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**ON HARMONIC MORPHISMS PROJECTING
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M. T. MUSTAFA

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ABSTRACT. For Riemannian manifolds M and N , admitting a submersive harmonic morphism ϕ with compact fibres, we introduce the vertical and horizontal components of a real-valued function f on $U \subset M$. By comparing the Laplacians on M and N , we determine conditions under which a harmonic function on $U = \phi^{-1}(V) \subset M$ projects down, via its horizontal component, to a harmonic function on $V \subset N$.

1. INTRODUCTION AND PRELIMINARIES

Harmonic morphisms are the maps between Riemannian manifolds which preserve germs of harmonic functions i.e. these (locally) pull back harmonic functions to harmonic functions. The aim of this note is to analyse the converse situation and to investigate the class of harmonic morphisms that (locally) projects or pushes forward harmonic functions to harmonic functions, in the sense of Definition 2.4. If such a class exists, another interesting question arises “*to what extent does the pull back of the projected function preserve the original function*”.

The formal theory of harmonic morphisms between Riemannian manifolds began with the work of Fuglede [6] and Ishihara [10].

Definition 1.1. A smooth map $\phi : M^m \rightarrow N^n$ between Riemannian manifolds is called a *harmonic morphism* if, for every real-valued function f which is harmonic on an open subset V of N with $\phi^{-1}(V)$ non-empty, $f \circ \phi$ is a harmonic function on $\phi^{-1}(V)$.

These maps are related to horizontally (weakly) conformal maps which are a natural generalization of Riemannian submersions.

For a smooth map $\phi : M^m \rightarrow N^n$, let $C_\phi = \{x \in M | \text{rank} d\phi_x < n\}$ be its *critical set*. The points of the set $M \setminus C_\phi$ are called *regular points*. For each $x \in M \setminus C_\phi$, the *vertical space* at x is defined by $T_x^V M = \text{Ker} d\phi_x$. The *horizontal space* $T_x^H M$ at x is given by the orthogonal complement of $T_x^V M$ in $T_x M$.

Definition 1.2. A smooth map $\phi : (M^m, \mathbf{g}) \rightarrow (N^n, \mathbf{h})$ is called *horizontally (weakly) conformal* if $d\phi = 0$ on C_ϕ and the restriction of ϕ to $M \setminus C_\phi$ is a conformal submersion,

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that is, for each $x \in M \setminus C_\phi$, the differential $d\phi_x : T_x^H M \rightarrow T_{\phi(x)}N$ is conformal and surjective. This means that there exists a function $\lambda : M \setminus C_\phi \rightarrow \mathbb{R}^+$ such that

$$\mathbf{h}(d\phi(X), d\phi(Y)) = \lambda^2 \mathbf{g}(X, Y) \quad \forall X, Y \in T_x^H M.$$

By setting $\lambda = 0$ on C_ϕ , we can extend $\lambda : M \rightarrow \mathbb{R}_0^+$ to a continuous function on M such that λ^2 is smooth. The extended function $\lambda : M \rightarrow \mathbb{R}_0^+$ is called the *dilation* of the map.

Recall that a map $\phi : M^m \rightarrow N^n$ is said to be *harmonic* if it extremizes the associated energy integral $E(\phi) = \frac{1}{2} \int_\Omega \|\phi_*\|^2 dv^M$ for every compact domain $\Omega \subset M$. It is well-known that a map ϕ is harmonic if and only if its tension field vanishes.

Harmonic morphisms can be viewed as a subclass of harmonic maps in the light of the following characterization, obtained in [6, 10].

A smooth map is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal.

What is special about this characterization of harmonic morphism is that it equips them with geometric as well as analytic features. For instance, the following result of Baird-Eells [2, Riemannian case] and Gudmundsson [8, semi-Riemannian case] reflects such properties of harmonic morphisms.

Theorem 1.3. *Let $\phi : M^m \rightarrow N^n$ be a horizontally conformal submersion with dilation λ . If*

- (1) $n = 2$, then ϕ is a harmonic map if and only if it has minimal fibres.
- (2) $n \geq 3$, then two of the following imply the other,
 - (a) ϕ is a harmonic map
 - (b) ϕ has minimal fibres
 - (c) $\mathcal{H}\mathbf{grad}\lambda^2 = 0$ where \mathcal{H} denotes the projection on the horizontal subbundle of TM .

For the fundamental results and properties of harmonic morphisms, the reader is referred to [1, 4, 6, 11] and for an updated **online** bibliography to [9].

2. HARMONIC MORPHISMS PROJECTING HARMONIC FUNCTIONS

Given a submersive harmonic morphism $\phi : M^m \rightarrow N^n$ with compact fibres, we can define the horizontal and vertical components of every integrable function f on $U \subset M$, using fibre integration.

Definition 2.1. Let $\phi : M^m \rightarrow N^n$ be a submersive harmonic morphism between Riemannian manifolds with compact fibres. Define the horizontal component of an

integrable function f , on M , at x as the average of f taken over the fibre $\phi^{-1}(\phi(x))$. Precisely, for any $V \subset N$ and integrable function $f : U = \phi^{-1}(V) \subset M \rightarrow \mathbb{R}$, the *horizontal component* of f at x is defined as

$$(\mathcal{H}f)(x) = \frac{1}{\text{vol}(\phi^{-1}(y))} \left(\int_{\phi^{-1}(y)} f v^{\phi^{-1}(y)} \right) (\phi(x))$$

where $x \in U$, $\phi(x) = y$, $v^{\phi^{-1}(y)}$ is the volume element of the fibre $\phi^{-1}(y)$, $\text{vol}(\phi^{-1}(y))$ is the volume of the fibre $\phi^{-1}(y)$ and $\left(\int_{\phi^{-1}(y)} f v^{\phi^{-1}(y)} \right) (\phi(x))$ denotes the integral of $f|_{\phi^{-1}(\phi(x))}$.

The *vertical component* of f is given by

$$(\mathcal{V}f)(x) = (f - \mathcal{H}f)(x).$$

Definition 2.2. A function $f : U \subset M \rightarrow \mathbb{R}$ is called *horizontally homothetic* if the horizontal component of $\mathbf{grad}(f)$ vanishes i.e. $\mathcal{H}\mathbf{grad}(f) = 0$ where \mathcal{H} denotes the orthogonal projection on the horizontal subbundle.

The components $\mathcal{H}f$ and $\mathcal{V}f$ have the following basic properties.

Lemma 2.3. *Let $\phi : M^m \rightarrow N^n$ be a submersive harmonic morphism with compact minimal fibres. Consider $x \in U$ and a function $f : U \subset M \rightarrow \mathbb{R}$.*

- (1) *If f is horizontally homothetic at x then $\mathcal{H}f$ is also horizontally homothetic at x .*
- (2) *If $\mathcal{H}f$ is horizontally homothetic at x and either $X_i(f) \geq 0$ or $X_i(f) \leq 0$ (for all i) on the fibre through x then f is horizontally homothetic, where $\{X_i\}_{i=1}^n$ is a local orthonormal frame for the horizontal distribution.*
- (3) *If f is constant along the fibre through x then $\mathcal{V}f = 0$.*

Proof. The proof can be completed by following the calculations in Proposition 2.5 (below). \square

Definition 2.4. Given a submersive harmonic morphism $\phi : M^m \rightarrow N^n$ with compact fibres, and let $f : U = \phi^{-1}(V) \subset M \rightarrow \mathbb{R}$ be an integrable function. The horizontal component of f defines a function $\tilde{f} : V \subset N \rightarrow \mathbb{R}$ as

$$\tilde{f}(y) = (\mathcal{H}f)(x)$$

where $x \in U$ and $y = \phi(x)$. The function \tilde{f} is called the *projection* of f on N , via the map ϕ .

The conditions under which the above projection takes harmonic functions to harmonic functions can be obtained by employing an identity relating the Laplacian on the fibre with the Laplacians on the domain and target manifolds.

Proposition 2.5. *Let $\phi : (M^m, \mathbf{g}) \rightarrow (N^n, \mathbf{h})$ ($n > 2$) be a non-constant submersive harmonic morphism with dilation λ , having compact, connected and minimal fibres. Then for any $V \subset N$ and $f : U = \phi^{-1}(V) \subset M \rightarrow \mathbb{R}$,*

$$\begin{aligned} \Delta^N \tilde{f} &= \frac{1}{\text{vol}(\phi^{-1}(y))} \int_{\phi^{-1}(y)} \frac{1}{\lambda^2} \left(\Delta^M f - \Delta^{\phi^{-1}(y)} f \right) v^{\phi^{-1}(y)} \\ &+ \frac{1}{\text{vol}(\phi^{-1}(y))} \sum_{i=1}^n \int_{\phi^{-1}(y)} (\mathcal{V} \nabla_{X_i}^M X_i) f v^{\phi^{-1}(y)} \end{aligned}$$

where $x \in U$, $\phi(x) = y$, \tilde{f} is as defined in Definition 2.4 and Δ^M , Δ^N , $\Delta^{\phi^{-1}(y)}$ are the Laplacians on M , N , $\phi^{-1}(y)$ respectively, and $\{X_i\}_{i=1}^n$ denote the horizontal lift of a local orthonormal frame $\{X'_i\}_{i=1}^n$ for TN .

Proof. First notice from Theorem 1.3 that λ is horizontally homothetic; a fact which will be used repeatedly in the proof.

Choose a local orthonormal frame $\{X'_i\}_{i=1}^n$ for TN . If X_i denotes the horizontal lift of X'_i for $i = 1, \dots, n$ then $\{\lambda X_i\}_{i=1}^n$ is a local orthonormal frame for the horizontal distribution. Let $\{X_\alpha\}_{\alpha=n+1}^m$ be a local orthonormal frame for the vertical distribution. Then we can write the Laplacian Δ^M on M as

$$\begin{aligned} \Delta^M &= \sum_{i=1}^n \{ \lambda X_i \circ \lambda X_i - \nabla_{\lambda X_i}^M \lambda X_i \} + \sum_{\alpha=n+1}^m \{ X_\alpha \circ X_\alpha - \nabla_{X_\alpha}^M X_\alpha \} \\ (2.1) \quad &= \lambda^2 \sum_{i=1}^n \{ X_i \circ X_i - \nabla_{X_i}^M X_i \} + \sum_{\alpha=n+1}^m \{ X_\alpha \circ X_\alpha - \nabla_{X_\alpha}^M X_\alpha \}. \end{aligned}$$

Now the Laplacian of the fibre $\phi^{-1}(y)$ is

$$\Delta^{\phi^{-1}(y)} = \sum_{\alpha=n+1}^m \{ X_\alpha \circ X_\alpha - \nabla_{X_\alpha}^{\phi^{-1}(y)} X_\alpha \}.$$

Therefore, from Equation 2.1 we obtain

$$\begin{aligned} \Delta^M &= \lambda^2 \sum_{i=1}^n \{ X_i \circ X_i - \mathcal{H} \nabla_{X_i}^M X_i \} + \Delta^{\phi^{-1}(y)} - H - \lambda^2 \sum_{i=1}^n \mathcal{V} \nabla_{X_i}^M X_i \\ (2.2) \quad &= \lambda^2 \sum_{i=1}^n \{ X_i \circ X_i - \mathcal{H} \nabla_{X_i}^M X_i \} + \Delta^{\phi^{-1}(y)} - \lambda^2 \sum_{i=1}^n \mathcal{V} \nabla_{X_i}^M X_i \end{aligned}$$

where H is $(m - n)$ times the mean curvature vector field of the fibres and \mathcal{H} , \mathcal{V} denote the orthogonal projections on the horizontal and vertical subbundles of TM , respectively.

Since X_i is horizontal lift of X'_i for $i = 1, \dots, n$, therefore for the function \tilde{f} we have

$$\begin{aligned}
X'_i(\tilde{f}) &= \frac{1}{\text{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f)v^{\phi^{-1}(y)} + \int_{\phi^{-1}(y)} fL_{X_i}(v^{\phi^{-1}(y)}) \right\} \\
&= \frac{1}{\text{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f)v^{\phi^{-1}(y)} + \sum_{\alpha=n+1}^m \int_{\phi^{-1}(y)} f\mathbf{g}(\nabla_{X_\alpha}^M X_i, X_\alpha)v^{\phi^{-1}(y)} \right\} \\
(2.3) \quad &= \frac{1}{\text{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f)v^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} f\mathbf{g}(H, X_i)v^{\phi^{-1}(y)} \right\}.
\end{aligned}$$

The volume of the fibres does not vary in the horizontal direction because of the relation $X'_i(\text{vol}(\phi^{-1}(y))) = -\int_{\phi^{-1}(y)} \mathbf{g}(H, X_i)v^{\phi^{-1}(y)}$ and the fact that the fibres are minimal.

Similarly, we obtain

$$\begin{aligned}
X'_i \circ X'_i(\tilde{f}) &= \frac{1}{\text{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i \circ X_i(f)v^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} X_i(f) \cdot \mathbf{g}(H, X_i)v^{\phi^{-1}(y)} \right\} \\
(2.4) \quad &- \frac{1}{\text{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f\mathbf{g}(H, X_i))v^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} f(\mathbf{g}(H, X_i))^2v^{\phi^{-1}(y)} \right\}
\end{aligned}$$

The horizontal homothety of the dilation implies that $\mathcal{H}\nabla_{X_i}^M X_i$ is the horizontal lift of $\nabla_{X'_i}^N X'_i$, cf. [3, Lemma 3.1], therefore, we have

$$(2.5) \quad \nabla_{X'_i}^N X'_i(\tilde{f}) = \frac{1}{\text{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} (\mathcal{H}\nabla_{X_i}^M X_i)(f)v^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} f \cdot \mathbf{g}(H, \mathcal{H}\nabla_{X_i}^M X_i)v^{\phi^{-1}(y)} \right\}.$$

Now using Equations 2.3, 2.4, 2.5, along with the condition that the fibres are minimal, in Equation 2.2 completes the proof. \square

From the above Proposition, we see that it suffices to take λ constant to have both f and \tilde{f} harmonic on M and N respectively.

Theorem 2.6. *Let $\phi : M^m \rightarrow N^n$ ($n \geq 2$) be a non-constant harmonic morphism with constant dilation and compact, connected fibres. Then the projection $\tilde{f} : V \subset N \rightarrow \mathbb{R}$ (via ϕ) of any harmonic function $f : U = \phi^{-1}(V) \subset M \rightarrow \mathbb{R}$ is a harmonic function. Moreover, $\mathcal{H}f = \tilde{f} \circ \phi$. If $[f_{\mathcal{H}}]$ denotes the class of harmonic functions on $U = \phi^{-1}(V)$ having same horizontal component then each class $[f_{\mathcal{H}}]$ has a unique representative in the space of harmonic functions on V .*

Proof. Since the fibres are compact and dilation is constant, the harmonicity of \tilde{f} follows from Proposition 2.5 because

$$\int_{\phi^{-1}(y)} \Delta^{\phi^{-1}(y)} f v^{\phi^{-1}(y)} = 0$$

and

$$\mathcal{V}\nabla_{X_i}^M X_i = -\frac{\lambda^2}{2}\mathcal{V}\mathbf{grad}\left(\frac{1}{\lambda^2}\right) = 0.$$

Rest of the proof follows from the construction of \tilde{f} . \square

As an application, we give a description of harmonic functions on manifolds admitting harmonic Riemannian submersions with compact fibres.

Corollary 2.7. *Let M^m be a Riemannian manifold admitting a harmonic Riemannian submersion $\phi : M^m \rightarrow N^n$ with compact fibres. Then*

- (1) *Every horizontally homothetic (in particular constant) harmonic function on $U \subset M$ is horizontal i.e. $\mathcal{V}f = 0$.*
- (2) *Every non-horizontally-homothetic harmonic function f on $U \subset M$ satisfies one of the following:*
 - (a) $\mathcal{V}f \neq 0$.
 - (b) $\mathcal{V}f = 0$ and $X_i(\mathcal{H}f) \neq 0$ for at least one $i \in \{1, \dots, n\}$.
 - (c) $\mathcal{V}f = 0$, $X_i(\mathcal{H}f) = 0$ (for all i) and $X_i(f)$ changes sign on the fibre, for at least one $i \in \{1, \dots, n\}$.

Proof. Equation 2.2 implies that a horizontally homothetic harmonic function on M is harmonic on the fibre and hence is constant on the fibre. Now using Lemma 2.3 we get the proof. \square

Remark 2.8.

- (1) Since an \mathbb{R}^N -valued map $f = (f^1, \dots, f^N)$ is harmonic if and only if each of its component is harmonic, we see that Riemannian submersions with compact fibres project \mathbb{R}^N -valued harmonic maps from $\phi^{-1}(V)$ to \mathbb{R}^N -valued harmonic maps from V .
- (2) Given a Lie group G and a compact subgroup H of G , the standard projection $\phi : G \rightarrow G/H$ with G -invariant metric provides many examples satisfying the hypothesis of Theorem 2.6. Further examples can be obtained from Bergery's construction $\phi : G/K \rightarrow G/H$ with $K \subset H \subset G$ and K, H compact; see [5] for the details of the metrics for which ϕ is a harmonic morphism. Another reference for such examples is [7, Ch. 6].

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DEPARTMENT OF MATHEMATICAL SCIENCES, KING FAHD UNIVERSITY OF PETROLEUM & MINERALS, DHAHRAN 31261, SAUDI ARABIA.

E-mail address: `tmustafa@kfupm.edu.sa`