King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 336

October 2005

Common Fixed Points of Weakly Compatible Maps on Balls and Applications

A.R. Khan and A.A.Domlo

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

Common Fixed Points of Weakly Compatible Maps on Balls and Applications

A.R. Khan* and A.A. Domlo
Department of Mathematical Sciences
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia

Abstract

In this paper we prove some common fixed point theorems for weakly compatible maps satisfying different contractive conditions on closed balls of a normed space. As applications, we demonstrate the existence of: (i) common fixed points of the maps from the set of best approximations, (ii) solutions to nonlinear eigenvalue problems for operators defined on closed balls. Our work sets analogues, unifies and improves the earlier results of a number of authors.

2000 Mathematics subject classification: Primary 47H10, 47A75, 41A65; Secondary 54H25.

Key words: Common fixed point; weakly compatible maps; closed ball; best approximation; eigenvalue; normed space.

1 Introduction

In 1982, Sessa [20] introduced the concept of weakly commuting maps to generalize commutativity. Jungck [10], in 1986, generalized weak commutativity to the notion of compatible maps. In 1996, Jungck [12] further weakened compatibility to the concept of weak compatibility. Since then, many interesting fixed point theorems of compatible and weakly compatible maps have been obtained by a number of authors (see, for example, Aamri and El Moutawakil

* Corresponding author.

E-mail: arahim@kfupm.edu.sa

[1], Jachymski [8], Jungck [11], Pant [15] and Popa [17-18]). Pant and Pant [16], in 2000, obtained a unique common fixed point of two noncompatible pointwise R-weakly commuting selfmaps under a strict contractive condition on a metric space (for the case of a sequence of maps, we refer to [15]). Aamri and El Moutawakil [1], in 2002, defined the property $(E \cdot A)$ for selfmaps on a metric space X which always holds for any two noncompatible selfmaps on X and proved two common fixed point theorems for weakly compatible maps satisfying the property $(E \cdot A)$ and certain strict contractive conditions. In 2004, Kamran [13] proved some coincidence and common fixed point theorems for maps satisfying the property $(E \cdot A)$ and hybrid strict contractive conditions (see also, Chang [4] and Sastry et al. [19]).

The existence of fixed points of maps defined on closed balls has been studied by several authors; for example, see Baskaran and Subrahmanyam [3], Delbosco [5], Granas and Dugundji [6], Liu [14] and Shahzad and Latif [22].

In [9], Jungck generalized the Banach contraction principle to the case of two commuting selfmaps on a metric space. Baskaran and Subrahmanyam [3] noted that the commutativity of the maps in Jungck's theorem can be replaced by weak commutativity and then they obtained some common fixed point theorems for two maps on the closed ball of a Banach space. They also provided a solution to a nonlinear eigenvlaue problem for operators on the closed ball of a Banach space.

In this paper, we continue with the above theme and establish results related to common fixed point, best approximation and eigenvalue problems for weakly compatible maps under certain contractive conditions.

The paper is organized as follows: In section 3, motivated by Baskaran and Subrahmanyam [3] and Aamri and El Moutawakil [1], we demonstrate the existence of common fixed points of weakly compatible maps which satisfy the property $(E \cdot A)$ and certain contractive conditions on closed balls of a normed space. Section 4 deals with common fixed points, of weakly compatible maps satisfying the property $(E \cdot A)$, from the set of best approximations. Finally, in Section 5, we find solutions to nonlinear eigenvalue problems for operators defined on closed balls of a normed space and reflexive Banach space.

2 Preliminaries

Let f and g be selfmaps of a metric space (X, d). The maps f and g are

(1) weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$, for all $x \in X$,

(2) compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$,

(3) weakly compatible if they commute at their coincidence points; i.e., if $fu = gu$ for some u in X, then $fgu = gfu$,

(4) satisfying the property $(E \cdot A)$ if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

The map f is nonexpansive if $d(fx, fy) \leq d(x, y)$, for all $x, y \in X$. We say f is g-nonexpansive if $d(fx, fy) \leq d(gx, gy)$, for all $x, y \in X$.

Note that weakly commuting maps are compatible and compatible maps are weakly compatible but the converse in each case does not hold (for examples and counter-examples, see [1], [10], [12], and [18]). It is easy to see that two noncompatible maps satisfy the property $(E \cdot A)$ (see [1], Remark 1). Some fixed point results for noncompatible maps are obtained in [13], [15] and [16].

Let M be a subset of X and $u \in X$. We denote by $P_M(u)$, the set of best approximations to u from M; that is,

$$P_M(u) = \{y \in M : d(y, u) = d(u, M)\},$$

where $d(u, M) = \inf\{d(u, m) : m \in M\}$. The existence of common fixed points in $P_M(u)$ has been studied by various authors; see Al-Thagafi [2], Hussain and Khan [7], Kamran [13] and Shahzad [21].

The use of the following two results by Aamri and El-Moutawakil [1] will be two fold in this work:

(i) to establish common fixed point results on closed balls of a normed space X,

(ii) to find common fixed points from $P_M(u)$.

Theorem A [1, Theorem 1]. *Let f and g be weakly compatible selfmaps of a metric space (X,d) satisfying the property $(E \cdot A)$. Assume that fX or gX is complete and $fX \subset gX$. For all $x \neq y$ in X, f and g satisfy the following contractive condition:*

$$d(fx,fy) < \max\left\{d(gx,gy), \frac{1}{2}[d(fx,gx)+d(fy,gy)], \frac{1}{2}[d(fy,gx)+d(fx,gy)]\right\}. \tag{2.1}$$

Then f and g have a unique common fixed point.

Suppose that $F : \mathbb{R}^+ \to \mathbb{R}^+$ (set of all nonnegative reals) satisfies the following conditions:

(i) F is nondecreasing,

(ii) $0 < F(t) < t$, for each $t \in (0, \infty)$.

Various conditions on F have been studied by different authors (see [4], [8], [15], [16] and [19]). In the next theorem the above function F is employed.

Theorem B [1, Theorem 2]. *Let f, g, p and q be selfmaps of a metric space (X,d) such that (i) the pairs (f,p) and (g,q) are weakly compatible, and the pair (f,p) or (g,q) satisfies the property $(E \cdot A)$, (ii) $fX \subset qX$, $gX \subset pX$ and the range of one of the maps f, g, p or q is complete. For all $x, y \in X$, the maps f, g, p and q satisfy the following contractive condition:*

$$d(fx,gy) \leq F(\max\{d(px,qy), d(px,gy), d(qy,gy)\}). \tag{2.2}$$

Then f, g, p and q have a unique common fixed point.

We remark that Theorem A is a generalization of a theorem of Jungck in [9] and Theorem B generalizes Theorem 3.1 of Jungck [11] for weakly compatible maps under a different contractive condition.

A number of authors have studied the existence of common fixed points under various contractive conditions (see [4], [8], [10], [11], [13], [15], [16], [18] and [19]).

Throughout this paper, we shall employ the following conventions:

(i) the closed ball of a normed space X with center 0 and radius R, $B(0,R) = \{x \in X : \|x\| \leq R\}$ will be simply denoted by B,

(ii) for any selfmap f on X or B, we set $f_n = \left(1 - \frac{1}{n}\right) f,\ n = 2, 3, 4, \dots$,

(iii) f satisfies condition (α) if $f(\alpha x) = \alpha f x$ for each $\alpha \in (\alpha_0, 1)$, $\alpha_0 \geq 0$, and x in B.

3 Common Fixed Point Results

In this section, we obtain some common fixed point theorems for weakly compatible maps satisfying certain contractive conditions on the closed ball of a normed space. Our methods are based on those used by Baskaran and Subrahmanyam [3].

We begin with the following analogue of Theorem 2.1 in [3].

Theorem 3.1 *Let f and g be continuous and weakly compatible selfmaps of the closed ball B of a normed space X. Assume that*

(i) f or g is compact and f satisfies condition (α),

(ii) f and g_n satisfy the property $(E \cdot A)$,

(iii) fB or g_nB is complete and $fB \subset g_nB$,

(iv) for all $x \neq y$ in B, the following contractive condition is satisfied:

$$\|fx - fy\| < \frac{1}{2} \max\{\|gx - gy\|, \|fx - g_nx\| + \|fy - g_ny\|,\ \|fy - g_nx\| + \|fx - g_ny\|\}. \quad (3.1)$$

Then f and g have a common fixed point in B.

Proof. Assume that $u \in B$ is a coincidence point of f and g_n; i.e., $fu = g_nu = \left(1 - \frac{1}{n}\right) gu$. Taking limit as $n \to \infty$, we get $fu = gu$. Since f and g are weakly compatible, therefore $fgu = gfu$. By this and condition (α), we get $fg_nu = f\left(1 - \frac{1}{n}\right) gu = \left(1 - \frac{1}{n}\right) fgu = \left(1 - \frac{1}{n}\right) gfu = g_nfu$. Thus f and g_n are weakly compatible for each n. From $\frac{1}{2}\|gx - gy\| \leq \|g_nx - g_ny\|$ and (3.1), we get

$$\|fx-fy\| < \max\left\{\|g_nx - g_ny\|, \frac{1}{2}(\|fx - g_nx\| + \|fy - g_ny\|), \frac{1}{2}(\|fy - g_nx\| + \|fx - g_ny\|)\right\}.$$

Hence by Theorem A, there exists $x_n \in B$ such that

$$x_n = fx_n = g_n x_n = \left(1 - \frac{1}{n}\right) gx_n.$$

Assume that g is compact (same concerns if f is compact). As $\{x_n\}$ is bounded so $\{gx_n\}$ has a subsequence $\{gx_{n_k}\}$ converging to b in B. Now, $x_{n_k} = fx_{n_k} = \left(1 - \frac{1}{n_k}\right) gx_{n_k}$. Proceeding to the limit as $k \to \infty$ and using the continuity of f and g, we get $b = fb = gb$. Thus f and g have a common fixed point b in B. ∎

The following exmaple verifies Theorem 3.1.

Example 3.2 *Let X be the usual space of reals. Define $f(x) = \frac{1}{4}x,\ g(x) = \frac{1}{2}(x^3 + x)$ where $x \in B = [-1, 1]$. Clearly, f and g are continuous, $fB \subset g_n B$ and f satisfies condition (α). From $gB = [-1, 1]$, it follows that g is compact. It is easy to verify that f and g are compatible. Also, f and g_n satisfy the property $(E \cdot A)$ for the sequence $x_m = \frac{1}{m},\quad m = 1, 2, \ldots$. Finally, for all $x \neq y$ in B, (3.1) holds because $|f(x) - f(y)| < \frac{1}{4}|x^3 - y^3 + x - y| = \frac{1}{2}|g(x) - g(y)|$. Thus all the conditions of Theorem 3.1 are satisfied and 0 is the common fixed point of f and g.*

We replace the property $(E \cdot A)$ in Theorem 3.1 by a map ϕ satisfying a contractive condition and use the argument of Corollary 2 in [1] to obtain:

Corollary 3.3 *Let f and g be continuous and weakly compatible selfmaps of the closed ball B of a normed space X such that*

(i) f or g is compact and f satisfies condition (α),

(ii) fB or $g_n B$ is complete and $fB \subset g_n B$,

(iii) for all $x \neq y$ in B, (3.1) holds,

(iv) there exsits a map $\phi : B \to \mathbb{R}^+$ such that $\|g_n x - fx\| < \phi(g_n x) - \phi(fx)$ for all x in B.

Then f and g have a common fixed point in B.

Proof. For a fixed $n > 1$, let $x_0 \in B$ and choose $x_1 \in B$ such that $fx_0 = g_n x_1$. For $m = 1, 2, \ldots$, choose $x_m \in B$ such that $fx_{m-1} = g_n x_m$. Then

$$\|g_n x_m - g_n x_{m+1}\| = \|g_n x_m - f x_m\| < \phi(g_n x_m) - \phi(f x_m) = \phi(g_n x_m) - \phi(g_n x_{m+1}).$$

Define a real sequence $\{a_m\}$ by $a_m = \phi(g_n x_m), \quad m = 1, 2, \ldots$. Clearly, $\{a_m\}$ is nonnegative, nonincreasing and bounded below by 0. Thus, $\{a_m\}$ is a convergent sequence. We also have

$$\|g_n x_m - g_n x_{m+k}\| \leq a_m - a_{m+k}$$

which implies that $\{g_n x_m\}$ is a Cauchy sequence in $g_n B$. Suppose that $g_n B$ is complete. Then there exists $t \in g_n B$ such that $\lim_{m \to \infty} g_n x_m = t$. So, $\lim_{m \to \infty} f x_m = t$. Hence, f and g_n satisfy the property $(E \cdot A)$. Since n is arbitrary, therefore f and g_n satisfy the property $(E \cdot A)$ for each $n > 1$.

All the conditions of Theorem 3.1 are satisfied and hence the conclusion. ■

The next theorem deals with three weakly compatible maps under a contractive condition in terms of the function F.

Theorem 3.4 *Let f, g and p be continuous selfmaps of the closed ball B of a normed space X such that*

(i) the pairs (f, p) and (g, p) are weakly compatible and one of the maps f, g or p is compact,

(ii) the pair (f, p_n) or (g, p_n) satisfies the property $(E \cdot A)$,

(iii) f and g satisfy condition (α),

(iv) $fB \subset p_n B$, $\;gB \subset p_n B$ and the range of one of the maps f, g or p_n is complete,

(v) for all $x, y \in B$, (2.2) holds with p and q both replaced by p_n.

Then f, g and p have a common fixed point in B.

Proof. As in the proof of Theorem 3.1, we can prove that the pairs (f, p_n) and (g, p_n) are weakly compatible. Hence, by Theorem B, there exists $x_n \in B$ such that $fx_n = gx_n =$

$p_n x_n = x_n$. Let p be compact (same concerns the cases when f or g is compact). As $\{x_n\}$ is bounded, so $\{px_n\}$ has a subsequence $\{px_{n_k}\}$ converging to u in B. Now, $x_{n_k} = fx_{n_k} = gx_{n_k} = \left(1 - \dfrac{1}{n_k}\right) px_{n_k}$. Proceeding to the limit as $k \to \infty$ and using the continuity of f, g and p, we get $u = fu = gu = pu$. Thus f, g and p have a common fixed point u in B. ■

Recall that a Banach space X is said to satisfy Opial's condition if for each x in X and each sequence $\{x_n\}$ weakly convergent to x, the following condition holds for all $y \neq x$:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

An extension of the above theorem to the case of four weakly compatible maps is presented below. Our result extends Theorem 2.2 by Baskaran and Subrahmanyam [3] under a different contractive condition.

Theorem 3.5 *Let f, g, p and q be selfmaps of the closed ball B of a reflexive Banach space X satisfying Opial's condition. Assume that*

(i) f and g are weakly continuous and the pairs (f, p) and (g, q) are weakly compatible,

(ii) $fB \subset q_n B$, $gB \subset p_n B$, the pair (f, p_n) or (g, q_n) satisfies the property $(E \cdot A)$, and the range of one of the maps f, g, p_n or q_n is complete,

(iii) f and g satisfy condition (α),

(iv) p is f-nonexpansive and q is g-nonexpansive,

(v) for all x, y in B, (2.2) holds with p and q replaced by p_n and q_n, respectively.

Then f, g, p and q have a common fixed point in B.

Proof. As before, we can show that the pairs (f, p_n) and (g, q_n) are weakly compatible. Thus, by Theorem B, there exists $x_n \in B$ such that

$$x_n = fx_n = gx_n = p_n x_n = q_n x_n.$$

Since X is a reflexive Banach space, therefore B is weakly compact. Hence, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging weakly to $u \in B$. Now

$$x_{n_k} = fx_{n_k} = gx_{n_k} = \left(1 - \frac{1}{n}\right) px_{n_k} = \left(1 - \frac{1}{n}\right) qx_{n_k}.$$

Taking limit as $k \to \infty$ and using the weak continuity of f and g, we get $u = fu = gu$. Also, px_{n_k} and qx_{n_k} converge weakly to $u = fu = gu$. If $fu \neq pu$, then by Opial's condition and p being f-nonexpansvie, we get

$$\begin{aligned}
\liminf \|fx_{n_k} - fu\| &< \liminf \|fx_{n_k} - pu\| \\
&\leq \liminf(\|fx_{n_k} - px_{n_k}\| + \|px_{n_k} - pu\|) \\
&= \liminf \|px_{n_k} - pu\| \\
&\leq \liminf \|fx_{n_k} - fu\|;
\end{aligned}$$

a contradiction. Hence $fu = pu = u$.

Similarly, we can prove that $gu = qu = u$. Thus u is a common fixed point of f, g, p and q. ■

4 Invariant Approximation

For further applications of Theorems A and B, we obtain common fixed points of best approximation. Our work provides analogues of most of the well-known results for the class of weakly compatible maps on a metric space.

An analogue of Theorem 3.1 of Hussain and Khan [7] is obtained in the following:

Theorem 4.1 *Let M be a subset of a metric space (X, d) and f and g be selfmaps of X. Assume that u is a common fixed point of f and g, and $D = P_M(u)$. Suppose that*

(i) f and g are weakly compatible and satisfy the property $(E \cdot A)$,

(ii) $gD = D$, $f(\partial M) \subseteq M$ (here ∂M denotes the boundary of M), and fD or gD is complete,

(iii) f *is* g*-nonexpansive on* $D \cup \{u\}$,

(iv) *for all* $x \neq y$ *in* D, *(2.1) holds.*

Then f *and* g *have a unique common fixed point in* $P_M(u)$.

Proof. Let $y \in D$. Then $gy \in D$. By the definition of $P_M(u)$, $y \in \partial M$ and since $f(\partial M) \subseteq M$, it follows that $fy \in M$. As f is g-nonexpansive on $D \cup \{u\}$, so

$$d(fy, u) = d(fy, fu) \leq d(gy, gu) = d(gy, u).$$

Now, $fy \in M$ and $gy \in D$ imply that $fy \in D$; consequently, f and g are selfmaps of D. By Theorem A, there exists a unique $b \in D$ such that $b = fb = gb$. ∎

The following example illustrates our theorem.

Example 4.2 *Let* $X = \mathbb{R}$ *and* $M = [1, 4]$. *Define* $f(x) = \frac{1}{3}(x+2)$ *and* $g(x) = \frac{1}{2}(x+1)$. *The mappings* f *and* g *being commuting are weakly compatible and satisfy the property* $(E \cdot A)$ *for the sequence* $x_n = 1 + \frac{1}{n}$, $n = 1, 2, \ldots$. *Also,* $|fx - fy| = \frac{1}{3}|x-y| < \frac{1}{2}|x-y| = |gx - gy|$. *All the conditions of Theorem 4.1 are satisfied. Clearly,* $P_M(0) = \{1\}$ *and 1 is the common fixed point of* f *and* g.

The following result concerning four weakly compatible maps seems to be new in the literature.

Theorem 4.3 *Let* f, g, p *and* q *be selfmaps of a metric space* (X, d) *and* M *be a subset of* X. *Assume that* u *is a common fixed point of* f, g, p *and* q, *and* $D = P_M(u)$. *Suppose that*

(i) *the pairs* (f, p) *and* (g, q) *are weakly compatible, and the pair* (f, p) *or* (g, q) *satisfies the property* $(E \cdot A)$,

(ii) $pD = D$, $qD = D$, $f(\partial M) \subseteq M$, $g(\partial M) \subseteq M$, *and* D, fD, *or* gD *is complete,*

(iii) f *is* p*-nonexpansive and* g *is* q*-nonexpansive on* $D \cup \{u\}$,

(iv) *for all* $x, y \in D$, *(2.2) holds.*

Then f, g, p and q have a unique common fixed point in $P_M(u)$.

Proof. Let $y \in D$. Then $py \in D$ and $qy \in D$. By the definition of $P_M(u)$, $y \in \partial M$. Since $f(\partial M) \subseteq M$ and $g(\partial M) \subseteq M$, it follows that $fy, gy \in M$. As f is p-nonexpansive on $D \cup \{u\}$, so

$$d(fy, u) = d(fy, fu) \leq d(py, pu) = d(py, u).$$

Now, $fy \in M$ and $py \in D$ imply that $fy \in D$. Similarly, $gy \in D$. Thus f, g, p and q are selfmaps of D. Therefore, by Theorem B, there exists a unique $b \in D$ such that b is a common fixed point of f, g, p and q. ■

Following Al-Thagafi [2], we define for $g : M \to X$, $C_M^g(u) = \{x \in M : gx \in P_M(u)\}$ and $D_M^g(u) = P_M(u) \cap C_M^g(u)$. Note that $D_M^g(u) = P_M(u) = C_M^g(u)$ whenever g is the identity map on M.

We establish the analogues of Theorem 2.3 (iii) by Shahzad [21] and Theorem 3.3 due to Hussain and Khan [7] in the following result.

Theorem 4.4 *Let f and g be selfmaps of a metric space (X, d) and M be a subset of X. Assume that u is a common fixed point of f and g, and $D^* = D_M^g(u)$. Suppose that*

(i) f and g are weakly compatible and satisfy the property $(E \cdot A)$,

(ii) g is nonexpansive on $P_M(u) \cup \{u\}$ and f is g-nonexpansive on $D^* \cup \{u\}$,

(iii) $gD^* = D^*, f(\partial M) \subseteq M$, and fD^* or D^* is complete,

(iv) for all $x \neq y$ in D^*, (2.1) holds.

Then f and g have a common fixed point in $P_M(u)$.

Proof. Let $y \in D^*$. Then $gy \in D^*$. By the definition of D^*, $y \in \partial M$ and so $fy \in M$. As f is g-nonexpansive on $D^* \cup \{u\}, d(fy, u) = d(fy, fu) \leq d(gy, u)$. Therefore, $fy \in P_M(u)$. Since g is nonexpansive on $P_M(u) \cup \{u\}$, therefore

$$d(gfy, u) = d(gfy, gu) \leq d(fy, u) = d(fy, fu) \leq d(gy, gu) = d(gy, u).$$

Thus, $gfy \in P_M(u)$ and so $fy \in C_M^g(u)$. Therefore, $fy \in D^*$. Hence f and g are selfmaps of D^*. Thus, by Theorem A, there exists $b \in D^* \subset P_M(u)$ such that $b = fb = gb$. ■

5 Eigenvlaue Problems

The aim of this section is to seek solutions of certain nonlinear eigenvlaue problems for operators defined on a normed space and closed balls of a reflexive Banach space; for this, we follow the arguments of Baskaran and Subrahmanyam [3].

We now apply Theorem A to solve an eigenvalue problem.

Theorem 5.1 *Let X be a normed space and f be a selfmap of X with $f(0) \neq 0$. Suppose that*

(i) there exists a sequence $\{x_m\}$ such that $\lim_{m\to\infty} f_n x_m = \lim_{m\to\infty} x_m = t$ for some $t \in X$,

(ii) X or fX is complete,

(iii) for all $x \neq y$ in X, the following condition holds:

$$\|fx - fy\| \leq \max\left\{\|x-y\|, \frac{1}{2}(\|f_nx - x\| + \|f_ny - y\|), \frac{1}{2}(\|f_ny - x\| + \|f_nx - y\|)\right\}. \tag{5.1}$$

Then $M_n = 1/\left(1 - \frac{1}{n}\right)$ is an eigenvalue of f for each $n > 1$.

Proof. Clearly, f_n and I (the identity map on X) are commuting and satisfy the property $(E \cdot A)$. Note that $\|f_nx - f_ny\| < \|fx - fy\|$ for each $n > 1$. By this and (iii), for all $x \neq y$ in X and each $n > 1$, (2.1) is satisfied for the maps f_n and I. By Theorem A, there exsists $x_n \in X$ such that $x_n = f_nx_n$; that is, $fx_n = M_nx_n$ for each $n > 1$. This and $f(0) \neq 0$ imply that $x_n \neq 0$ for each $n > 1$. Thus, for each $n > 1$, x_n is an eigenvector and M_n is an eigenvalue for f. ■

The above theorem can be similarly proved for selfmaps of the closed ball B of a normed space.

Example 5.2 *Let* $X = \mathbb{R}^2$ *and* f *be defined by* $f(x,y) = (x-1, y-1)$*. Clearly,* $f(0,0) \neq (0,0)$ *and* X *is complete. Also, (5.1) holds because*

$$|f(x_1, y_1) - f(x_2, y_2)| = |(x_1, y_1) - (x_2, y_2)|.$$

Now, for the sequence $(x_n, y_n) = \left(\frac{1}{n} - 1, \frac{1}{n} - 1\right), \quad n = 1, 2, \ldots,$

$$\lim_{n\to\infty} f_2(x_n, y_n) = \lim_{n\to\infty} \frac{1}{2} f(x_n, y_n) = (-1,-1) = \lim_{n\to\infty} (x_n, y_n).$$

Thus, by Theorem 5.1, $M_2 = 2$ *is an eigenvalue of* f*. The corresponding eigenvector is* $(-1,-1)$*.*

For the solution of eigenvalue problems of nonself-maps on the closed balls, we need the following known result.

Theorem C ([5], p. 92). *Let* X *be a reflexive Banach space and* $f : B \to X$ *be a weakly continuous map. Suppose that for each* $x \in \partial B$*, one of the following conditions holds:*

(i) $\|fx\| \leq \max\{\|fx - x\|, \|x\|\}$,

(ii) there exists $p > 1$ *such that* $\|fx\|^p \leq \|fx - x\|^p + \|x\|^p$.

Then f *has a fixed point in* B*.*

Theorem 5.3 *Let* X *be a reflexive Banach space and* $f : B \to X$ *be weakly continuous with* $f(0) \neq 0$*. Suppose that for each* $x \in \partial B$ *and for* $k \in (0,1]$*, one of the following conditions holds:*

(i) $\|fx\| \leq \max\{\|kfx - x\|, \|x\|\}$,

(ii) there exists $p > 1$ *such that* $\|fx\|^p \leq \|kfx - x\|^p + \|x\|^p$.

Then $M = \dfrac{1}{k}$ *is an eigenvalue for* f*.*

Proof. Let $M = \dfrac{1}{k}, \quad k \in (0,1]$. Define, $f_k = kf$. Assume that (i) or (ii) is satisfied; then we get one of the following:

(a) $\|f_k x\| \leq \|fx\| \leq \max\{\|f_k x - x\|, \|x\|\}$,

(b) $\|f_k x\|^p \leq \|fx\|^p \leq \{\|f_k x - x\|^p, \|x\|^p\}$.

By Theorem C, there exists $u \in B$ such that $f_k u = u$. So $fu = Mu$. This and $f(0) \neq 0$ imply that $u \neq 0$. Thus u is an eigenvector for f and so M is an eigenvalue for f as desired. ■

As an application of Theorem 5.3, we obtain the following result for nonlinear operators on L^p, $p > 1$.

Theorem 5.4 *Let C be a closed and bounded subset of $\mathbb{R}^n$ and $T : L^p(C) \to L^p(C)$. Suppose that*

(i) $H = H(t,s) : C \times \mathbb{R} \to \mathbb{R}$ is weakly continuous with respect to s uniformly in t,

(ii) $x(t) \in L^p(C)$ implies $H(t, Tx(t)) \in L^p(C)$,

(iii) for $x(t) \in L^p(C)$ with $\|x(t)\|_p = 1$, $\|H(t,Tx(t))\|_p \leq \max\{1, \|kH(t,Tx(t)) - x(t)\|_p\}$, where $k \in (0,1]$,

(iv) $H(t, T(0)) \neq 0$.

Then the operator equation

$$H(t, Tx(t)) = ux(t) \tag{5.2}$$

has a solution in B_1, the closed unit ball of $L^p(C)$, for each $u = \dfrac{1}{k}$, $k \in (0,1]$.

Proof. We know that L^p $(1 < p < \infty)$ is a reflexive Banach space. The operator S defined by

$$Sx(t) = H(t, Tx(t))$$

maps $L^p(C)$ into itself by (ii). In view of (iii), for the operator $S : B_1 \to L^p(C)$,

$$\|Sx(t)\|_p \leq \max\{\|x(t)\|_p, \|kSx(t) - x(t)\|_p\}$$

for each $x(t) \in \partial B_1$ and $k \in (0,1]$. By (iv), we have $S(0) \neq 0$. Now, let $\{x_n(t)\}$ converges weakly to $x(t)$. This implies, by (i), that $\{H(t, x_n(t))\}$ converges weakly to $H(t, x(t))$; that is, $\{Sx_n(t)\}$ converges weakly to $Sx(t)$. Thus S is weakly continuous. Hence, by Theorem 5.3, for each $u = \frac{1}{k}, \quad k \in (0,1]$, we get, $Sx(t) = ux(t)$; that is, the operator equation (5.2) has a solution in B for each $u = \frac{1}{k}, \quad k \in (0,1]$. ■

The following example supports the above theorem.

Example 5.5 *The eigenvalue problem*

$$e^t - \|x(t)\| = ux(t)$$

has a nontrivial solution in the closed unit ball B_1 of $L^2([0,1])$.

Solution *Set $H(t,s) = e^t - s$, $Tx(t) = \|x(t)\|$, $C = [0,1]$ and $p = 2$ in Theorem 5.4.*

(i) Assume that $\{s_n\}$ converges weakly to s. Then, for any continuous linear functional f, we have

$$|f(H(t,s_n)) - f(H(t,s))| = |f(s_n - s)| = |f(s) - f(s_n)|.$$

Thus $H(t,s_n)$ converges weakly to $H(t,s)$ and so H is weakly continuous with respect to s uniformly on t.

(ii) If $x(t) \in L^2$, then clearly $H(t, Tx(t)) = e^t - \|x(t)\| \in L^2([0,1])$.

(iii) For $x(t) \in L^2([0,1])$ with $\|x(t)\|_2 = 1$, we get $H(t, Tx(t)) = e^t - 1$. So, $\|H(t, Tx(t))\|_2 = \left(\int_0^1 |e^t - 1|^2 dt\right)^{1/2} < 1$.

(iv) $\|H(t, T(0))\|_2 = \|e^t\|_2 = \left(\int_0^1 e^{2t} dt\right)^{1/2} > 1$. So, H does not map B_1 into itself and $H(t, T(0)) \neq 0$.

Now, the conclusion is clear by Theorem 5.4.

We need the following theorem to proceed further.

Theorem D [6, Corollary 1.6, p. 54]. *Let H be a Hilbert space and $f : B \to H$ a nonexpansive map. Assume that for all $x \in \partial B$, one of the following conditions holds:*

(i) $\|fx\| \leq \|x\|$,

(ii) $\|fx\| \leq \|fx - x\|$,

(iii) $\|fx\|^2 \leq \|x\|^2 + \|fx - x\|^2$,

(iv) $\langle x, fx \rangle \leq \|x\|^2$,

(v) $fx = -f(-x)$.

Then f has a fixed point in B.

Now, we demonstrate the existence of a solution to an eigenvalue problem in the context of a Hilbert space, under a number of boundary conditions, different from the ones considered earlier.

Theorem 5.6 *Let H be a Hilbert space and $f : B \to H$ nonexpansive with $f(0) \neq 0$. Assume that for each $x \in \partial B$ and $k \in (0, 1]$, one of the following conditions holds:*

(a) $\|fx\| \leq \|x\|$,

(b) $\|fx\| \leq \|kfx - x\|$,

(c) $\|fx\|^2 \leq \|x\|^2 + \|kfx - x\|^2$,

(d) $\langle x, kfx \rangle \leq \|x\|^2$,

(e) $fx = -f(-x)$.

Then $M = \frac{1}{k}$ is an eigenvalue for f.

Proof. Define $f_k : B \to H$ by $f_k = kf$ where $k \in (0, 1]$. Clearly, f_k is nonexpansive Now for each $x \in \partial B$ and for $k \in (0, 1]$, one of the conditions (i) − (v) in Theorem D holds for f_k.

Thus, there exists u in B such that $f_k u = u$. So, $fu = Mu$. This and $f(0) \neq 0$ imply that $u \neq 0$. This completes the proof. ■

Acknowledgment. The authors gratefully acknowledge the support provided by the King Fahd University of Petroleum & Minerals during this research.

References

[1] M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270(2002), 181–188.

[2] M.A. Al-Thagafi, Common fixed points and best approximation, J. Approx. Theory 85(1996), 318–323.

[3] R. Baskaran and P.V. Subrahmanyam, Common fixed points in closed balls, Atti. Sem. Mat. Fis. Univ. Modena XXXVI (1988), 1–5.

[4] T.H. Chang, Common fixed point theorems for multivalued mappings, Math. Japonica 41(2) (1995), 311–320.

[5] D. Delbosco, Fixed points of mappings defined on a ball of a reflexive Banach space, Nonlinear Anal. Forum 6(1)(2001), 91–95.

[6] A. Granas and J. Dugundji, Fixed Point Theory, Springer, New York, 2003.

[7] N. Hussain and A.R. Khan, Common fixed-point results in best approximation theory, Appl. Math. Letters 16(2003), 575–580.

[8] J. Jachymski, Common fixed point theorems for some families of maps, Indian J. Pure Appl. Math. 25(9)(1994), 925–937.

[9] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly 83(1976), 261–263.

[10] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. & Math. Sci. 9(4)(1986), 771–779.

[11] G. Jungck, Common fixed points for commuting and compatible maps on compacta, Proc. Amer. Math. Soc. 103(3)(1988), 977–983.

[12] G. Jungck, Common fixed points for noncontinuous nonself mappings on a nonmetric space, Far East J. Math. Sci. 4(2)(1996), 199–212.

[13] T. Kamran, Coincidence and fixed points for hybrid strict contractions, J. Math. Anal. Appl. 299(2004), 235–241.

[14] L.S. Liu, Approximation theorems and fixed point theorems for various classes of 1-set-contractive mappings in Banach spaces, Acta Math. Sinica, English Series 17(1)(2001), 103–112.

[15] R.P. Pant, Common fixed point theorems for contractive maps, J. Math. Anal. Appl. 226(1998), 251–258.

[16] R.P. Pant and V. Pant, Common fixed points under strict contractive conditions, J. Math. Anal. Appl. 248(2000), 327–332.

[17] V. Popa, Some fixed point theorems for weakly compatible mappings, Radovi Mat. 10(2001), 245–252.

[18] V. Popa, A general common fixed point theorem of Meir and Keeler type for noncontinuous weak compatible mappings, Filomat (Nis) 18(2004), 33–40.

[19] K.P.R. Sastry, S.V.R. Naidu and J.R. Prasad, Common fixed points for multimaps in a metric space, Nonlinear Anal. 13(3)(1989), 221–229.

[20] S. Sessa, On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. 32(1982), 149–153.

[21] N. Shahzad, Invariant approximations and R-subweakly commuting maps, J. Math. Anal. Appl. 257(2001), 39–45.

[22] N. Shahzad and S. Latif, Random fixed points for several classes of 1-ball-contractive and 1-set-contractive random maps, J. Math. Anal. Appl. 237(1999), 83–92.

(rah2042)