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Absract A correlated bivariate chisquare distribution is derived from the bivariate Wishart distribution. Several theorems are established for deriving the product moments of the distribution. Closed form expressions for product moments are obtained for some orders of interest.

1. Introduction

Let $X_1, X_2, \cdots X_N \ (N > 2)$ be a two-dimensional independent normal random vector with mean vector $\bar{X}$ so that the sums of squares and cross product matrix is given by $\sum_{j=1}^{N}(X_j - \bar{X})(X_j - \bar{X})' = A$. The random symmetric positive definite matrix A is said to have a Wishart distribution with parameters $m = N - 1$ and $\Sigma(2\times 2) > 0$, written as $A \sim W_2(m, \Sigma)$ if its probability density function is given by

$$f_*(A) = \frac{|A|^{(m-3)/2} \exp\left(-\frac{1}{2} tr\Sigma^{-1}A\right)}{2^m \sqrt{\pi}\, |\Sigma|^{m/2} \prod_{i=1}^{p} \Gamma\left(\frac{1}{2}(m+1-i)\right)}, \quad A > 0, m > 2$$

(See e.g. Anderson, 2003, 252).

In this paper we consider deriving a bivariate chisquare distribution based on the the bivariate Wishart distribution. Let the matrix A for the bivariate case be denoted by $A = (a_{ik}), i = 1,2; k = 1,2$ where $a_{ii} = ms_i^2 = \sum_{j=1}^{N}(X_{ij} - \bar{X}_i)^2, m = N - 1, (i = 1,2)$ and $a_{12} = \sum_{j=1}^{N}(X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) = mrs_1s_2$. Also let the elements of the matrix $\Sigma = (\sigma_{ik})$, $i = 1,2; k = 1,2$ where $\sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2, \sigma_{12} = \rho\sigma_1\sigma_2$ where $\sigma_1 > 0, \sigma_2 > 0$ and ρ, $(-1 < \rho < 1)$, is the product moment correlation coefficient between X_1 and X_2.

Fisher (1915) derived the distribution of the bivariate Wishart matrix A for $p = 2$ in order to study the distribution of correlation coefficient from a normal sample. Wishart (1928) obtained the distribution of p-variate Wishart matrix as the joint distribution of sample variances and covariances from multivariate normal population. Because of its important role in multivariate statistical analysis, various authors have given different derivations. See the references in Gupta and Nagar (200, 87-88) who has also a good update on the moments of Wishart distribution.

The primary objective of the paper is to derive the joint moments of the bivariate chisquare distribution derived in Section 2. We refer to Kotz, Balakrishnan and Johnson (2000) for a historic development of the distribution. The following notation will be used in sequel:

$a_{(i)} = a(a-1)(a-2)\cdots(a-i+1)$

2. The Density Function of the Bivariate Chisquare Distribution

Theorem 2.1 The random variable U and V is said to have a correlated bivariate chisquare distribution with m degrees of freedom if its probability density function is given by

$$f(u,v) = \frac{\left(1-\rho^2\right)^{-m/2}}{2^m\ \Gamma(m/2)} (uv)^{m/2-1} e^{-\frac{u+v}{2(1-\rho^2)}} \sum_{k=0}^{\infty} \left(\frac{\rho\sqrt{uv}}{2\left(1-\rho^2\right)} \right)^k \frac{1}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{k+m}{2}\right)},$$

$m = N-1 > 2$, $-1 < \rho < 1$.

Proof. The pdf of the elements of A can be written as

$$f_1(a_{11},a_{22},a_{12}) = \frac{\left(1-\rho^2\right)^{-m/2} (\sigma_1\sigma_2)^{-m}}{2^m \sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)} \left(a_{11}a_{22}-a_{12}^2\right)^{(m-3)/2}$$
$$\times \exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2}\right) \exp\left(-\frac{a_{22}}{2(1-\rho^2)\sigma_2^2}\right) \exp\left(-\frac{\rho a_{12}}{2(1-\rho^2)\sigma_1\sigma_2}\right)$$

$a_{11} > 0, a_{22} > 0, -\infty < a_{12} < \infty, m > 2, -1 < \rho < 1$. Under the transformation $a_{11} = ms_1^2, a_{22} = ms_2^2, a_{12} = mrs_1s_2$ with Jacobian $m^3 s_1 s_2$, the pdf of S_1^2, S_2^2 and R is given by

$$f_2(s_1^2,s_2^2,r) = \frac{m^m\left(1-\rho^2\right)^{-m/2} (\sigma_1\sigma_2)^{-m}}{2^m \sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)} s_1^{m-2} \exp\left(-\frac{ms_1^2}{2(1-\rho^2)\sigma_1^2}\right)$$
$$\times s_2^{m-2} \exp\left(-\frac{ms_2^2}{2(1-\rho^2)\sigma_2^2}\right) \left(1-r^2\right)^{(m-3)/2} \exp\left(-\frac{m\rho r s_1 s_2}{2(1-\rho^2)\sigma_1\sigma_2}\right).$$

By expanding the last term we have

$$f_2(s_1^2,s_2^2,r) = \frac{m^m\left(1-\rho^2\right)^{-m/2} (\sigma_1\sigma_2)^{-m}}{2^m \sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)}$$
$$\times \sum_{k=0}^{\infty} \left(\frac{m\rho}{(1-\rho^2)\sigma_1\sigma_2}\right)^k \frac{1}{k!} s_1^{m-2+k} \exp\left(-\frac{ms_1^2}{2(1-\rho^2)\sigma_1^2}\right)$$
$$\times s_2^{m-2+k} \exp\left(-\frac{ms_2^2}{2(1-\rho^2)\sigma_2^2}\right) r^k \left(1-r^2\right)^{(m-3)/2}.$$

By the use of $z = m-1$ in the duplication formula of gamma function

$\Gamma(z)=\frac{2^{z-1}}{\sqrt{\pi}}\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{m}{2}\right)$ we have

$$f_2(s_1^2,s_2^2,r)=\frac{m^m\left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}}{4\pi\ \Gamma(m-1)}$$
$$\times\sum_{k=0}^{\infty}\frac{(m\rho)^k}{(1-\rho^2)^k(\sigma_1\sigma_2)^k\,k\,!}\,s_1^{m-2+k}\exp\left(-\frac{ms_1^2}{2(1-\rho^2)\sigma_1^2}\right) \tag{2.1}$$
$$\times s_2^{m-2+k}\exp\left(-\frac{ms_2^2}{2(1-\rho^2)\sigma_2^2}\right)r^k\left(1-r^2\right)^{(m-3)/2}$$

The probability density function of S_1^2 and S_2^2 is given by

$$f_3(s_1^2,s_2^2)=\frac{m^m\left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}}{4\pi\ \Gamma(m-1)}$$
$$\times\sum_{k=0}^{\infty}\frac{(m\rho)^k}{(1-\rho^2)^k(\sigma_1\sigma_2)^k\,k\,!}\,s_1^{m-2+k}\exp\left(-\frac{ms_1^2}{2(1-\rho^2)\sigma_1^2}\right)$$
$$\times s_2^{m-2+k}\exp\left(-\frac{ms_2^2}{2(1-\rho^2)\sigma_2^2}\right)\frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{k+m}{2}\right)}.$$

Then the probability density function of $U=\frac{mS_1^2}{\sigma_1^2}$ and $V=\frac{mS_2^2}{\sigma_2^2}$ with Jacobian $J(s_1^2,s_2^2\to u,v)=m^{-2}\sigma_1^2\sigma_2^2$ is given by

$$f\,(u,v\,)=\frac{\left(1-\rho^2\right)^{-m/2}\Gamma\left(\frac{m-1}{2}\right)}{4\pi\ \Gamma(m-1)}(uv\,)^{m/2-1}e^{-\frac{1}{2}\frac{(u+v\,)}{(1-\rho^2)}}\sum_{k=0}^{\infty}\left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^k\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(k+1)\Gamma\left(\frac{k+m}{2}\right)}\ . \tag{2.2}$$

By the use of the duplication formula of gamma function $\Gamma(2z\,)=2^{2z-1}\ \Gamma(z\,)\ \Gamma(z\,+1/2)/\sqrt{\pi}$ with $2z=m-1$ and $2z=k$ respectively we have

$$\frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(m-1\right)}=\frac{\sqrt{\pi}}{2^{m-2}\Gamma(m/2)}\quad\text{and}\quad\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(k+1\right)}=\frac{\sqrt{\pi}}{2^k\,\Gamma\left(k/2+1\right)}$$

so that an equivalent form of the pdf in (2.2) is given by

$$f(u,v)=\frac{\left(1-\rho^2\right)^{-m/2}}{2^m\ \Gamma(m/2)}\ (uv)^{m/2-1}e^{-\frac{u+v}{2(1-\rho^2)}}\sum_{k=0}^{\infty}\left(\frac{\rho\sqrt{uv}}{2\left(1-\rho^2\right)}\right)^k\frac{1}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{k+m}{2}\right)} \tag{2.3}$$

It can be checked that the product moment correlation between U and V is given by $Corr(U,\ V)=\rho^2$. Note that if $\rho=0$, the above pdf becomes the product of that of the two independent chisquare distributions each with m degrees of freedom. By integrating out v we have the pdf of U as

$$g(u)=\frac{u^{m/2-1}e^{-\frac{u}{2(1-\rho^2)}}}{2^{m/2}\ \Gamma(m/2)}\ \sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(k/2+1\right)},\ u>0 \tag{2.4}$$

where $z=\dfrac{u\rho^2}{2\left(1-\rho^2\right)}$. The random variables U and V are identically distributed. When $\rho=0$, the above reduces to the pdf of chisquare distribution with m degrees of freedom. Note that $\sum_{k=0}^{\infty}\dfrac{z^k}{\Gamma\left(k/2+1\right)}$ is the Mittag-Leffler function $E_\alpha(z)$ with $\alpha=1/2$ which admits the following representation

$E_{1/2}(z)=e^{z^2}\left(1+\text{erf}(z)\right)$ with $\text{erf}(z)=\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z e^{-t^2}dt$ (See Saxena, Mathai and Haubold , 2002).

Theorem 2.2 The (i,j) th moment of the distribution of U and V, denoted by $\mu(i,j;\rho)$, is given by

$$\mu'(i,j;\rho)=\frac{2^{i+j}\left(1-\rho^2\right)^{i+j}}{L(m,\rho)}\sum_{k=0}^{\infty}\frac{(2\rho)^k}{k\,!}\Gamma\left(\frac{k+m}{2}+i\right)\Gamma\left(\frac{k+m}{2}+j\right)\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+m}{2}\right)}$$

where $L(m,\rho)=\dfrac{4\pi\ \Gamma(m-1)}{2^m\Gamma\left(\frac{m-1}{2}\right)\left(1-\rho^2\right)^{m/2}}$, $m>2,-1<\rho<1$.

Proof. It follows from (2.2) that the (i,j) th moment of the distribution of U and V, denoted by $\mu'(i,j;\rho)$, is given by

$$E\left(U^iV^j\right)=\frac{\left(1-\rho^2\right)^{-m/2}\Gamma\left(\frac{m-1}{2}\right)}{4\pi\;\Gamma(m-1)}\,e^{-\frac{(u+v)}{2(1-\rho^2)}}$$

$$\times\sum_{k=0}^{\infty}\left(\frac{\rho}{1-\rho^2}\right)^k\frac{1}{k\,!}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+m}{2}\right)}\int_0^{\infty}\int_0^{\infty}u^{\frac{m+k}{2}+i-1}\,v^{\frac{m+k}{2}+j-1}dudv\quad,$$

where $i\,,j$ are integers. The theorem then follows by having evaluated the integrals.

Note that $\mu'(i\,,j\,;\rho)=\mu'(j\,,i\,;\rho)$. As an example let us try to calculate $\mu'(4,4;\rho)$. It can be checked that

$$\mu'(4,4;\rho)=\frac{2^4\left(1-\rho^2\right)^8}{L(m,\rho)}\sum_{k=0}^{\infty}\rho^k\,\left((k+m)(k+m+2)(k+m+4)(k+m+6)\right)$$
$$\frac{2^k}{k\,!}\Gamma\left(\frac{k+m}{2}+4\right)\Gamma\left(\frac{k+1}{2}\right).$$

Since

$$\begin{aligned}&(k+m)(k+m+2)(k+m+4)(k+m+6)\\&=k^4+4(m+3)k^3+2(3m^2+18m+22)k^2\\&+4(m^3+9m^2+22m+12)k+m(m^3+12m^2+44m+48),\end{aligned}$$

it is obvious that the moment can be calculated in closed forms if the infinite series of the type

$$\sum_{k=0}^{\infty}k^i\rho^k\;\frac{2^k}{k\,!}\Gamma\left(\frac{k+m}{2}+4\right)\Gamma\left(\frac{k+1}{2}\right),(i=1,2,...,)$$

are evaluated first. Prompted by the idea, we have the following theorems.

3. Some Product Moments $\mu'(i\,,j\,;\rho)$

Theorem 3.1 Let $b_{k,m}=\frac{2^k}{k\,!}\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k+m}{2}\right),m>1$. Then for $L(m,\rho)$, $m>2,\ -1<\rho<1$ defined in Theorem 2.2, we have

(i) $\sum_{k=0}^{\infty}\rho^k b_{k,m}=L(m,\rho)$

(ii) $\sum_{k=0}^{\infty}k\,b_{k,m}=m\rho^2\left(1-\rho^2\right)^{-1}L(m,\rho)=w_1(m,\rho)L(m,\rho)$

(iii) $\sum_{k=0}^{\infty} k^{(2)}\rho^k b_{k,m} = \left(m(m+1)\rho^4 + m\rho^2\right)\left(1-\rho^2\right)^{-2} L(m,\rho) = w_{(1)}(m,\rho)L(m,\rho)$

(iv) $\sum_{k=0}^{\infty} k^{(3)}\rho^k b_{k,m} = w_{(3)}(m,\rho)L(m,\rho)$ where

$$w_3(m,\rho) = \left((m^3+3m^2+2m)\rho^6 + (3m^2+6m)\rho^4\right)\left(1-\rho^2\right)^{-3}$$
$$= \left(m(m+1)(m+2)\rho^6 + 3m(m+2)\rho^4\right)\left(1-\rho^2\right)^{-3} L(m,\rho) = w_{(3)}(m)L(m,\rho)$$

(v) $\sum_{k=0}^{\infty} k^{(4)}\rho^k \; b_{k,m} = w_{(4)}(m,\rho)L(m,\rho)$ where

$$w_4(m,\rho) = \left[(m^4+6m^3+11m^2+6m)\rho^8 + (6m^3+30m^2+36m)\rho^6 + (3m^2+6m)\rho^4\right]\left(1-\rho^2\right)^{-4}$$
$$= \left[m(m+1)(m+2)(m+3)\rho^8 + 6m(m+2)(m+3)\rho^6 + 3m(m+2)\rho^4\right]\left(1-\rho^2\right)^{-4}$$

(vi) $\sum_{k=0}^{\infty} k^2\rho^k b_{k,m} = \left(m^2\rho^4 + 2m\rho^2\right)\left(1-\rho^2\right)^{-2} L(m,\rho) = w_2(m,\rho)L(m,\rho)$

(vii) $\sum_{k=0}^{\infty} k^3\rho^k b_{k,m} = \left(m^3\rho^6 + (6m^2+4m)\rho^4 + 4m\rho^2\right)\left(1-\rho^2\right)^{-3} L(m,\rho) = w_3(m,\rho)L(m,\rho)$

(viii) $\sum_{k=0}^{\infty} k^4\rho^k \; b_{k,m} = w_4(m,\rho)L(m,\rho)$ where

$$w_4(m,\rho) = \left[m^4\rho^8 + (12m^3+16m^2+8m)\rho^6 + (28m^2+32m)\rho^4 + 8m\rho^2\right]\left(1-\rho^2\right)^{-4}$$

Proof.

The proof of (i) is obvious by virtue of $\mu(0,0;\rho)=1$ where $\mu(i,j;\rho)$ is defined in (2.6). The identity in (i) can be rewritten as

$$\sum_{k=0}^{\infty} \rho^k b_{k,m} = L(m,0)\left(1-\rho^2\right)^{-m/2} \tag{3.1}$$

Differentiating both sides of the identity in (3.1) with respect to ρ we have

$$\sum_{k=0}^{\infty} k\rho^{k-1} b_{k,m} = mL(m,0)\rho\left(1-\rho^2\right)^{-m/2-1} \tag{3.2}$$

which yields the identity in (ii). Differentiating the identity in (3.2) again we have

$$\sum_{k=0}^{\infty} k(k-1)\rho^{k-2}\ b_{k,m} = mL(m,0)\left[\rho\left\{(-m/2-1)\left(1-\rho^2\right)^{-m/2-2}(-2\rho)\right\}+\left(1-\rho^2\right)^{-m/2-1}\right] \\ = mL(m,0)\left[1+(m+1)\rho^2\right]\left(1-\rho^2\right)^{-m/2-2} \tag{3.3}$$

which yields the identity in (iii). Differentiating the identity in (3.3) we have

$$\sum_{k=0}^{\infty} k(k-1)(k-2)\rho^{k-3}\ b_{k,m} \\ = mL(m,0)\left[\left(1+(m+1)\rho^2\right)\left\{(-m/2-2)\left(1-\rho^2\right)^{-m/2-3}(-2\rho)\right\}+\left\{(m+1)2\rho\right\}\left(1-\rho^2\right)^{-m/2-2}\right] \\ = mL(m,0)\left[(m^2+3m+2)\rho^3+(3m+6)\rho\right]\left(1-\rho^2\right)^{-m/2-3} \tag{3.4}$$

which yields the identity in (v). Differentiating the identity in (3.4) we have

$$\sum_{k=0}^{\infty} k(k-1)(k-2)(k-3)\rho^{k-4}\ b_{k,m} \\ = mL(m,0)\left[\left((m^2+3m+2)\rho^3+(3m+6)\rho\right)\left\{(-m/2-3)\left(1-\rho^2\right)^{-m/2-4}(2\rho)\right\}\right. \\ \left.+\left\{(m^2+3m+2)3\rho^2+(3m+6)\right\}\left(1-\rho^2\right)^{-m/2-3}\right] \\ = mL(m,0)\left[(m^3+6m^2+11m+6)\rho^4+(6m^2+30m+36)\rho^2+(3m+6)\right]\left(1-\rho^2\right)^{-m/2-4}$$

which yields the identity in (v).

The identity in (vi) can be proved as follows:

$$\sum_{k=0}^{\infty} k^2\rho^k b_{k,m} = \sum_{k=0}^{\infty}\left(k^{(2)}+k\right)\rho^k b_{k,m} \\ = \left(w_{(2)}(m,\rho)+w_{(1)}(m,\rho)\right)L(m,\rho) \\ = \left[w_{(2)}(m,\rho)\left(1-\rho^2\right)^2+w_{(1)}(m,\rho)\left(1-\rho^2\right)^2\right]L(m,\rho)\left(1-\rho^2\right)^{-2} \\ = \left[\left((m^2+m)\rho^4+m\rho^2\right)+m\rho^2\left(1-\rho^2\right)^2\right]L(m,\rho)\left(1-\rho^2\right)^{-2}$$

The identity in (vii) can be proved as follows:

$$
\begin{aligned}
\sum_{k=0}^{\infty} k^3 \rho^k b_{k,m} &= \sum_{k=0}^{\infty}\left(k^{(3)} + 3k^{(2)} + k\right)\rho^k b_{k,m} \\
&= \left(w_{(3)}(m,\rho) + 3w_{(2)}(m,\rho) + w_{(1)}(m,\rho)\right)L(m,\rho) \\
&= \left[w_{(3)}\left(1-\rho^2\right)^3 + 3w_{(2)}(m,\rho)\left(1-\rho^2\right)^3 + w_{(1)}(m,\rho)\left(1-\rho^2\right)^3\right]L(m,\rho)\left(1-\rho^2\right)^{-2} \\
&= \Big[\left((m^3+3m^2+2m)\rho^6 + (3m^2+6m)\rho^4\right) + 3\left((m^2+m)\rho^4 + m\rho^2\right)\left(1-\rho^2\right) \\
&\quad + m\rho^2\left(1-\rho^2\right)^2\Big]L(m,\rho)\left(1-\rho^2\right)^{-3}
\end{aligned}
$$

The identity in (vii) can be proved as follows:

$$
\begin{aligned}
\sum_{k=0}^{\infty} k^4 \rho^k b_{k,m} &= \sum_{k=0}^{\infty}\left(k^{(4)} + 6k^{(3)} 7k^{(2)} + k\right)\rho^k b_{k,m} \\
&= \left(w_{(4)}(m,\rho) + 6w_{(3)}(m,\rho) + 7w_{(2)}(m,\rho) + w_{(1)}(m,\rho)\right)L(m,\rho) \\
&= \Big[w_{(4)}(m,\rho)\left(1-\rho^2\right)^4 + 6w_{(3)}(m,\rho)\left(1-\rho^2\right)^4 + 7w_{(2)}(m,\rho)\left(1-\rho^2\right)^4 \\
&\quad + w_{(1)}(m,\rho)\left(1-\rho^2\right)^4\Big]L(m,\rho)\left(1-\rho^2\right)^{-4} \\
&= \Big[\left((m^4+6m^3+11m^2+6m)\rho^8 + (6m^3+30m^2+36m)\rho^6 + (3m^2+6m)\rho^4\right) \\
&\quad + 6\left(m^3+3m^2+2m)\rho^6 + (3m^2+6m)\rho^4\right)\left(1-\rho^2\right) \\
&\quad + 7\left((m^2+m)\rho^4 + m\rho^2\right)\left(1-\rho^2\right)^2 + m\rho^2\left(1-\rho^2\right)^3\Big]L(m,\rho)\left(1-\rho^2\right)^{-4} \\
&= w_4(m,\rho)L(m,\rho)
\end{aligned}
$$

where $w_{(i)}(m,\rho)$ and $w_i(m,\rho),\ (i=1,2,3,4)$ are defined in the lemma.

Note that

$$
\frac{L(m+c,\rho)}{L(m,\rho)} = \frac{\Gamma\left((m+c)/2\right)}{\Gamma\left(m/2\right)}\ (1-\rho^2)^{-c/2} \tag{3.4}
$$

Theorem 3.2 For $m>0, -1<\rho<1, 1\le i\le 4$ and $-4\le j\le 4$, let

$$
\psi(i,j;\rho) = \frac{\left(1-\rho^2\right)^{i+j}}{L(m,\rho)}\frac{\Gamma(m/2)}{\Gamma(m/2+j)}\ W(i,j;\rho) \tag{3.5}
$$

where

$$W(i,j;\rho)=2^i\sum_{k=0}^{\infty}\frac{(2\rho)^k}{k!}\frac{\Gamma\left(\frac{k+m}{2}+i\right)}{\Gamma\left(\frac{k+m}{2}\right)}\Gamma\left(\frac{k+m}{2}+j\right)\Gamma\left(\frac{k+1}{2}\right). \tag{3.6}$$

Then

$$\begin{aligned}
&(i)\ \psi(1,j;\rho)=m+2j\rho^2\\
&(ii)\ \psi(2,j;\rho)=4j(j-1)\rho^4+4j(m+2)\rho^2+m(m+2)\\
&(iii)\ \psi(3,j;\rho)=8j(j-1)(j-2)\rho^6+12j(j-1)(m+4)\rho^4\\
&\qquad\qquad +6j(m+2)(m+4)\rho^2+m(m+2)(m+4)\\
&(iv)\ \psi(4,j;\rho)=16j(j-1)(j-2)(j-3)\rho^8+32j(j-1)(j-2)(m+6)\rho^6\\
&\qquad\qquad +24j(j-1)(m+4)(m+6)\rho^4+8j(m+2)(m+4)(m+6)\rho^2\\
&\qquad\qquad +m(m+2)(m+4)(m+6)
\end{aligned}$$

Proof. (i) By virtue of Theorem 3.2, it follows from (3.6) that

$$\begin{aligned}
W(1,j;\rho)&=\sum_{k=0}^{\infty}(k+m)\rho^k b_{k,m+2j}\\
&=\sum_{k=0}^{\infty}k\rho^k b_{k,m+2j}+m\sum_{k=0}^{\infty}\rho^k b_{k,m+2j}\\
&=w_1(m+2j,\rho)L(m+2j,\rho)+mL(m+2j,\rho)
\end{aligned}$$

and

$$\begin{aligned}
\psi(1,j;\rho)&=\left(w_1(m+2j,\rho)+m\right)L(m+2j,\rho)\\
&=\left((m+2j)\rho^2\left(1-\rho^2\right)^{-1}+m\right)L(m+2j,\rho)\\
&=\left((m+2j)\rho^2\left(1-\rho^2\right)^{-1}+m\right)\left(1-\rho^2\right).
\end{aligned}$$

(ii) By virtue of Theorem 3.2, it follows from (3.6) that

$$\begin{aligned}
W(2,j;\rho)&=\sum_{k=0}^{\infty}(k+m+2)(k+m)\rho^k b_{k,m+2j}\\
&=\sum_{k=0}^{\infty}\left(k^2+2(m+1)k+(m^2+2m)\right)\rho^k b_{k,m+2j}\\
&=\left(w_2(m+2j,\rho)+2(m+1)w_1(m+2j,\rho)+(m^2+2m)\right)L(m+2j,\rho)
\end{aligned}$$

and

$$\psi(2,j;\rho)=\left(w_2(m+2j,\rho)+2(m+1)w_1(m+2j,\rho)+(m^2+2m)\right)\left(1-\rho^2\right)$$
$$=\Big[\left((m+2j)^2\rho^4+2(m+2j)\rho^2\right)\left(1-\rho^2\right)^{-2}$$
$$+2(m+1)(m+2j)\rho^2\left(1-\rho^2\right)^{-2}+\left(m^2+m\right)\Big]\left(1-\rho^2\right)^2$$

(iii) By virtue of Theorem 3.2, it follows from (3.6) that

$$W(3,j;\rho)=\sum_{k=0}^{\infty}(k+m+4)(k+m+2)(k+m)\rho^k b_{k,m+2j}$$
$$=\sum_{k=0}^{\infty}\left(k^3+3(m+2)k^2+3(m^2+12m+8)+(m^3+6m^2+8m\right)\rho^k b_{k,m+2j}$$
$$=\left(w_3(m+2j,\rho)+3(m+2)w_2(m+2j,\rho)+3(m^2+12m+8)w_1(m+2j,\rho)\right.$$
$$\left.+(m^3+6m^2+8m)\right)L(m+2j,\rho)$$

and

$$\psi(3,j;\rho)=\left(w_3(m+2j,\rho)+3(m+2)w_2(m+2j,\rho)+3(m^2+12m+8)w_1(m+2j,\rho)\right.$$
$$\left.+(m^3+6m^2+8m)\right)\left(1-\rho^2\right)^3$$

where quantities $w_i(m+2j,\rho),(i=1,2,3)$ are defined in Theorem 3.1.

(iv) By virtue of Theorem 3.2, it follows from (3.6) that

$$W(4,j;\rho)=\sum_{k=0}^{\infty}(k+m+6)(k+m+4)(k+m+2)(k+m)\rho^k b_{k,m+2j}$$
$$=\sum_{k=0}^{\infty}\left(k^4+4(m+3)k^3+2(3m^2+18m+22)k^2+4(m^3+9m^2+22m+12)k\right.$$
$$\left.+(m^4+12m^3+44m^2+48m)\right)\rho^k b_{k,m+2j}$$
$$=\left(w_4(m+2j,\rho)+4(m+3)w_3(m+2j,\rho)+2(3m^2+18m+22)w_2(m+2j,\rho)\right.$$
$$\left.+4(m^3+9m^2+22m+12)w_1(m+2j,\rho)+(m^4+12m^3+44m^2+48m)\right)L(m+2j,\rho)$$

and that

$$\psi(4,j;\rho)=\Big(w_4(m+2j,\rho)+4(m+3)w_3(m+2j,\rho)+2(3m^2+18m+22)w_2(m+2j,\rho)$$
$$+4(m^3+9m^2+22m+12)w_1(m+2j,\rho)+(m^4+12m^3+44m^2+48m)\Big)\left(1-\rho^2\right)^4$$
$$+(m^4+12m^3+44m^2+48m)\Big)\left(1-\rho^2\right)^4.$$

In general, for $m>0$, $-1<\rho<1$, $1\le i\le 4$ and $-4\le j\le 4$

$$\psi(i,j;\rho)=2^i\sum_{k=0}^{i}\binom{i}{k} i_{(i-k)}\left(\frac{m}{2}+i-1\right)_{(k)}\rho^{2(i-k)};\ k\le i \qquad (3.7)$$

Theorem 3.3 For $m>0$, $-1<\rho<1$, $1\le i\le 4$ and $-4\le j\le 4$ the joint moment given by (2.6) can be expressed as

$$\mu'(i,j;\rho)=\mu'(0,j;0)\ \psi(i,j;\rho)$$

where $\mu'(0,j;0)=\dfrac{2^j\,\Gamma(m/2+j)}{\Gamma(m/2)}$ is the j th moment of usual univariate chisquare distribution with m degrees of freedom, and $\psi(i,j;\rho)$ is defined by (3.7) .

Proof. The proof is immediate from Theorem 2.2 by virtue of Theorem 3.2.

Corollary 3.1 Let $\mu(i,j,l;\rho)$ be the product moments of the bivariate Wishart distribution with pdf given by (2.1). Then

$$\mu'(i,j,0;\rho)=\frac{\sigma_1^{2i}\sigma_2^{2j}}{m^{i+j}}\mu'(i,j;\rho)$$

Note that $\mu'(i,j;\rho)=\mu'(j,i;\rho)$, and that $\psi(i,0;0)$ is the i th moment of usual univariate chisqaure distribution with m degrees of freedom. When both orders i and j are negative, it is difficult to get closed form expressions for joint moments $\mu'(i,j;\rho)$. Some other special cases are evaluated below.

Case I (one of the orders is positive):

$$\mu'(1,-4;\rho)=\frac{2^{-4}\Gamma(m/2-4)}{\Gamma(m/2)}(m-8\rho^2),\ m>8$$
$$\mu'(1,-3;\rho)=\frac{2^{-3}\Gamma(m/2-4)}{\Gamma(m/2)}(m-6\rho^2),\ m>8$$
$$\mu'(1,-2;\rho)=\frac{(m-4\rho^2)}{(m-2)(m-4)},\ m>4$$
$$\mu'(1,-1;\rho)=\frac{m-2\rho^2}{m-2},\ m>2$$

$$\mu'(2,-4;\rho)=\frac{2^{-4}\Gamma(m/2-4)}{\Gamma(m/2)}\left[80\rho^4-16(m+2)\rho^2+m(m+2)\right],\ m>8$$

$$\mu'(2,-3;\rho)=\frac{2^{-3}\Gamma(m/2-3)}{\Gamma(m/2)}\left[48\rho^4-12(m+2)\rho^2+m(m+2)\right],\ m>6$$

$$\mu'(2,-2;\rho)=\frac{1}{(m-2)(m-4)}\left(24\rho^4-8(m+2)\rho^2+m(m+2)\right),\ m>4$$

$$\mu'(2,-1;\rho)=\frac{1}{m-2}\left[8\rho^4-4(m+2)\rho^2+m(m+2)\right],\ m>2$$

$$\mu'(3,-4;\rho)=\frac{2^{-4}\Gamma(m/2-4)}{\Gamma(m/2)}$$
$$\times\left[-960\rho^6+240(m+4)\rho^4-24(m+2)(m+4)\rho^2+m(m+2)(m+4)\right],\ m>8$$

$$\mu'(3,-3;\rho)=\frac{2^{-3}\Gamma(m/2-3)}{\Gamma(m/2)}$$
$$\times\left[-480\rho^6+144(m+4)\rho^4-18(m+2)(m+4)\rho^2+m(m+2)(m+4)\right],\ m>6$$

$$\mu'(3,-2;\rho)=\frac{2^{-2}\Gamma(m/2-2)}{\Gamma(m/2)}$$
$$\times\left[-192\rho^6+72(m+4)\rho^4-12(m+2)(m+4)\rho^2+m(m+2)(m+4)\right],\ m>4$$

$$\mu'(3,-1;\rho)=\frac{2^{-1}\Gamma(m/2-1)}{\Gamma(m/2)}$$
$$\times\left[-48\rho^6+24(m+4)\rho^4-6(m+2)(m+4)\rho^2+m(m+2)(m+4)\right],m>2$$

$$\mu'(4,-4;\rho)=\frac{2^{-4}\Gamma(m/2-4)}{\Gamma(m/2)}\Big[13440\rho^8-3840(m+6)\rho^6+480(m+4)(m+6)\rho^4$$
$$-32(m+2)(m+4)(m+6)\rho^2+m(m+2)(m+4)\Big],\ m>8$$

$$\mu'(4,-3;\rho)=\frac{2^{-3}\Gamma(m/2-3)}{\Gamma(m/2)}\Big[5760\rho^8-1920(m+6)\rho^6+288(m+4)(m+6)\rho^4$$
$$-24(m+2)(m+4)(m+6)\rho^2+m(m+2)(m+4)(m+6)\Big],\ m>6$$

$$\mu'(4,-2;\rho)=\frac{2^{-2}\Gamma(m/2-2)}{\Gamma(m/2)}\Big[1920\rho^8-768(m+6)\rho^6+144(m+4)(m+6)\rho^4$$
$$-16(m+2)(m+4)(m+6)\rho^2+m(m+2)(m+4)(m+6)\Big],m>4$$

$$\mu'(4,-1;\rho)=\frac{2^{-1}\Gamma(m/2-1)}{\Gamma(m/2)}\Big[384\rho^8-192(m+6)\rho^6+48(m+4)(m+6)\rho^4$$
$$-8(m+2)(m+4)(m+6)\rho^2+m(m+2)(m+4)(m+6)\Big],\ m>2$$

Case II (when both orders are positive):

$$\mu'(1,1;\rho)=m\left(m+2\rho^2\right)$$

$$\mu'(1,2;\rho) = m(m+2)\left(m+4\rho^2\right)$$
$$\mu'(1,3;\rho) = m(m+2)(m+4)\left(m+6\rho^2\right)$$
$$\mu'(1,4;\rho) = m(m+2)(m+4)(m+6)\left(m+8\rho^2\right)$$
$$\mu'(2,2;\rho) = m(m+2)\left[8\rho^4+8(m+2)\rho^2+m(m+2)\right]$$
$$\mu'(2,3;\rho) = m(m+2)(m+4)\left[24\rho^4+12(m+2)\rho^2+m(m+2)\right]$$
$$\mu'(2,4;\rho) = m(m+2)(m+4)(m+6)\left[48\rho^4+16(m+2)\rho^2+m(m+2)\right]$$
$$\mu'(3,3;\rho) = m(m+2)(m+4)\left[48\rho^6+72(m+4)\rho^4+18(m+2)(m+4)\rho^2+m(m+2)(m+4)\right]$$
$$\mu'(3,4;\rho) = m(m+2)(m+4)(m+6)$$
$$\times\left[192\rho^6+144(m+4)\rho^4+24(m+2)(m+4)\rho^2+m(m+2)(m+4)\right]$$
$$\mu'(4,4;\rho) = m(m+2)(m+4)(m+6)\left[384\rho^8+768(m+6)\rho^6+288(m+4)(m+6)\rho^4\right.$$
$$\left.+32(m+2)(m+4)(m+6)\rho^2+m(m+2)(m+4)(m+6)\right]$$

If $\rho=0$, then $\mu(4,4;\rho)=m^2(m+2)^2(m+4)^2(m+6)^2$ which is the joint moment of (4,4) order of bivariate independent chisquare distribution each with m degrees of freedom. It is observed that $\mu'(1,j;\rho)=2^j(m/2+3)_j(m+2j\rho^2),\ j=1,2,3,\cdots$.

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