Common Fixed Points of Weakly Compatible Maps on Balls and Applications

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Abstract

In this paper we prove some common fixed point theorems for weakly compatible maps satisfying different contractive conditions on closed balls of a normed space. As applications, we demonstrate the existence of: (i) common fixed points of the maps from the set of best approximations, (ii) solutions to nonlinear eigenvalue problems for operators defined on closed balls. Our work sets analogues, unifies and improves the earlier results of a number of authors.

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1 Introduction

In 1982, Sessa [20] introduced the concept of weakly commuting maps to generalize commutativity. Jungck [10], in 1986, generalized weak commutativity to the notion of compatible maps. In 1996, Jungck [12] further weakened compatibility to the concept of weak compatibility. Since then, many interesting fixed point theorems of compatible and weakly compatible maps have been obtained by a number of authors (see, for example, Aamri and El Moutawakil

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Jachymski [8], Jungck [11], Pant [15] and Popa [17-18]). Pant and Pant [16], in 2000, obtained a unique common fixed point of two noncompatible pointwise $R$-weakly commuting selfmaps under a strict contractive condition on a metric space (for the case of a sequence of maps, we refer to [15]). Aamri and El Moutawakil [1], in 2002, defined the property $(E \cdot A)$ for selfmaps on a metric space $X$ which always holds for any two noncompatible selfmaps on $X$ and proved two common fixed point theorems for weakly compatible maps satisfying the property $(E \cdot A)$ and certain strict contractive conditions. In 2004, Kamran [13] proved some coincidence and common fixed point theorems for maps satisfying the property $(E \cdot A)$ and hybrid strict contractive conditions (see also, Chang [4] and Sastry et al. [19]).

The existence of fixed points of maps defined on closed balls has been studied by several authors; for example, see Baskaran and Subrahmanyam [3], Delbosco [5], Granas and Dugundji [6], Liu [14] and Shahzad and Latif [22].

In [9], Jungck generalized the Banach contraction principle to the case of two commuting selfmaps on a metric space. Baskaran and Subrahmanyam [3] noted that the commutativity of the maps in Jungck’s theorem can be replaced by weak commutativity and then they obtained some common fixed point theorems for two maps on the closed ball of a Banach space. They also provided a solution to a nonlinear eigenvalue problem for operators on the closed ball of a Banach space.

In this paper, we continue with the above theme and establish results related to common fixed point, best approximation and eigenvalue problems for weakly compatible maps under certain contractive conditions.

The paper is organized as follows: In section 3, motivated by Baskaran and Subrahmanyam [3] and Aamri and El Moutawakil [1], we demonstrate the existence of common fixed points of weakly compatible maps which satisfy the property $(E \cdot A)$ and certain contractive conditions on closed balls of a normed space. Section 4 deals with common fixed points, of weakly compatible maps satisfying the property $(E \cdot A)$, from the set of best approximations. Finally, in Section 5, we find solutions to nonlinear eigenvalue problems for
operators defined on closed balls of a normed space and reflexive Banach space.

2 Preliminaries

Let $f$ and $g$ be selfmaps of a metric space $(X, d)$. The maps $f$ and $g$ are

1. weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$, for all $x \in X$,

2. compatible if $\lim_{n \to \infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$,

3. weakly compatible if they commute at their coincidence points; i.e., if $fu = gu$ for some $u$ in $X$, then $fgu = gfu$,

4. satisfying the property $(E \cdot A)$ if there exists a sequence $\{x_n\}$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$.

The map $f$ is nonexpansive if $d(fx, fy) \leq d(x, y)$, for all $x, y \in X$. We say $f$ is $g$-nonexpansive if $d(fx, fy) \leq d(gx, gy)$, for all $x, y \in X$. Note that weakly commuting maps are compatible and compatible maps are weakly compatible but the converse in each case does not hold (for examples and counter-examples, see [1], [10], [12], and [18]). It is easy to see that two noncompatible maps satisfy the property $(E \cdot A)$ (see [1], Remark 1). Some fixed point results for noncompatible maps are obtained in [13], [15] and [16].

Let $M$ be a subset of $X$ and $u \in X$. We denote by $P_M(u)$, the set of best approximations to $u$ from $M$; that is,

$$P_M(u) = \{y \in M : d(y, u) = d(u, M)\},$$

where $d(u, M) = \inf\{d(u, m) : m \in M\}$. The existence of common fixed points in $P_M(u)$ has been studied by various authors; see Al-Thagafi [2], Hussain and Khan [7], Kamran [13] and Shahzad [21].

The use of the following two results by Aamri and El-Moutawakil [1] will be two fold in this work:
(i) to establish common fixed point results on closed balls of a normed space $X$,

(ii) to find common fixed points from $P_M(u)$.

**Theorem A** [1, Theorem 1]. Let $f$ and $g$ be weakly compatible selfmaps of a metric space $(X,d)$ satisfying the property $(E \cdot A)$. Assume that $fX$ or $gX$ is complete and $fX \subset gX$.

For all $x \neq y$ in $X$, $f$ and $g$ satisfy the following contractive condition:

$$d(fx, fy) < \max \left\{ d(gx, gy), \frac{1}{2}[d(fx, gx) + d(fy, gy)], \frac{1}{2}[d(fy, gx) + d(fx, gy)] \right\}. \quad (2.1)$$

Then $f$ and $g$ have a unique common fixed point.

Suppose that $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ (set of all nonnegative reals) satisfies the following conditions:

(i) $F$ is nondecreasing,

(ii) $0 < F(t) < t$, for each $t \in (0, \infty)$.

Various conditions on $F$ have been studied by different authors (see [4], [8], [15], [16] and [19]). In the next theorem the above function $F$ is employed.

**Theorem B** [1, Theorem 2]. Let $f, g, p$ and $q$ be selfmaps of a metric space $(X,d)$ such that

(i) the pairs $(f, p)$ and $(g, q)$ are weakly compatible, and the pair $(f, p)$ or $(g, q)$ satisfies the property $(E \cdot A)$,

(ii) $fX \subset qX$, $gX \subset pX$ and the range of one of the maps $f, g, p$ or $q$ is complete.

For all $x, y \in X$, the maps $f, g, p$ and $q$ satisfy the following contractive condition:

$$d(fx, gy) \leq F(\max\{d(px, qy), d(px, gy), d(qy, gy)\}). \quad (2.2)$$

Then $f, g, p$ and $q$ have a unique common fixed point.

We remark that Theorem A is a generalization of a theorem of Jungck in [9] and Theorem B generalizes Theorem 3.1 of Jungck [11] for weakly compatible maps under a different contractive condition.

A number of authors have studied the existence of common fixed points under various contractive conditions (see [4], [8], [10], [11], [13], [15], [16], [18] and [19]).

Throughout this paper, we shall employ the following conventions:
(i) the closed ball of a normed space $X$ with center 0 and radius $R$, $B(0, R) = \{ x \in X : \| x \| \leq R \}$ will be simply denoted by $B$,

(ii) for any selfmap $f$ on $X$ or $B$, we set $f_n = (1 - \frac{1}{n}) f$, $n = 2, 3, 4, \ldots$ ,

(iii) $f$ satisfies condition $(\alpha)$ if $f(\alpha x) = \alpha f x$ for each $\alpha \in (\alpha_0, 1)$, $\alpha_0 \geq 0$, and $x$ in $B$.

3 Common Fixed Point Results

In this section, we obtain some common fixed point theorems for weakly compatible maps satisfying certain contractive conditions on the closed ball of a normed space. Our methods are based on those used by Baskaran and Subrahmanyam [3].

We begin with the following analogue of Theorem 2.1 in [3].

**Theorem 3.1** Let $f$ and $g$ be continuous and weakly compatible selfmaps of the closed ball $B$ of a normed space $X$. Assume that

(i) $f$ or $g$ is compact and $f$ satisfies condition $(\alpha)$,

(ii) $f$ and $g_n$ satisfy the property $(E \cdot A)$,

(iii) $fB$ or $g_nB$ is complete and $fB \subset g_nB$,

(iv) for all $x \neq y$ in $B$, the following contractive condition is satisfied:

$$\| fx - fy \| < \frac{1}{2} \max\{ \| gx - gy \|, \| fx - g_nx \| + \| fy - g_ny \|, \| fy - g_nx \| + \| fx - g_ny \| \}. \quad (3.1)$$

Then $f$ and $g$ have a common fixed point in $B$.

**Proof.** Assume that $u \in B$ is a coincidence point of $f$ and $g_n$; i.e., $fu = g_nu = (1 - \frac{1}{n}) gu$. Taking limit as $n \to \infty$, we get $fu = gu$. Since $f$ and $g$ are weakly compatible, therefore $fgu = gfu$. By this and condition $(\alpha)$, we get $fg_nu = f (1 - \frac{1}{n}) gu = (1 - \frac{1}{n}) f gu = (1 - \frac{1}{n}) gfu = g_n fu$. Thus $f$ and $g_n$ are weakly compatible for each $n$. From $\frac{1}{2} \| gx - gy \| \leq \| g_n x - g_n y \|$ and (3.1), we get

$$\| fx - fy \| < \max \left\{ \| g_n x - g_n y \|, \frac{1}{2} (\| fx - g_n x \| + \| fy - g_n y \|), \frac{1}{2} (\| fy - g_n x \| + \| fx - g_n y \|) \right\}.$$
Hence by Theorem A, there exists \( x_n \in B \) such that

\[
x_n = f x_n = g_n x_n = \left(1 - \frac{1}{n} \right) g x_n.
\]

Assume that \( g \) is compact (same concerns if \( f \) is compact). As \( \{ x_n \} \) is bounded so \( \{ g x_n \} \) has a subsequence \( \{ g x_{n_k} \} \) converging to \( b \) in \( B \). Now, \( x_{n_k} = f x_{n_k} = \left(1 - \frac{1}{n_k} \right) g x_{n_k} \). Proceeding to the limit as \( k \to \infty \) and using the continuity of \( f \) and \( g \), we get \( b = f b = g b \). Thus \( f \) and \( g \) have a common fixed point \( b \) in \( B \). ■

The following example verifies Theorem 3.1.

**Example 3.2** Let \( X \) be the usual space of reals. Define \( f(x) = \frac{1}{2} x \), \( g(x) = \frac{1}{2} (x^3 + x) \) where \( x \in B = [-1, 1] \). Clearly, \( f \) and \( g \) are continuous, \( fB \subset g_n B \) and \( f \) satisfies condition \((\alpha)\).

From \( gB = [-1, 1] \), it follows that \( g \) is compact. It is easy to verify that \( f \) and \( g \) are compatible. Also, \( f \) and \( g_n \) satisfy the property \((E \cdot A)\) for the sequence \( x_m = \frac{1}{m}, \ m = 1,2, \ldots \) Finally, for all \( x \neq y \) in \( B \), \((3.1)\) holds because \(|f(x) - f(y)| < \frac{1}{4} |x^3 - y^3 + x - y| = \frac{1}{2} |g(x) - g(y)|\).

Thus all the conditions of Theorem 3.1 are satisfied and 0 is the common fixed point of \( f \) and \( g \).

We replace the property \((E \cdot A)\) in Theorem 3.1 by a map \( \phi \) satisfying a contractive condition and use the argument of Corollary 2 in [1] to obtain:

**Corollary 3.3** Let \( f \) and \( g \) be continuous and weakly compatible selfmaps of the closed ball \( B \) of a normed space \( X \) such that

(i) \( f \) or \( g \) is compact and \( f \) satisfies condition \((\alpha)\),

(ii) \( fB \) or \( g_n B \) is complete and \( fB \subset g_n B \),

(iii) for all \( x \neq y \) in \( B \), \((3.1)\) holds,

(iv) there exists a map \( \phi : B \to \mathbb{R}^+ \) such that \( \| g_n x - f x \| < \phi(g_n x) - \phi(f x) \) for all \( x \) in \( B \).

Then \( f \) and \( g \) have a common fixed point in \( B \).
Proof. For a fixed $n > 1$, let $x_0 \in B$ and choose $x_1 \in B$ such that $f x_0 = g_n x_1$. For $m = 1, 2, \ldots$, choose $x_m \in B$ such that $f x_{m-1} = g_n x_m$. Then

$$\|g_n x_m - g_n x_{m+1}\| = \|g_n x_m - f x_m\| < \phi(g_n x_m) - \phi(f x_m) = \phi(g_n x_m) - \phi(g_n x_{m+1}).$$

Define a real sequence $\{a_m\}$ by $a_m = \phi(g_n x_m), \ m = 1, 2, \ldots$ Clearly, $\{a_m\}$ is nonnegative, nonincreasing and bounded below by 0. Thus, $\{a_m\}$ is a convergent sequence. We also have

$$\|g_n x_m - g_n x_{m+k}\| \leq a_m - a_{m+k}$$

which implies that $\{g_n x_m\}$ is a Cauchy sequence in $g_n B$. Suppose that $g_n B$ is complete. Then there exists $t \in g_n B$ such that $\lim_{m \to \infty} g_n x_m = t$. So, $\lim_{m \to \infty} f x_m = t$. Hence, $f$ and $g_n$ satisfy the property $(E \cdot A)$. Since $n$ is arbitrary, therefore $f$ and $g_n$ satisfy the property $(E \cdot A)$ for each $n > 1$.

All the conditions of Theorem 3.1 are satisfied and hence the conclusion. □

The next theorem deals with three weakly compatible maps under a contractive condition in terms of the function $F$.

Theorem 3.4 Let $f, g$ and $p$ be continuous selfmaps of the closed ball $B$ of a normed space $X$ such that

(i) the pairs $(f, p)$ and $(g, p)$ are weakly compatible and one of the maps $f, g$ or $p$ is compact,

(ii) the pair $(f, p_n)$ or $(g, p_n)$ satisfies the property $(E \cdot A)$,

(iii) $f$ and $g$ satisfy condition ($\alpha$),

(iv) $f B \subset p_n B$, $g B \subset p_n B$ and the range of one of the maps $f, g$ or $p_n$ is complete,

(v) for all $x, y \in B$, (2.2) holds with $p$ and $q$ both replaced by $p_n$.

Then $f, g$ and $p$ have a common fixed point in $B$.

Proof. As in the proof of Theorem 3.1, we can prove that the pairs $(f, p_n)$ and $(g, p_n)$ are weakly compatible. Hence, by Theorem B, there exists $x_n \in B$ such that $f x_n = g x_n = \ldots$
Let $p$ be compact (same concerns the cases when $f$ or $g$ is compact). As $\{x_n\}$ is bounded, so $\{px_n\}$ has a subsequence $\{px_{n_k}\}$ converging to $u$ in $B$. Now, $x_{n_k} = fx_{n_k} = gx_{n_k} = \left(1 - \frac{1}{n_k}\right) px_{n_k}$. Proceeding to the limit as $k \to \infty$ and using the continuity of $f, g$ and $p$, we get $u = fu = gu = pu$. Thus $f, g$ and $p$ have a common fixed point $u$ in $B$. \hfill \blacksquare$

Recall that a Banach space $X$ is said to satisfy Opial’s condition if for each $x$ in $X$ and each sequence $\{x_n\}$ weakly convergent to $x$, the following condition holds for all $y \neq x$:

$$\lim \inf_{n \to \infty} \|x_n - x\| < \lim \inf_{n \to \infty} \|x_n - y\|.$$ 

An extension of the above theorem to the case of four weakly compatible maps is presented below. Our result extends Theorem 2.2 by Baskaran and Subrahmanyam [3] under a different contractive condition.

**Theorem 3.5** Let $f, g, p$ and $q$ be selfmaps of the closed ball $B$ of a reflexive Banach space $X$ satisfying Opial’s condition. Assume that

(i) $f$ and $g$ are weakly continuous and the pairs $(f, p)$ and $(g, q)$ are weakly compatible,

(ii) $fB \subset q_nB$, $gB \subset p_nB$, the pair $(f, p_n)$ or $(g, q_n)$ satisfies the property $(E \cdot A)$, and the range of one of the maps $f, g, p_n$ or $q_n$ is complete,

(iii) $f$ and $g$ satisfy condition $(\alpha)$,

(iv) $p$ is $f$-nonexpansive and $q$ is $g$-nonexpansive,

(v) for all $x, y$ in $B$, (2.2) holds with $p$ and $q$ replaced by $p_n$ and $q_n$, respectively.

Then $f, g, p$ and $q$ have a common fixed point in $B$.

**Proof.** As before, we can show that the pairs $(f, p_n)$ and $(g, q_n)$ are weakly compatible. Thus, by Theorem B, there exists $x_n \in B$ such that

$$x_n = fx_n = gx_n = p_nx_n = q_nx_n.$$ 

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Since $X$ is a reflexive Banach space, therefore $B$ is weakly compact. Hence, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging weakly to $u \in B$. Now

$$x_{n_k} = fx_{n_k} = gx_{n_k} = \left(1 - \frac{1}{n}\right) px_{n_k} = \left(1 - \frac{1}{n}\right) qx_{n_k}.$$ 

Taking limit as $k \to \infty$ and using the weak continuity of $f$ and $g$, we get $u = fu = gu$. Also, $px_{n_k}$ and $qx_{n_k}$ converge weakly to $u = fu = gu$. If $fu \neq pu$, then by Opial’s condition and $p$ being $f$-nonexpansive, we get

$$\lim \inf \|fx_{n_k} - fu\| < \lim \inf \|fx_{n_k} - pu\| \leq \lim \inf (\|fx_{n_k} - px_{n_k}\| + \|px_{n_k} - pu\|) = \lim \inf \|px_{n_k} - pu\| \leq \lim \inf \|fx_{n_k} - fu\|;$$

a contradiction. Hence $fu = pu = u$.

Similarly, we can prove that $gu = qu = u$. Thus $u$ is a common fixed point of $f, g, p$ and $q$. ■

4 Invariant Approximation

For further applications of Theorems A and B, we obtain common fixed points of best approximation. Our work provides analogues of most of the well-known results for the class of weakly compatible maps on a metric space.

An analogue of Theorem 3.1 of Hussain and Khan [7] is obtained in the following:

**Theorem 4.1** Let $M$ be a subset of a metric space $(X, d)$ and $f$ and $g$ be selfmaps of $X$. Assume that $u$ is a common fixed point of $f$ and $g$, and $D = P_M(u)$. Suppose that

(i) $f$ and $g$ are weakly compatible and satisfy the property $(E \cdot A)$,

(ii) $gD = D$, $f(\partial M) \subseteq M$ (here $\partial M$ denotes the boundary of $M$), and $fD$ or $gD$ is complete,
(iii) $f$ is $g$-nonexpansive on $D \cup \{u\}$,

(iv) for all $x \neq y$ in $D$, (2.1) holds.

Then $f$ and $g$ have a unique common fixed point in $P_M(u)$.

**Proof.** Let $y \in D$. Then $gy \in D$. By the definition of $P_M(u)$, $y \in \partial M$ and since $f(\partial M) \subseteq M$, it follows that $fy \in M$. As $f$ is $g$-nonexpansive on $D \cup \{u\}$, so

$$d(fy,u) = d(fy, fu) \leq d(gy, gu) = d(gy, u).$$

Now, $fy \in M$ and $gy \in D$ imply that $fy \in D$; consequently, $f$ and $g$ are selfmaps of $D$. By Theorem A, there exists a unique $b \in D$ such that $b = fb = gb$. ■

The following example illustrates our theorem.

**Example 4.2** Let $X = \mathbb{R}$ and $M = [1, 4]$. Define $f(x) = \frac{1}{3}(x + 2)$ and $g(x) = \frac{1}{2}(x + 1)$. The mappings $f$ and $g$ being commuting are weakly compatible and satisfy the property $(E \cdot A)$ for the sequence $x_n = 1 + \frac{1}{n}$, $n = 1, 2, \ldots$. Also, $|fx - fy| = \frac{1}{3}|x - y| < \frac{1}{2}|x - y| = |gx - gy|$. All the conditions of Theorem 4.1 are satisfied. Clearly, $P_M(0) = \{1\}$ and 1 is the common fixed point of $f$ and $g$.

The following result concerning four weakly compatible maps seems to be new in the literature.

**Theorem 4.3** Let $f, g, p$ and $q$ be selfmaps of a metric space $(X, d)$ and $M$ be a subset of $X$. Assume that $u$ is a common fixed point of $f, g, p$ and $q$, and $D = P_M(u)$. Suppose that

(i) the pairs $(f, p)$ and $(g, q)$ are weakly compatible, and the pair $(f, p)$ or $(g, q)$ satisfies the property $(E \cdot A)$,

(ii) $pD = D$, $qD = D$, $f(\partial M) \subseteq M$, $g(\partial M) \subseteq M$, and $D, fD$, or $gD$ is complete,

(iii) $f$ is $p$-nonexpansive and $g$ is $q$-nonexpansive on $D \cup \{u\}$,

(iv) for all $x, y \in D$, (2.2) holds.
Then $f, g, p$ and $q$ have a unique common fixed point in $P_M(u)$.

**Proof.** Let $y \in D$. Then $py \in D$ and $qy \in D$. By the definition of $P_M(u)$, $y \in \partial M$. Since $f(\partial M) \subseteq M$ and $g(\partial M) \subseteq M$, it follows that $fy, gy \in M$. As $f$ is $p$-nonexpansive on $D \cup \{u\}$, so

$$d(fy, u) = d(fy, fu) \leq d(py, pu) = d(py, u).$$

Now, $fy \in M$ and $py \in D$ imply that $fy \in D$. Similarly, $gy \in D$. Thus $f, g, p$ and $q$ are selfmaps of $D$. Therefore, by Theorem B, there exists a unique $b \in D$ such that $b$ is a common fixed point of $f, g, p$ and $q$. ■

Following Al-Thagafi [2], we define for $g : M \to X$, $C^g_M(u) = \{ x \in M : gx \in P_M(u) \}$ and $D^g_M(u) = P_M(u) \cap C^g_M(u)$. Note that $D^g_M(u) = P_M(u) = C^g_M(u)$ whenever $g$ is the identity map on $M$.

We establish the analogues of Theorem 2.3 (iii) by Shahzad [21] and Theorem 3.3 due to Hussain and Khan [7] in the following result.

**Theorem 4.4** Let $f$ and $g$ be selfmaps of a metric space $(X,d)$ and $M$ be a subset of $X$. Assume that $u$ is a common fixed point of $f$ and $g$, and $D^* = D^g_M(u)$. Suppose that

1. $(i)$ $f$ and $g$ are weakly compatible and satisfy the property $(E \cdot A)$,
2. $(ii)$ $g$ is nonexpansive on $P_M(u) \cup \{u\}$ and $f$ is $g$-nonexpansive on $D^* \cup \{u\}$,
3. $(iii)$ $gD^* = D^*$, $f(\partial M) \subseteq M$, and $fD^*$ or $D^*$ is complete,
4. $(iv)$ for all $x \neq y$ in $D^*$, (2.1) holds.

Then $f$ and $g$ have a common fixed point in $P_M(u)$.

**Proof.** Let $y \in D^*$. Then $gy \in D^*$. By the definition of $D^*$, $y \in \partial M$ and so $fy \in M$. As $f$ is $g$-nonexpansive on $D^* \cup \{u\}$, $d(fy, u) = d(fy, fu) \leq d(gy, u)$. Therefore, $fy \in P_M(u)$. Since $g$ is nonexpansive on $P_M(u) \cup \{u\}$, therefore

$$d(gfy, u) = d(gfy, gu) \leq d(fy, u) = d(fy, fu) \leq d(gy, gu) = d(gy, u).$$

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Thus, $gfy \in P_M(u)$ and so $fy \in C^2_M(u)$. Therefore, $fy \in D^*$. Hence $f$ and $g$ are selfmaps of $D^*$. Thus, by Theorem A, there exists $b \in D^* \subset P_M(u)$ such that $b = fb = gb$. □

5 Eigenvlaue Problems

The aim of this section is to seek solutions of certain nonlinear eigenvlaue problems for operators defined on a normed space and closed balls of a reflexive Banach space; for this, we follow the arguments of Baskaran and Subrahmanyam [3].

We now apply Theorem A to solve an eigenvalue problem.

**Theorem 5.1** Let $X$ be a normed space and $f$ be a selfmap of $X$ with $f(0) \neq 0$. Suppose that

(i) there exists a sequence $\{x_m\}$ such that $\lim_{m \to \infty} f_nx_m = \lim_{m \to \infty} x_m = t$ for some $t \in X$,

(ii) $X$ or $fX$ is complete,

(iii) for all $x \neq y$ in $X$, the following condition holds:

$$\|fx - fy\| \leq \max \left\{ \|x - y\|, \frac{1}{2}(\|f_nx - x\| + \|f_ny - y\|), \frac{1}{2}(\|f_ny - x\| + \|f_nx - y\|) \right\}.$$  

(5.1)

Then $M_n = 1/ (1 - 1/n)$ is an eigenvalue of $f$ for each $n > 1$.

**Proof.** Clearly, $f_n$ and $I$ (the identity map on $X$) are commuting and satisfy the property $(E \cdot A)$. Note that $\|f_n x - f_n y\| < \|fx - fy\|$ for each $n > 1$. By this and (iii), for all $x \neq y$ in $X$ and each $n > 1$, (2.1) is satisfied for the maps $f_n$ and $I$. By Theorem A, there exists $x_n \in X$ such that $x_n = f_n x_n$; that is, $fx_n = M_n x_n$ for each $n > 1$. This and $f(0) \neq 0$ imply that $x_n \neq 0$ for each $n > 1$. Thus, for each $n > 1$, $x_n$ is an eigenvector and $M_n$ is an eigenvalue for $f$. □

The above theorem can be similarly proved for selfmaps of the closed ball $B$ of a normed space.
Example 5.2 Let $X = \mathbb{R}^2$ and $f$ be defined by $f(x, y) = (x-1, y-1)$. Clearly, $f(0, 0) \neq (0, 0)$ and $X$ is complete. Also, (5.1) holds because

$$|f(x_1, y_1) - f(x_2, y_2)| = |(x_1, y_1) - (x_2, y_2)|.$$ 

Now, for the sequence $(x_n, y_n) = \left(\frac{1}{n} - 1, \frac{1}{n} - 1\right)$, $n = 1, 2, \ldots$,

$$\lim_{n \to \infty} f_2(x_n, y_n) = \lim_{n \to \infty} \frac{1}{2} f(x_n, y_n) = (-1, -1) = \lim_{n \to \infty} (x_n, y_n).$$

Thus, by Theorem 5.1, $M_2 = 2$ is an eigenvalue of $f$. The corresponding eigenvector is $(-1, -1)$.

For the solution of eigenvalue problems of nonself-maps on the closed balls, we need the following known result.

Theorem C ([5], p. 92). Let $X$ be a reflexive Banach space and $f : B \to X$ be a weakly continuous map. Suppose that for each $x \in \partial B$, one of the following conditions holds:

(i) $\|fx\| \leq \max\{\|fx - x\|, \|x\|\}$,

(ii) there exists $p > 1$ such that $\|fx\|^p \leq \|fx - x\|^p + \|x\|^p$.

Then $f$ has a fixed point in $B$.

Theorem 5.3 Let $X$ be a reflexive Banach space and $f : B \to X$ be weakly continuous with $f(0) \neq 0$. Suppose that for each $x \in \partial B$ and for $k \in (0, 1]$, one of the following conditions holds:

(i) $\|fx\| \leq \max\{\|kf x - x\|, \|x\|\}$,

(ii) there exists $p > 1$ such that $\|fx\|^p \leq \|kf x - x\|^p + \|x\|^p$.

Then $M = \frac{1}{k}$ is an eigenvalue for $f$.

Proof. Let $M = \frac{1}{k}$, $k \in (0, 1]$. Define, $f_k = kf$. Assume that (i) or (ii) is satisfied; then we get one of the following:
(a) \( \| f_k x \| \leq \| f x \| \leq \max \{ \| f_k x - x \|, \| x \| \} \);

(b) \( \| f_k x \|^p \leq \| f x \|^p \leq \{ \| f_k x - x \|^p, \| x \|^p \} \).

By Theorem C, there exists \( u \in B \) such that \( f_k u = u \). So \( fu = Mu \). This and \( f(0) \neq 0 \) imply that \( u \neq 0 \). Thus \( u \) is an eigenvector for \( f \) and so \( M \) is an eigenvalue for \( f \) as desired.

As an application of Theorem 5.3, we obtain the following result for nonlinear operators on \( L^p, \ p > 1 \).

**Theorem 5.4** Let \( C \) be a closed and bounded subset of \( \mathbb{R}^n \) and \( T : L^p(C) \rightarrow L^p(C) \). Suppose that

(i) \( H = H(t, s) : C \times \mathbb{R} \rightarrow \mathbb{R} \) is weakly continuous with respect to \( s \) uniformly in \( t \),

(ii) \( x(t) \in L^p(C) \) implies \( H(t, Tx(t)) \in L^p(C) \),

(iii) for \( x(t) \in L^p(C) \) with \( \| x(t) \|_p = 1 \), \( \| H(t, Tx(t)) \|_p \leq \max \{ 1, k \| H(t, T^p x(t)) - x(t) \|_p \} \), where \( k \in (0, 1] \),

(iv) \( H(t, T(0)) \neq 0 \).

Then the operator equation

\[
H(t, Tx(t)) = ux(t)
\]  

has a solution in \( B_1, \) the closed unit ball of \( L^p(C) \), for each \( u = \frac{1}{k}, \ k \in (0, 1] \).

**Proof.** We know that \( L^p \ (1 < p < \infty) \) is a reflexive Banach space. The operator \( S \) defined by

\[
Sx(t) = H(t, Tx(t))
\]

maps \( L^p(C) \) into itself by (ii). In view of (iii), for the operator \( S : B_1 \rightarrow L^p(C) \),

\[
\| Sx(t) \|_p \leq \max \{ \| x(t) \|_p, \| kSx(t) - x(t) \|_p \}
\]

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for each $x(t) \in \partial B_1$ and $k \in (0, 1]$. By (iv), we have $S(0) \neq 0$. Now, let $\{x_n(t)\}$ converges weakly to $x(t)$. This implies, by (i), that $\{H(t, x_n(t))\}$ converges weakly to $H(t, x(t))$; that is, $\{Sx_n(t)\}$ converges weakly to $Sx(t)$. Thus $S$ is weakly continuous. Hence, by Theorem 5.3, for each $u = \frac{1}{k}, \ k \in (0, 1]$, we get, $Sx(t) = ux(t)$; that is, the operator equation (5.2) has a solution in $B$ for each $u = \frac{1}{k}, \ k \in (0, 1]$. \[\blacksquare\]

The following example supports the above theorem.

**Example 5.5** The eigenvalue problem

\[e^t - \|x(t)\| = ux(t)\]

has a nontrivial solution in the closed unit ball $B_1$ of $L^2([0, 1])$.

**Solution** Set $H(t, s) = e^t - s$, $Tx(t) = \|x(t)\|$, $C = [0, 1]$ and $p = 2$ in Theorem 5.4.

(i) Assume that $\{s_n\}$ converges weakly to $s$. Then, for any continuous linear functional $f$, we have

\[|f(H(t, s_n)) - f(H(t, s))| = |f(s_n - s)| = |f(s) - f(s_n)|.\]

Thus $H(t, s_n)$ converges weakly to $H(t, s)$ and so $H$ is weakly continuous with respect to $s$ uniformly on $t$.

(ii) If $x(t) \in L^2$, then clearly $H(t, Tx(t)) = e^t - \|x(t)\| \in L^2([0, 1])$.

(iii) For $x(t) \in L^2([0, 1])$ with $\|x(t)\|_2 = 1$, we get $H(t, Tx(t)) = e^t - 1$. So, $\|H(t, Tx(t))\|_2 = \left(\int_0^1 |e^t - 1|^2 dt\right)^{1/2} < 1$.

(iv) $\|H(t, T(0))\|_2 = \|e^t\|_2 = \left(\int_0^1 e^{2t} dt\right)^{1/2} > 1$. So, $H$ does not map $B_1$ into itself and $H(t, T(0)) \neq 0$.

Now, the conclusion is clear by Theorem 5.4.

We need the following theorem to proceed further.
**Theorem D** [6, Corollary 1.6, p. 54]. Let $H$ be a Hilbert space and $f : B \to H$ a nonexpansive map. Assume that for all $x \in \partial B$, one of the following conditions holds:

(i) $\|fx\| \leq \|x\|$,  
(ii) $\|fx\| \leq \|fx - x\|$,  
(iii) $\|fx\|^2 \leq \|x\|^2 + \|fx - x\|^2$,  
(iv) $\langle x, fx \rangle \leq \|x\|^2$,  
(v) $fx = -f(-x)$.

Then $f$ has a fixed point in $B$.

Now, we demonstrate the existence of a solution to an eigenvalue problem in the context of a Hilbert space, under a number of boundary conditions, different from the ones considered earlier.

**Theorem 5.6** Let $H$ be a Hilbert space and $f : B \to H$ nonexpansive with $f(0) \neq 0$. Assume that for each $x \in \partial B$ and $k \in (0, 1]$, one of the following conditions holds:

(a) $\|fx\| \leq \|x\|$,  
(b) $\|fx\| \leq \|kfx - x\|$,  
(c) $\|fx\|^2 \leq \|x\|^2 + \|kfx - x\|^2$,  
(d) $\langle x, kfx \rangle \leq \|x\|^2$,  
(e) $fx = -f(-x)$.

Then $M = \frac{1}{k}$ is an eigenvalue for $f$.

**Proof.** Define $f_k : B \to H$ by $f_k = kf$ where $k \in (0, 1]$. Clearly, $f_k$ is nonexpansive. Now for each $x \in \partial B$ and for $k \in (0, 1]$, one of the conditions (i) – (v) in Theorem D holds for $f_k$. 

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Thus, there exists $u$ in $B$ such that $f_k u = u$. So, $f u = Mu$. This and $f(0) \neq 0$ imply that $u \neq 0$. This completes the proof. ■

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References


