*irresolute if the inverse image of each* $\gamma-$*open set of* Y *is a* $\gamma-$*open set in* X.

Theorem 2.8. [*Min*, 2002]. *Let* $(X, \tau) \to (Y, \sigma)$ *be a function between topological spaces*. *Then the following are equivalent*:

(1) f *is* $\gamma-$*irresolute*;

(2) *the inverse image of each* $\gamma-$*closed set in* Y *is a* $\gamma-$*closed set*;

(3) $Cl_\gamma[f^{-1}(V)] \subseteq f^{-1}[Cl_\gamma(V)]$ *for every* $V \subseteq Y$;

(4) $f[Cl_\gamma(U)] \subseteq Cl_\gamma[f(U)]$ *for every* $U \subseteq X$;

(5) $f^{-1}[Int_\gamma(B)] \subseteq Int_\gamma[f^{-1}(B)]$ *for every* $B \subseteq Y$.

Theorem 2.9. *Prove that a function* $f : (X, \tau) \to (Y, \sigma)$ *is* $\gamma-$*irresolute if and only if for each* *point p in X and each* $\gamma-$ *open set B in Y with* $f(p) \in B$, *there is a* $\gamma-$ *open set A in X such that* $p \in A, f(A) \subseteq B$.

Proof *Necessity*. *Let* $p \in X$ *and* $B \in \sigma^{\gamma}$ *such that* $f(p) \in B$. *Let* $A = f^{-1}(B)$. *Since f is* $\gamma-$ *irresolute,* A *is* $\gamma-$ *open in X. Also* $p \in f^{-1}(B) = A$ *as* $f(p) \in B$. *Thus we have* $f(A) = f[f^{-1}(B)] \subseteq B$.

Sufficiency. *Let* $B \in \sigma^{\gamma}$, *let* $A = f^{-1}(B)$. *We show that A is* $\gamma-$ *open in X. For this let* $x \in A$. *It implies that* $f(x) \in B$. *Then by hypothesis, there exists* $A_x \in \tau^{\gamma}$ *such that* $x \in A_x$ *and* $f(A_x) \subseteq B$. *Then* $A_x \subseteq f^{-1}[f(A_x)] \subseteq f^{-1}(B) = A$. *Thus* $A = \cup\{A_x : x \in A\}$. *It follows that A is* $\gamma-$ *open in X. Hence f is* $\gamma-$ *irresolute.*

Definition 2.10. *Let* (X, τ) *be a topological space. Let* $x \in X$ *and* $N \subseteq X$. *We say that N is a* $\gamma-$ *neighbourhood of x if there exists a* $\gamma-$ *open set M of X such that* $x \in M \subseteq N$.

Theorem 2.11. *Prove that a function* $f : (X, \tau) \rightarrow (Y, \sigma)$ *is* $\gamma-$ *irresolute if and only if for each x in X, the inverse image of every* $\gamma-$ *neighbourhood of* $f(x)$, *is a* $\gamma-$ *neighbourhood of x.*

Proof *Necessity*. *Let* $x \in X$ *and let B be a* $\gamma-$ *neighbourhood of* $f(x)$. *Then there exists* $U \in \sigma^{\gamma}$ *such that* $f(x) \in U \subseteq B$. *This implies that* $x \in f^{-1}(U) \subseteq f^{-1}(B)$. *Since f is* $\gamma-$ *irresolute, so* $f^{-1}(U) \in \tau^{\gamma}$. *Hence* $f^{-1}(B)$ *is a* $\gamma-$ *neighbourhood of x.*

Sufficiency. *Let* $B \in \sigma^{\gamma}$. *Put* $A = f^{-1}(B)$. *Let* $x \in A$. *Then* $f(x) \in B$. *But then, B being* $\gamma-$ *open set, is a* $\gamma-$ *neighborhood of* $f(x)$. *So by hypothesis* , $A = f^{-1}(B)$ *is a* $\gamma-$ *neighborhood of x. Hence by definition, there exists* $A_x \in \tau^{\gamma}$ *such that* $x \in A_x \subseteq A$. *Thus* $A = \cup\{A_x : x \in A\}$. *It follows that A is a* $\gamma-$ *open set in X. Therefore f is* $\gamma-$ *irresolute.*

Theorem 2.12. *Prove that a function* $f : (X, \tau) \rightarrow (Y, \sigma)$ *is* $\gamma-$ *irresolute if and only if for each x in X, and each* $\gamma-$ *neighborhood U of* $f(x)$, *there is a* $\gamma-$ *neighborhood V of x such that* $f(V) \subseteq U$.

Proof *Necessity*. *Let* $x \in X$ *and let U be* $\gamma-$ *neighbourhood of* $f(x)$. *Then there exists* $O_{f(x)} \in \sigma^{\gamma}$ *such that* $f(x) \in O_{f(x)} \subseteq U$. *It follows that* $x \in f^{-1}[O_{f(x)}] \subseteq f^{-1}(U)$. *By hypothesis,* $f^{-1}[O_{f(x)}] \in \tau^{\gamma}$. *Let* $V = f^{-1}(U)$. *Then it follows that V is a* $\gamma-$ *neighbourhood of x and* $f(V) = f[f^{-1}(U)] \subseteq U$.

Sufficiency. *Let* $B \in \sigma^{\gamma}$. *Put* $O = f^{-1}(B)$. *Let* $x \in O$. *Then* $f(x) \in B$. *Thus B is a* $\gamma-$ *neighbourhood of* $f(x)$. *So by hypothesis, there exists a* $\gamma-$ *neighbourhood* V_x *of x such that* $f(V_x) \subseteq B$. *Thus it follows that* $x \in V_x \subseteq f^{-1}[f(V_x)] \subseteq f^{-1}(B) = O$. *Since* V_x *is a* $\gamma-$ *neighbourhood of x, so there exists an* $O_x \in \tau^{\gamma}$ *such that* $x \in O_x \subseteq V_x$. *Hence* $x \in O_x \subseteq O$, $O_x \in \tau^{\gamma}$. *Thus* $O = \cup\{O_x : x \in O\}$. *It follows that O is* $\gamma-$ *open in X. Therefore, f is* $\gamma-$ *irresolute.*

Theorem 2.13. *Prove that a function* $f : (X, \tau) \rightarrow (Y, \sigma)$ *is* $\gamma-$ *irresolute if and only if* $f[D_{\gamma}(A)] \subseteq f(A) \cup D_{\gamma}[f(A)]$, *for all* $A \subseteq X$.

Proof *Necessity*. *Let* $f : X \rightarrow Y$ *be* $\gamma-$ *irresolute. Let* $A \subseteq X$, *and* $a_0 \in D_{\gamma}(A)$. *Assume that* $f(a_0) \notin f(A)$ *and let V denote a* $\gamma-$ *neighborhood of* $f(a_0)$. *Since f is* $\gamma-$ *irresolute, so by Theorem* 2.12, *there exists a* $\gamma-$ *neighborhood U of* a_0 *such that* $f(U) \subseteq V$. *From* $a_0 \in D_{\gamma}(A)$, *it follows that* $U \cap A \neq \phi$; *there exists, therefore, at least one element* $a \in U \cap A$ *such that* $f(a) \in f(A)$ *and* $f(a) \in V$. *Since* $f(a_0) \notin f(A)$, *we have* $f(a) \neq f(a_0)$. *Thus every* $\gamma-$ *neighbourhood of* $f(a_0)$ *contains an element of* $f(A)$ *different from* $f(a_0)$, *consequently,* $f(a_0) \in D_{\gamma}[f(A)]$. *This proves necessity of the condition.*

Sufficiency. *Assume that f is not* $\gamma-$ *irresolute. Then by Theorem* 2.12, *there exists* $a_0 \in X$

and a $\gamma-$ *neighbourhood* V *of* $f(a_0)$ *such that every* $\gamma-$ *neighbourhood* U *of* a_0 *contains at least one element* $a \in U$ *for which* $f(a) \notin V$. *Put* $A = \{a \in X : f(a) \notin V\}$. *Then* $a_0 \notin A$ *since* $f(a_0) \in V$, *and therefore* $f(a_0) \notin f(A)$; *also* $f(a_0) \notin D_\gamma[f(A)]$ *since* $f(A) \cap (V - \{f(a_0)\}) = \phi$. *It follows that* $f(a_0) \in f[D_\gamma(A)] - [f(A) \cup D_\gamma(f(A))] \neq \phi$, *which is a contradiction to the given condition. The condition of the Theorem is therefore sufficient and the theorem is proved.*

Theorem 2.14. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be a* $1-1$ *function. Then* f *is* $\gamma-$ *irresolute if and only if* $f[D_\gamma(A)] \subseteq D_\gamma[f(A)]$, *for all* $A \subseteq X$.

Proof *Necessity. Let* f *be* $\gamma-$ *irresolute. Let* $A \subseteq X$, $a_0 \in D_\gamma(A)$ *and* V *be a* $\gamma-$ *neighbourhood of* $f(a_0)$. *Since* f *is* $\gamma-$ *irresolute, so by Theorem* 2.12, *there exists a* $\gamma-$ *neighbourhood* U *of* a_0 *such that* $f(U) \subseteq V$. *But* $a_0 \in D_\gamma(A)$; *hence there exists an element* $a \in U \cap A$ *such that* $a \neq a_0$; *then* $f(a) \in f(A)$ *and, since* f *is* $1-1$, $f(a) \neq f(a_0)$. *Thus every* $\gamma-$ *neighbourhood* V *of* $f(a_0)$ *contains an element of* $f(A)$ *different from* $f(a_0)$; *consequently* $f(a_0) \in D_\gamma[f(A)]$. *We have therefore* $f[D_\gamma(A)] \subseteq D_\gamma[f(A)]$.

Sufficiency. Follows from Theorem 2.13.

C. γ –Open Functions

In [*Min*, 2002] Min defined $\gamma-$ *open* mappings as a generalization of open mappings and investigated some properties of such mappings. The purpose of this section is to add some more characterizations of $\gamma-$ *open* mappings.

Definition 2.15. *Let* (X,τ) *and* (Y,σ) *be topological spaces. A function* $f : X \to Y$ *is called* $\gamma-$ *open if for every open set* G *in* X, $f(G)$ *is a* $\gamma-$ *open set in* Y.

Theorem 2.16. *Prove that a mapping* $f : (X,\tau) \to (Y,\sigma)$ *is* $\gamma-$ *open if and only if for each* $x \in X$, *and* $U \in \tau$ *such that* $x \in U$, *there exists a* $\gamma-$ *open set* $W \subseteq Y$ *containing* $f(x)$ *such that* $W \subseteq f(U)$.

Proof *Follows immediately from Definition* 2.15.

Theorem 2.17. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be* $\gamma-$ *open. If* $W \subseteq Y$ *and* $F \subseteq X$ *is a closed set containing* $f^{-1}(W)$, *then there exists a* $\gamma-$ *closed* $H \subseteq Y$ *containing* W *such that* $f^{-1}(H) \subseteq F$.

Proof *Let* $H = Y - f(X-F)$. *Since* $f^{-1}(W) \subseteq F$, *we have* $f(X-F) \subseteq (Y-W)$. *Since* f *is* $\gamma-$ *open, then* H *is* $\gamma-$ *closed and* $f^{-1}(H) = X - f^{-1}[f(X-F)] \subseteq X-(X-F) = F$.

Theorem 2.18. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be* $\gamma-$ *open and let* $B \subseteq Y$. *Then* $f^{-1}[Cl_\gamma(Int_\gamma(Cl_\gamma(B)))] \subseteq Cl[f^{-1}(B)]$.

Proof $Cl[f^{-1}(B)]$ *is closed in* X *containing* $f^{-1}(B)$. *By Theorem* 2.17, *there exists a* $\gamma-$ *closed set* $B \subseteq H \subseteq Y$, *such that* $f^{-1}(H) \subseteq Cl[f^{-1}(B)]$. *Thus,*
$f^{-1}[Cl_\gamma(Int_\gamma(Cl_\gamma(B)))] \subseteq f^{-1}[Cl_\gamma(Int_\gamma(Cl_\gamma(H)))] \subseteq f^{-1}(H) \subseteq Cl[f^{-1}(B)]$.

Theorem 2.19. *Prove that a function* $f : (X,\tau) \to (Y,\sigma)$ *is* $\gamma-$ *open if and only if* $f[Int(A)] \subseteq Int_\gamma[f(A)]$, *for all* $A \subseteq X$.

Proof *Necessity. Let* $A \subseteq X$. *Let* $x \in Int(A)$. *Then there exists* $U_x \in \tau$ *such that* $x \in U_x \subseteq A$. *So* $f(x) \in f(U_x) \subseteq f(A)$ *and by hypothesis,* $f(U_x) \in \sigma^\gamma$. *Hence* $f(x) \in Int_\gamma[f(A)]$. *Thus* $f[Int(A)] \subseteq Int_\gamma[f(A)]$.

Sufficiency. Let $U \in \tau$. Then by hypothesis, $f[Int(U)] \subseteq Int_\gamma[f(U)]$. Since $Int(U) = U$ as U is open. Also $Int_\gamma[f(U)] \subseteq f(U)$. Hence $f(U) = Int_\gamma[f(U)]$. Thus $f(U)$ is $\gamma-$ open in Y. So f is $\gamma-$ open.

We remark that the equality does not hold in the preceding Theorem as the following example shows.

Example 2.20. *Let* $X = Y = \{1,2\}$. *Suppose* τ *be antidiscrete topology on* X *and* σ *be discrete topology on* Y. *Then* $\tau^\gamma = \tau$ *and* $\sigma^\gamma = \sigma$. *Let* $f = Id.$, $A = \{1\}$. *Then* $\phi = f[Int(A)] \neq Int_\gamma[f(A)] = \{1\}$.

Theorem 2.21. *Prove that a function* $f: (X,\tau) \to (Y,\sigma)$ *is* $\gamma-$ *open if and only if* $Int[f^{-1}(B)] \subseteq f^{-1}[Int_\gamma(B)]$, *for all* $B \subseteq Y$.

Proof *Necessity*. Let $B \subseteq Y$. Since $Int[f^{-1}(B)]$ is open in X and f is $\gamma-$ open, $f[Int(f^{-1}(B))]$ is $\gamma-$ open in Y. Also we have $f[Int(f^{-1}(B))] \subseteq f[f^{-1}(B)] \subseteq B$. Hence, $f[Int(f^{-1}(B))] \subseteq Int_\gamma(B)$. Therefore $Int[f^{-1}(B)] \subseteq f^{-1}[Int_\gamma(B)]$.

Sufficiency. Let $A \subseteq X$. Then $f(A) \subseteq Y$. Hence by hypothesis, we obtain $Int(A) \subseteq Int[f^{-1}(f(A))] \subseteq f^{-1}[Int_\gamma(f(A))]$. Thus $f[Int(A)] \subseteq Int_\gamma[f(A)]$, for all $A \subseteq X$. Hence, by Theorem 2.19, f is $\gamma-$ open.

Theorem 2.22. *Let* $f: (X,\tau) \to (Y,\sigma)$ *be a mapping. Then a necessary and sufficient condition for* f *to be* $\gamma-$ *open is that* $f^{-1}[Cl_\gamma(B)] \subseteq Cl[f^{-1}(B)]$ *for every subset* B *of* Y.

Proof *Necessity*. Assume f is $\gamma-$ open. Let $B \subseteq Y$. Let $x \in f^{-1}[Cl_\gamma(B)]$. Then $f(x) \in Cl_\gamma(B)$. Let $U \in \tau$ such that $x \in U$. Since f is $\gamma-$ open, then $f(U)$ is a $\gamma-$ open set in Y. Therefore, $B \cap f(U) \neq \phi$. Then $U \cap f^{-1}(B) \neq \phi$. Hence $x \in Cl[f^{-1}(B)]$. We conclude that $f^{-1}[Cl_\gamma(B)] \subseteq Cl[f^{-1}(B)]$.

Sufficiency. Let $B \subseteq Y$. Then $(Y - B) \subseteq Y$. By hypothesis, $f^{-1}[Cl_\gamma(Y-B)] \subseteq Cl[f^{-1}(Y-B)]$. This implies $X - Cl[f^{-1}(Y-B)] \subseteq X - f^{-1}[Cl_\gamma(Y-B)]$. Hence $X - Cl[X - f^{-1}(B)] \subseteq f^{-1}[Y - Cl_\gamma(Y-B)]$. By applying Theorem 10 [Latif, 1993], $Int[f^{-1}(B)] \subseteq f^{-1}[Int_\gamma(B)]$. Now form Theorem 2.21, it follows that f is $\gamma-$ open.

D. $\gamma-$ Closed Functions

In this section we introduce $\gamma-$ *closed* functions and study certain properties and characterizations of this type of functions.

Definition 2.23. *A mapping* $f: (X,\tau) \to (Y,\sigma)$ *is called* $\gamma-$ *closed if the image of each closed set in* X *is a* $\gamma-$ *closed set.*

Theorem 2.24. *Prove that a mapping* $f: (X,\tau) \to (Y,\sigma)$ *is* $\gamma-$ *closed if and only* $Cl_\gamma[f(A)] \subseteq f[Cl(A)]$ *for each* $A \subseteq X$.

Proof *Necessity*. Let f be $\gamma-$ closed and let $A \subseteq X$. Then $f(A) \subseteq f[Cl(A)]$ and $f[Cl(A)]$ is a $\gamma-$ closed set in Y. Thus $Cl_\gamma[f(A)] \subseteq f[Cl(A)]$.

Conversely, suppose that $Cl_\gamma[f(A)] \subseteq f[Cl(A)]$, for each $A \subseteq X$. Let $A \subseteq X$ be a closed set.

Then $Cl_\gamma[f(A)] \subseteq f[Cl(A)] = f(A)$. *This shows that* $f(A)$ *is a* $\gamma-$ *closed set. Hence* f *is* $\gamma-$ *closed.*

Theorem 2.25. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be* $\gamma-$ *closed. If* $V \subseteq Y$ *and* $E \subseteq X$ *is an open set containing* $f^{-1}(V)$, *then there exists a* $\gamma-$ *open set* $G \subseteq Y$ *containing* V *such that* $f^{-1}(G) \subseteq E$.

Proof *Let* $G = Y - f(X-E)$. *Since* $f^{-1}(V) \subseteq E$, *we have* $f(X-E) \subseteq Y - V$. *Since* f *is* $\gamma-$ *closed, then G is a* $\gamma-$ *open set and* $f^{-1}(G) = X - f^{-1}[f(X-E)] \subseteq X-(X-E) = E$.

Theorem 2.26. *Suppose that* $f : (X,\tau) \to (Y,\sigma)$ *is a* $\gamma-$ *closed mapping. Then* $Int_\gamma[Cl_\gamma(f(A))] \subseteq f[Cl(A)]$ *for every subset A of X.*

Proof *Suppose* f *is a* $\gamma-$ *closed mapping and A is an arbitrary subset of X. Then* $f[Cl(A)]$ *is* $\gamma-$ *closed in Y. Then* $Int_\gamma[Cl_\gamma(f(Cl(A)))] \subseteq f[Cl(A)]$. *But also* $Int_\gamma[Cl_\gamma(f(A))] \subseteq Int_\gamma[Cl_\gamma(f(Cl(A)))]$. *Hence* $Int_\gamma[Cl_\gamma(f(A))] \subseteq f[Cl(A)]$.

Theorem 2.27. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be a* $\gamma-$ *closed function, and* $B, C \subseteq Y$.

(1) If U is an open neighbourhood of $f^{-1}(B)$, then there exists a $\gamma-$ *open* neighbourhood V of B such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

(2) *If* f *is also onto, then if* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint open neighbourhoods, so have B and C.*

Proof (1) *Let* $V = Y - f(X-U)$. *Then* $V^c = Y - V = f(U^c)$. *Since* f *is* $\gamma-$ *closed, so V is a* $\gamma-$ *open set. Since* $f^{-1}(B) \subseteq U$, *we have* $V^c = f(U^c) \subseteq f[f^{-1}(B^c)] \subseteq B^c$. *Hence,* $B \subseteq V$, *and thus V is a* $\gamma-$ *open neighbourhood of B. Further* $U^c \subseteq f^{-1}[f(U^c)] = f^{-1}(V^c) = [f^{-1}(V)]^c$. *This proves that* $f^{-1}(V) \subseteq U$.

(2) *If* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint open neighbourhoods M and N, then by* (1), *we have* $\gamma-$ *open neighbourhoods U and V of B and C respectively such that* $f^{-1}(B) \subseteq f^{-1}(U) \subseteq Int_\gamma(M)$ *and* $f^{-1}(C) \subseteq f^{-1}(V) \subseteq Int_\gamma(N)$. *Since M and N are disjoint, so are* $Int_\gamma(M)$ *and* $Int_\gamma(N)$, *and hence so* $f^{-1}(U)$ *and* $f^{-1}(V)$ *are disjoint as well. It follows that U and V are disjoint too as f is onto.*

Theorem 2.28. *Prove that a surjective mapping* $f : (X,\tau) \to (Y,\sigma)$ *is* $\gamma-$ *closed if and only if for each subset B of Y and each open set U in X containing* $f^{-1}(B)$, *there exists a* $\gamma-$ *open set V in Y containing B such that* $f^{-1}(V) \subseteq U$.

Proof *Necessity. This follows from* (1) *of Theorem* 2.27.

Sufficiency. Suppose F is an arbitrary closed set in X. Let y be an arbitrary point in $Y - f(F)$. *Then* $f^{-1}(y) \subseteq X - f^{-1}[f(F)] \subseteq (X-F)$ *and* $(X-F)$ *is open in X. Hence by hypothesis, there exists a* $\gamma-$ *open set* V_y *containing y such that* $f^{-1}(V_y) \subseteq (X-F)$. *This implies that* $y \in V_y \subseteq [Y - f(F)]$. *Thus* $Y - f(F) = \cup\{V_y : y \in Y - f(F)\}$. *Hence* $Y - f(F)$, *being a union of* $\gamma-$ *open sets is* $\gamma-$ *open. Thus its complement* $f(F)$ *is* $\gamma-$ *closed. This shows that f is* $\gamma-$ *closed.*

Theorem 2.29. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be a bijection. Then the following are equivalent*:

(a) f *is* $\gamma-$ *closed.*

(b) f *is* $\gamma-$ *open.*

(c) f^{-1} *is* $\gamma-$ *continuous.*

Proof $(a) \Rightarrow (b)$: *Let* $U \in \tau$. *Then* $X - U$ *is closed in X. By* (a), $f(X-U)$ *is* $\gamma-$ *closed in Y. But* $f(X-U) = f(X) - f(U) = Y - f(U)$. *Thus* $f(U)$ *is* $\gamma-$ *open in Y. This shows that f is* $\gamma-$ *open.*

$(b) \Rightarrow (c)$: Let $U \subseteq X$ be an open set. Since f is $\gamma -$ open. So $f(U) = (f^{-1})^{-1}(U)$ is $\gamma -$ open in Y. Hence f^{-1} is $\gamma -$ continuous.

$(c) \Rightarrow (a)$: Let A be an arbitrary closed set in X. Then $X - A$ is open in X. Since f^{-1} is $\gamma -$ continuous, $(f^{-1})^{-1}(X - A)$ is $\gamma -$ open in Y. But $(f^{-1})^{-1}(X - A) = f(X - A) = Y - f(A)$. Thus $f(A)$ is $\gamma -$ closed in Y. This shows that f is $\gamma -$ closed.

Example 2.30. *Let $X = Y = \{x, y, z, \}$ and $\tau = \sigma = \{X, \phi, \{x, y\}, \{x\}\}$. Then, a mapping $f : X \to Y$ which is defined by $f(x) = x$, $f(y) = z$ and $f(z) = y$, is $\gamma -$ open and $\gamma -$ closed but neither open nor closed.*

E. Pre $-\gamma -$ Open Functions

The purpose of this section is to introduce and discuss certain properties and characterizations of *pre $- \gamma -$ open* functions.

Definition 2.31. *Let (X, τ) and (Y, σ) be topological spaces. Then a function $f : X \to Y$ is said to be pre $- \gamma -$ open if and only if for each $A \in \tau^\gamma$, $f(A) \in \sigma^\gamma$.*

Theorem 2.32. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$ be any two pre $- \gamma -$ open functions. Then the composition function $g \circ f : X \to Z$ is a pre $- \gamma -$ open function.*

Proof *Let $U \in \tau^\gamma$. Then $f(U) \in \sigma^\gamma$ since f is pre $- \gamma -$ open. But then $g(f(U)) \in \mu^\gamma$ as g is pre $- \gamma -$ open. Hence, $g \circ f$ is pre $- \gamma -$ open.*

Theorem 2.33. *Prove that a mapping $f : (X, \tau) \to (Y, \sigma)$ is pre $- \gamma -$ open if and only if for each $x \in X$ and for any $U \in \tau^\gamma$ such that $x \in U$, there exists $V \in \sigma^\gamma$ such that $f(x) \in V$ and $V \subseteq f(U)$.*

Proof *Routine.*

Theorem 2.34. *Prove that a mapping $f : (X, \tau) \to (Y, \sigma)$ is pre $- \gamma -$ open if and only if for each $x \in X$ and for any $\gamma -$ neighbourhood U of x in X, there exists a $\gamma -$ neighbourhood V of $f(x)$ in Y such that $V \subseteq f(U)$.*

Proof *Necessity. Let $x \in X$ and let U be a $\gamma -$ neighbourhood of x. Then there exists $W \in \tau^\gamma$ such that $x \in W \subseteq U$. Then $f(x) \in f(W) \subseteq f(U)$. But $f(W) \in \sigma^\gamma$ as f is pre $- \gamma -$ open. Hence $V = f(W)$ is a $\gamma -$ neighbourhood of $f(x)$ and $V \subseteq f(U)$.*

Sufficiency. Let $U \in \tau^\gamma$. Let $x \in U$. Then U is a $\gamma -$ neighbourhood of x. So by hypothesis, there exists a $\gamma -$ neighbourhood $V_{f(x)}$ of $f(x)$ such that $f(x) \in V_{f(x)} \subseteq f(U)$. It follows at once that $f(U)$ is a $\gamma -$ neighbourhood of each of its points. Therefore $f(U)$ is $\gamma -$ open. Hence f is pre $- \gamma -$ open.

Theorem 2.35. *Prove that a function $f : (X, \tau) \to (Y, \sigma)$ is pre $- \gamma -$ open if and only if $f[Int_\gamma(A)] \subseteq Int_\gamma[f(A)]$, for all $A \subseteq X$.*

Proof *Necessity. Let $A \subseteq X$. Let $x \in Int_\gamma(A)$. Then there exists $U_x \in \tau^\gamma$ such that $x \in U_x \subseteq A$. So $f(x) \in f(U_x) \subseteq f(A)$ and by hypothesis, $f(U_x) \in \sigma^\gamma$. Hence $f(x) \in Int_\gamma[f(A)]$. Thus $f[Int_\gamma(A)] \subseteq Int_\gamma[f(A)]$.*

Sufficiency. Let $U \in \tau^\gamma$. Then by hypothesis, $f[Int_\gamma(U)] \subseteq Int_\gamma[f(U)]$. Since $Int_\gamma(U) = U$ as U is $\gamma -$ open. Also $Int_\gamma[f(U)] \subseteq f(U)$. Hence $f(U) = Int_\gamma[f(U)]$. Thus $f(U)$ is $\gamma -$ open in Y. So f is

$pre-\gamma-open$.

We remark that the equality does not hold in Theorem 2.35 as the following example shows.

Example 2.36. *Let* $X = Y = \{1,2\}$. *suppose* X *is antidiscrete and* Y *is discrete. Let* $f = Id.$, $A = \{1\}$. *Then* $\phi = f[Int_\gamma(A)] \neq Int_\gamma[f(A)] = \{1\}$.

Theorem 2.37. *Prove that a function* $f:(X,\tau)\to(Y,\sigma)$ *is* $pre-\gamma-open$ *if and only if* $Int_\gamma[f^{-1}(B)] \subseteq f^{-1}[Int_\gamma(B)]$, *for all* $B \subseteq Y$.

Proof Necessity. Let $B \subseteq Y$. Since $Int_\gamma[f^{-1}(B)]$ is $\gamma-open$ in X and f is $pre-\gamma-open$, $f[Int_\gamma(f^{-1}(B))]$ is $\gamma-open$ in Y. Also we have $f[Int_\gamma(f^{-1}(B))] \subseteq f[f^{-1}(B)] \subseteq B$. Hence, $f[Int_\gamma(f^{-1}(B))] \subseteq Int_\gamma(B)$. Therefore $Int_\gamma[f^{-1}(B)] \subseteq f^{-1}[Int_\gamma(B)]$.

Sufficiency. Let $A \subseteq X$. Then $f(A) \subseteq Y$. Hence by hypothesis, we obtain $Int_\gamma(A) \subseteq Int_\gamma[f^{-1}(f(A))] \subseteq f^{-1}[Int_\gamma(f(A))]$. This implies that $f[Int_\gamma(A)] \subseteq f[f^{-1}(Int_\gamma(f(A)))] \subseteq Int_\gamma[f(A)]$. Thus $f[Int_\gamma(A)] \subseteq Int_\gamma[f(A)]$, for all $A \subseteq X$. Hence, by Theorem 2.35, f is $pre-\gamma-open$.

Theorem 2.38. *Prove that a mapping* $f:(X,\tau)\to(Y,\sigma)$ *is* $pre-\gamma-open$ *if and only if* $f^{-1}[Cl_\gamma(B)] \subseteq Cl_\gamma[f^{-1}(B)]$, *for every subset* B *of* Y.

Proof *Necessity. Let* $B \subseteq Y$. *Let* $x \in f^{-1}[Cl_\gamma(B)]$. *Then* $f(x) \in Cl_\gamma(B)$. *Let* $U \in \tau^\gamma$ *such that* $x \in U$. *By hypothesis*, $f(U) \in \sigma^\gamma$ *and* $f(x) \in f(U)$. *Thus* $f(U) \cap B \neq \phi$. *Hence* $U \cap f^{-1}(B) \neq \phi$. *Therefore*, $x \in Cl_\gamma[f^{-1}(B)]$. *So we obtain* $f^{-1}[Cl_\gamma(B)] \subseteq Cl_\gamma[f^{-1}(B)]$.

Sufficiency. Let $B \subseteq Y$. *Then* $(Y-B) \subseteq Y$. *By hypothesis*, $f^{-1}[Cl_\gamma(Y-B)] \subseteq Cl_\gamma[f^{-1}(Y-B)]$. *This implies that* $X - Cl_\gamma[f^{-1}(Y-B)] \subseteq X - f^{-1}[Cl_\gamma(Y-B)]$. *Hence* $X - Cl_\gamma[X - f^{-1}(B)] \subseteq f^{-1}[Y - Cl_\gamma(Y-B)]$. *By Theorem* 3.7(6) [*Latif*, 2005], $Int_\gamma[f^{-1}(B)] \subseteq f^{-1}[Int_\gamma(B)]$. *Now by Theorem* 2.37, *it follows that* f *is* $pre-\gamma-open$.

Theorem 2.39. *Let* $f:(X,\tau)\to(Y,\sigma)$ *and* $g:(Y,\sigma)\to(Z,\mu)$ *be two mappings such that* $g\circ f:(X,\tau)\to(Z,\mu)$ *is* $\gamma-irresolute$. *Then*

(1) *If* g *is a* $pre-\gamma-open$ *injection, then* f *is* $\gamma-irresolute$.

(2) *If* f *is a* $pre-\gamma-open$ *surjection, then* g *is* $\gamma-irresolute$.

Proof (1) *Let* $U \in \sigma^\gamma$. *Then* $g(U) \in \mu^\gamma$ *since* g *is* $pre-\gamma-open$. *Also* $g\circ f$ *is* $\gamma-irresolute$. *Therefore, we have* $(g\circ f)^{-1}[g(U)] \in \tau^\gamma$. *Since* g *is an injection, so we have* : $(g\circ f)^{-1}[g(U)] = (f^{-1}\circ g^{-1})[g(U)] = f^{-1}[g^{-1}(g(U))] = f^{-1}(U)$. *Consequently* $f^{-1}(U)$ *is* $\gamma-open$ *in* X. *This proves that* f *is* $\gamma-irresolute$.

(2) *Let* $V \in \mu^\gamma$. *Then* $(g\circ f)^{-1}(V) \in \tau^\gamma$ *since* $g\circ f$ *is* $\gamma-irresolute$. *Also* f *is pre-*$\gamma-open$, $f\left[(g\circ f)^{-1}(V)\right]$ *is* $\gamma-open$ *in* Y. *Since* f *is surjective, we note that* $f\left[(g\circ f)^{-1}(V)\right]$ $= \left[f\circ(g\circ f)^{-1}\right](V) = [f\circ(f^{-1}\circ g^{-1})](V) = [(f\circ f^{-1})\circ g^{-1}](V) = g^{-1}(V)$. *Hence* g *is* $\gamma-irresolute$.

F. Pre $-\gamma-$ Closed Functions

In this last section, we introduce and explore several properties and characterizations of $pre-\gamma-closed$ functions.

Definition 2.40. *A function* $f:(X,\tau)\to(Y,\sigma)$ *is said to be* $pre-\gamma-closed$ *if and only if the* *image set* $f(A)$ *is* $\gamma-$ *closed for each* $\gamma-$ *closed subset A of X.*

Theorem 2.41. *The composition of two pre* $-\gamma-$ *closed mappings is a pre* $-\gamma-$ *closed mapping.*

Proof *The straight forward proof is omitted.*

Theorem 2.42. *Prove that a mapping* $f:(X,\tau)\to(Y,\sigma)$ *is pre* $-\gamma-$ *closed if and only if* $Cl_\gamma[f(A)]\subseteq f[Cl_\gamma(A)]$ *for every subset A of X.*

Proof *Necessity. Suppose f is a pre* $-\gamma-$ *closed mapping and A is an arbitrary subset of X. Then* $f[Cl_\gamma(A)]$ *is* $\gamma-$ *closed in Y. Since* $f(A)\subseteq f[Cl_\gamma(A)]$, *we obtain* $Cl_\gamma[f(A)]\subseteq f[Cl_\gamma(A)]$.

Sufficiency. Suppose F is an arbitrary $\gamma-$ *closed set in X. By hypothesis, we obtain* $f(F)\subseteq Cl_\gamma[f(F)]\subseteq f[Cl_\gamma(F)]=f(F)$. *Hence* $f(F)=Cl_\gamma[f(F)]$. *Thus* $f(F)$ *is* $\gamma-$ *closed in Y. It follows that f is pre* $-\gamma-$ *closed.*

Theorem 2.43. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be a mapping such that* $Int[Cl(f(A))]\subseteq f[Cl_\gamma(A)]$ *for every subset A of X. Then f is pre* $-\gamma-$ *closed.*

Proof *Suppose A is an arbitrary* $\gamma-$ *closed set in X. Then by hypothesis, we have* $Int[Cl(f(A))]\subseteq f[Cl_\gamma(A)]=f(A)$. *Take* $B=Cl[f(A)]$. *Then B is closed in Y. Also it implies that* $Int(B)\subseteq f(A)\subseteq B$. *Hence* $f(A)$ *is semi* $-$ *closed in Y. Since* $\sigma\subseteq\sigma^\gamma$. *Thus* $f(A)$ *is* $\gamma-$ *closed in Y. This implies that f is pre* $-\gamma-$ *closed.*

Theorem 2.44. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be a pre* $-\gamma-$ *closed function, and* $B,C\subseteq Y$.

(1) If U is a $\gamma-$ *open* neighbourhood of $f^{-1}(B)$, then there exists a $\gamma-$ *open* neighbourhood V of B such that $f^{-1}(B)\subseteq f^{-1}(V)\subseteq U$.

(2) If f is also onto, then if $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint $\gamma-$ *open* neighbourhoods, so have B and C.

Proof (1) *Let* $V=Y-f(X-U)$. *Then* $V^c=Y-V=f(U^c)$. *Since f is pre* $-\gamma-$ *closed, so V is* $\gamma-$ *open. Since* $f^{-1}(B)\subseteq U$, *we have* $V^c=f(U^c)\subseteq f[f^{-1}(B^c)]\subseteq B^c$. *Hence,* $B\subseteq V$, *and thus V is a* $\gamma-$ *open neighbourhood of B. Further* $U^c\subseteq f^{-1}[f(U^c)]=f^{-1}(V^c)=[f^{-1}(V)]^c$. *This proves that* $f^{-1}(V)\subseteq U$.

(2) *If* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* $\gamma-$ *open neighbourhoods M and N, then by* (1), *we have* $\gamma-$ *open neighbourhoods U and V of B and C respectively such that* $f^{-1}(B)\subseteq f^{-1}(U)\subseteq Int_\gamma(M)$ *and* $f^{-1}(C)\subseteq f^{-1}(V)\subseteq Int_\gamma(N)$. *Since M and N are disjoint, so are* $Int_\gamma(M)$ *and* $Int_\gamma(N)$, *and hence so* $f^{-1}(U)$ *and* $f^{-1}(V)$ *are disjoint as well. It follows that U and V are disjoint too as f is onto.*

Theorem 2.45. *Prove that a surjective mapping* $f:(X,\tau)\to(Y,\sigma)$ *is pre* $-\gamma-$ *closed if and only if for each subset B of Y and each* $\gamma-$ *open set U in X containing* $f^{-1}(B)$, *there exists a* $\gamma-$ *open set V in Y containing B, such that* $f^{-1}(V)\subseteq U$.

Proof *Necessity. This follows from* (1) *of Theorem* 2.44.

Sufficiency. Suppose F is an arbitrary $\gamma-$ *closed set in X. Let y be an arbitrary point in* $Y-f(F)$. *Then* $f^{-1}(y)\subseteq X-f^{-1}[f(F)]\subseteq(X-F)$ *and* $(X-F)$ *is* $\gamma-$ *open in X. Hence by hypothesis, there exists a* $\gamma-$ *open set* V_y *containing y such that* $f^{-1}(V_y)\subseteq(X-F)$. *This implies that* $y\in V_y\subseteq[Y-f(F)]$. *Thus* $Y-f(F)=\cup\{V_y : y\in Y-f(F)\}$. *Hence* $Y-f(F)$, *being a union of* $\gamma-$ *open sets is* $\gamma-$ *open. Thus its complement* $f(F)$ *is* $\gamma-$ *closed. This shows that f is* $\gamma-$ *closed.*

Theorem 2.46. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be a bijection. Then the following are equivalent*:

(1) f *is* $pre-\gamma-closed$.

(2) f *is* $pre-\gamma-open$.

(3) f^{-1} *is* $\gamma-irresolute$.

Proof (1) $\Rightarrow$ (2) : *Let* $U \in \tau^{\gamma}$. *Then* $X - U$ *is* $\gamma-closed$ *in* X. *By* (1), $f(X-U)$ *is* $\gamma-closed$ *in* Y. *But* $f(X-U) = f(X) - f(U) = Y - f(U)$. *Thus* $f(U)$ *is* $\gamma-open$ *in* Y. *This shows that* f *is* $pre-\gamma-open$.

(2) $\Rightarrow$ (3) : *Let* $A \subseteq X$. *Since* f *is* $pre-\gamma-open$, *so by Theorem* 2.38, $f^{-1}[Cl_{\gamma}(f(A))] \subseteq Cl_{\gamma}[f^{-1}(f(A))]$. *It implies that* $Cl_{\gamma}[f(A)] \subseteq f[Cl_{\gamma}(A)]$. *Thus* $Cl_{\gamma}\left[(f^{-1})^{-1}(A)\right] \subseteq (f^{-1})^{-1}[Cl_{\gamma}(A)]$, *for all* $A \subseteq X$. *Then by Theorem* 2.8, *it follows that* f^{-1} *is* $\gamma-irresolute$.

(3) $\Rightarrow$ (1) : *Let* A *be an arbitrary* $\gamma-closed$ *set in* X. *Then* $X - A$ *is* $\gamma-open$ *in* X. *Since* f^{-1} *is a* $\gamma-irresolute$, $(f^{-1})^{-1}(X-A)$ *is* $\gamma-open$ *in* Y. *But* $(f^{-1})^{-1}(X-A) = f(X-A) = Y - f(A)$. *Thus* $f(A)$ *is* $\gamma-closed$ *in* Y. *This shows that* f *is* $pre-\gamma-closed$.

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