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Decay of solution energy of some viscoelastic equations of hyperbolic type

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Abstract

Consider two Hilbert spaces H and V such that $V \subset H \subset V'$ (dual of V). Our aim is to study the asymptotic behavior of solutions of the following problem

$$\begin{array}{l} u_{tt}(t) + Au(t) - \int_0^t g(t-s)Au(s)ds = 0, \qquad t > 0 \\ u(0) = u_0 \in V, \qquad u_t(0) = u_1 \in H, \end{array}$$

where $A : V \longrightarrow V'$ is a self-adjoint "differential" operator satisfying

$$< Au, v >_{V' \times V} = < A^{1/2}u, A^{1/2}v >_{H \times H}$$

and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive nonincreasing differentiable function. We will show that the dissipation induced by the integral term is strong enough to have a uniform stabilization. We also give some applications.
Keywords : exponential decay, hyperbolic, polynomial decay, relaxation function, viscoelastic.
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1 Introduction

Cavalcanti *et al.* [6] studied the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + |u|^\gamma u = 0, \quad \text{in } \Omega \times (0, \infty),$$

for $a : \Omega \to \mathbb{R}^+$, a function which may vanish on a part $\omega \subset \Omega$ of positive measure. Under some geometry restrictions on ω and

$$\begin{aligned} a(x) &\geq a_0 > 0, \qquad \forall x \in \omega, \\ -\xi_1 g(t) &\leq g'(t) \leq -\xi_2 g(t), \; t \geq 0, \end{aligned}$$

the authors established an exponential rate of decay. Berrimi and Messaoudi [2] improved Cavalcanti's result by introducing a different functional which allowed them to weaken the conditions on both a and g. In particular, the function a can vanish on the whole domain Ω and consequently the geometry condition has disappeared. In [7], Cavalcanti *et al* considered

$$u_{tt} - k_0\Delta u + \int_0^t div[a(x)g(t-\tau)\nabla u(\tau)]d\tau + b(x)h(u_t) + f(u) = 0,$$

under similar conditions on the relaxation function g and $a(x) + b(x) \geq \rho > 0$, for all $x \in \Omega$. They improved the result of [6] by establishing exponential stability for g decaying exponentially and h linear and polynomial stability for g decaying polynomially and h nonlinear. Their proof, based on the use of piecewise multipliers, is similar to the one in [6]. Though both results in [2] and [7] improve the earlier one in [6], the approaches as well as the functionals used are different. Another problem, where the dissipation induced by the integral term is cooperating with a damping acting on a part of the boundary was also discussed by Cavalcanti *et al* [4]. Also, Cavalcanti *et al* [5] studied, in a bounded domain, the following equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma\Delta u_t = 0, \quad \rho > 0,$$

and proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. This decay result was later pushed by Messaoudi and Tatar [10] to a situation where a source term is present. A related result is the work of Kawashima [8], in which he considered a one-dimensional model equation for viscoelastic materials of integral type where the memory function is allowed to have an integrable singularity. For small initial data, Muñoz Rivera and Baretto [13] proved that the first and the second-order energies of the solution to a viscoelastic plate, decay exponentially provided that the kernel of the memory decays exponentially. Kirane and Tatar [9] considered a mildly damped wave equation and proved that any small internal dissipation is sufficient to uniformly stabilize the solution by means of a nonlinear feedback of memory type acting on a part of the boundary. This result was established without any restriction on the space dimension or geometrical conditions on the domain or its boundary. Furthermore, Berrimi and Messaoudi [3] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^{p-2}\, u$$

in a bounded domain and $p > 2$. They established a local existence result and showed, under weaker conditions than those in [7], that the local solution is global and decays uniformly if the initial data are small enough.

Concerning nonexistence, Messaoudi [11] studied

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a\,|u_t|^{\alpha-2}\, u_t = b\,|u|^{p-2}\, u$$

and proved a blow up result for solutions with negative initial energy if $p > \alpha$ and a global result for $p \leq \alpha$. This result has been later improved by Messaoudi [12] to accommodate certain solutions with positive initial energy. By the end it is also worthmentionning the work of Aassila *et al* [1] in which an asymptotic stability and decay rates, for solutions of the wave equation in star-shaped domains, were established by combination of memory effect and damping mechanism.

In this paper, we consider an abstract viscoelastic problem of hyperbolic type of the form

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^t g(t-s)Au(s)ds = 0, & t > 0 \\ u(0) = u_0 \in V, \qquad u_t(0) = u_1 \in H, \end{cases} \tag{1.1}$$

where $A : V \longrightarrow V'$ is a self-adjoint "differential" operator satisfying

$$< Au, v >_{V' \times V} = < A^{1/2}u, A^{1/2}v >_{H \times H} \tag{1.2}$$

$$||v||^2 \leq C_p ||A^{1/2}v||^2, \qquad \forall v \in V, \tag{1.3}$$

$||.||$ denotes the norm in H, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l > 0 \tag{1.4}$$

$$g'(t) \leq -\xi g^p(t), \; t \geq 0, \quad 1 \leq p < \frac{3}{2}. \tag{1.5}$$

We show that the dissipation induced by the integral term is strong enough to stabilize the system. Precisely, we prove that the decay is exponential if $p = 1$ and polynomial if $p > 1$. As an application to our result we go over some problems related to the wave eqaution, the Petrovsky system, and the multi-dimensional wave eqaution.
Definition: By a weak solution of (1.1), we mean a function

$$u \in C([0, T); V) \cap C^1([0, T); H)$$

satisfying, for almost every $t \geq 0$ and for every $v \in V$

$$\begin{aligned} \frac{d}{dt} &< u_t(t), v > + < A^{1/2}u(t), A^{1/2}v > - \int_0^t g(t-s) < A^{1/2}u(s), A^{1/2}v > ds = 0 \\ u(0) &= u_0 \in V, \qquad u_t(0) = u_1 \in H. \end{aligned}$$

We also define the energy by

$$\mathcal{E}(t) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) ||A^{1/2}u(t)||^2 + \frac{1}{2}||u_t(t)||^2 + \frac{1}{2}(g \circ A^{1/2}u)(t), \tag{1.6}$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau)||v(t) - v(\tau)||^2 d\tau. \tag{1.7}$$

Remark.1.1. Condition $p < 3/2$ is made so that

$$\int_0^\infty g^{2-p}(s)ds < \infty.$$

2 Decay of solutions

In this section we state and prove our main result. For this purpose we set

$$F(t) := \mathcal{E}(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t), \tag{2.1}$$

where ε_1 and ε_2 are positive constants and

$$\begin{aligned} \Psi(t) \ &: \ =< u, u_t >_{H\times H} \\ \chi(t) \ &: \ = - < u_t, \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau >_{H\times H} . \end{aligned} \tag{2.2}$$

Lemma 2.1 *For $r > 1$ and $0 < \theta < 1$, we have*

$$\int_0^t g(t-s)||w(s)||^2 ds \leq \left(\int_0^t g^{1-\theta}(t-s)||w(s)||^2 ds \right)^{1/r} \left(\int_0^t g^{(r-1+\theta)/(r-1)}(t-s)||w(s)||^2 ds \right)^{(r-1)/r}$$

for any $w \in H$.
Proof. It suffice to note that

$$\int_0^t g(t-s)||w(s)||^2 ds = \int_0^t g^{(1-\theta)/r}(t-s)||w(s)||^{2/r} g^{(r-1+\theta)/r}(t-s)||w(s)||^{2(r-1)/r} ds$$

and apply Holder's inequality.
Lemma 2.2. *Let $v(t)$ be such that $A^{1/2}v \in L^\infty((0,T);H)$ and g be a continuous function on $[0,T]$ and suppose that $0 < \theta < 1$ and $p > 1$. Then, there exists a constant $C > 0$ such that*

$$\begin{aligned} \int_0^t g(t-s)||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds &\leq C \left(\sup_{0<s<T} ||A^{1/2}v(s)||^2 \int_0^t g^{1-\theta}(s) ds \right)^{\frac{p-1}{p-1+\theta}} \\ &\times \left(\int_0^t g^p(t-s)||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds \right)^{\frac{\theta}{p-1+\theta}} . \end{aligned} \tag{2.3}$$

Proof. By using lemma 2.1 with $r = (p-1+\theta)/(p-1)$, we obtain

$$\begin{aligned} \int_0^t g(t-s)||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds &\leq \left(\int_0^t g^{1-\theta}(t-s)||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds \right)^{\frac{p-1}{p-1+\theta}} \\ &\times \left(\int_0^t g^p(t-s)||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds \right)^{\frac{\theta}{p-1+\theta}} . \end{aligned} \tag{2.4}$$

It is easy to see that

$$\int_0^t g^{1-\theta}(t-s)||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds \leq C \sup_{0<s<T} ||A^{1/2}v(s)||^2 \int_0^t g^{1-\theta}(s) ds \tag{2.5}$$

By combining (2.4) and (2.5), the proof of the lemma is complete.

Lemma 2.3. *Let $v(t)$ be such that $A^{1/2}v \in L^{\infty}((0,T);H)$ and g be a continuous function on $[0,T]$ and suppose that $p > 1$. Then, there exists a constant $C > 0$ such that*

$$\int_0^t g(t-s)||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds \leq C \left(t||A^{1/2}v(t)||^2 + \int_0^t ||A^{1/2}v(s)||^2 ds \right)^{(p-1)/p} \times \left(\int_0^t g^p(t-s)||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds \right)^{1/p}. \tag{2.6}$$

Proof. We use (2.5), for $\theta = 1$ to arrive at

$$\int_0^t g(t-s)||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds \leq \left(\int_0^t ||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds \right)^{(p-1)/p} \times \left(\int_0^t g^p(t-s)||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds \right)^{1/p}. \tag{2.7}$$

It suffices to note that

$$\int_0^t ||A^{1/2}v(t) - A^{1/2}v(s)||^2 ds = t||A^{1/2}v(t)||^2 + \int_0^t ||A^{1/2}v(s)||^2 ds$$

to obtain (2.6). This completes the proof.

Lemma 2.4 *If u is the solution of* (1.1) *then the energy $\mathcal{E}$ satisfies*

$$\mathcal{E}'(t) = \frac{1}{2}(g' \circ A^{1/2}u)(t) - g(t)||A^{1/2}u(t)||^2 \leq \frac{1}{2}(g' \circ A^{1/2}u)(t) \leq 0. \tag{2.8}$$

Proof. By multiplying "scalarly" equation (1.1) by u_t, using (1.2)-(1.5) with some manipulations as in [11], we obtain (2.8).

Lemma 2.5. *For ε_1 and ε_2 small enough, we have*

$$\alpha_1 F(t) \leq \mathcal{E}(t) \leq \alpha_2 F(t), \tag{2.9}$$

holds for two positive constants α_1 and α_2.

Proof. Straightforward computations lead to

$$F(t) \leq \mathcal{E}(t) + (\varepsilon_1/2)\,||u_t||^2 + (\varepsilon_1/2)\,||u||^2 + (\varepsilon_2/2)\,||u_t||^2 + (\varepsilon_2/2)\,||\int_0^t g(t-\tau)(u(t) - u(\tau))d\tau||^2. \tag{2.10}$$

By using (1.2)-(1.4), we have

$$
\begin{aligned}
&||\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau|| = \int_0^t g(t-\tau)||(u(t)-u(\tau))||d\tau \\
&\le \left(\int_0^t \left(\sqrt{g(t-\tau)}\right)^2 ||(u(t)-u(\tau))||^2 d\tau\right)^{1/2} \left(\int_0^t \left(\sqrt{g(t-\tau)}\right)^2 d\tau\right)^{1/2} \\
&= \left(\int_0^t g(t-\tau)||(u(t)-u(\tau))||^2 d\tau\right)^{1/2} \left(\int_0^t g(t-\tau)d\tau\right)^{1/2} \\
&\le \left((1-l)(g\circ A^{1/2}u)(t)\right)^{1/2}.
\end{aligned}
\tag{2.11}
$$

Therefore (2.10) becomes

$$
\begin{aligned}
F(t) &\le \mathcal{E}(t) + [(\varepsilon_1+\varepsilon_2)/2]\,||u_t||^2 + (\varepsilon_1/2)\,C_p||A^{1/2}u||^2 \\
&+ (\varepsilon_2/2)\,C_p(1-l)(g\circ A^{1/2}u)(t) \le \alpha_2\mathcal{E}(t).
\end{aligned}
\tag{2.12}
$$

Similarly we have

$$
\begin{aligned}
F(t) &\ge \mathcal{E}(t) - (\varepsilon_1/2)\,||u_t||^2 - (\varepsilon_1/2)\,||u||^2 \\
&- (\varepsilon_2/2)\,||u_t||^2 - (\varepsilon_2/2)\,C_p(1-l)(g\circ A^{1/2}u)(t) \\
&\ge \tfrac{1}{2}l||A^{1/2}u(t)||^2 + \tfrac{1}{2}||u_t||^2 + \tfrac{1}{2}(g\circ A^{1/2}u)(t) - [(\varepsilon_1+\varepsilon_2)/2]\,||u_t||^2 \\
&- (\varepsilon_1/2)\,C_p||A^{1/2}u(t)||^2 - (\varepsilon_2/2)\,C_p(1-l)(g\circ A^{1/2}u)(t) \ge \alpha_1\mathcal{E}(t)
\end{aligned}
\tag{2.13}
$$

for ε_1 and ε_1 small enough.

Lemma 2.6 *Under the asumptions* (1.2)-(1.5), *the functional*

$$\Psi(t) := < u, u_t >_{H\times H}$$

satisfies, along the solution of (1.1),

$$\Psi'(t) \le ||u_t||^2 - \frac{l}{2}||A^{1/2}u||^2 + \frac{1}{l}\left[\int_0^t g^{2-p}(\tau)d\tau\right](g^p\circ A^{1/2}u)(t). \tag{2.14}$$

Proof By using equation (1.1), we easily see that

$$\Psi'(t) = ||u_t||^2 - ||A^{1/2}u||^2 + < A^{1/2}u, \int_0^t g(t-\tau)A^{1/2}u(\tau)d\tau >_{H\times H}. \tag{2.15}$$

We now estimate the third term in the right side of (2.15) as follows:

$$< A^{1/2}u, \int_0^t g(t-\tau)A^{1/2}u(\tau)d\tau >_{H\times H} \le \frac{1}{2}||A^{1/2}u||^2 + \frac{1}{2}||\int_0^t g(t-\tau)A^{1/2}u(\tau)d\tau||^2$$

$$\leq \frac{1}{2}||A^{1/2}u||^2 + \frac{1}{2}||\int_0^t g(t-\tau)A^{1/2}(u(\tau)-u(t)+u(t))d\tau||^2. \tag{2.16}$$

We then use Cauchy-Schwarz inequality, Young's inequality, and the fact that

$$\int_0^t g(\tau)d\tau \leq \int_0^\infty g(\tau)d\tau = 1-l,$$

to obtain, for any $\eta > 0$,

$$\begin{aligned} ||\int_0^t g(t-\tau)A^{1/2}(u(\tau)-u(t)+u(t))d\tau||^2 &\leq ||\int_0^t g(t-\tau)(A^{1/2}u(\tau)-A^{1/2}u(t))d\tau||^2 \\ +||\int_0^t g(t-\tau)A^{1/2}u(t)d\tau||^2 &+ 2 < \int_0^t g(t-\tau)A^{1/2}(u(\tau)-u(t))d\tau, \int_0^t g(t-\tau)A^{1/2}u(t)d\tau > \\ \leq (1+\eta)||\int_0^t g(t-\tau)A^{1/2}u(t)d\tau||^2 &+ (1+\tfrac{1}{\eta})||\int_0^t g(t-\tau)(A^{1/2}u(\tau)-A^{1/2}u(t))d\tau||^2. \end{aligned} \tag{2.17}$$

At this point, we exploit Cauchy-Schwarz inequality, to estimate

$$\begin{aligned} ||\int_0^t g(t-\tau)(A^{1/2}(u(\tau)-u(t))d\tau||^2 &= \left(\int_0^t g(t-\tau)||A^{1/2}u(\tau)-A^{1/2}u(t)||d\tau\right)^2 \\ = \quad &\left(\int_0^t g^{1-p/2}g^{p/2}(t-\tau)||A^{1/2}(u(\tau)-u(t))||d\tau\right)^2 \\ \leq \quad &\left(\int_0^t g^{2-p}(\tau)d\tau\right)\int_0^t g^p(t-\tau)||A^{1/2}(u(\tau)-u(t))||^2 d\tau. \end{aligned} \tag{2.18}$$

Thus (2.17) takes on the form

$$\begin{aligned} &||\int_0^t g(t-\tau)A^{1/2}(u(\tau)-u(t)+u(t))d\tau||^2 \\ &\leq (1+\eta)\left(\int_0^t g(t-\tau)d\tau\right)^2 ||A^{1/2}u(t)||^2 + (1+\tfrac{1}{\eta})\left(\int_0^t g^{2-p}(\tau)d\tau\right)(g^p \circ A^{1/2}u)(t) \\ &\leq (1+\eta)(1-l)^2||A^{1/2}u(t)||^2 + (1+\tfrac{1}{\eta})\left(\int_0^t g^{2-p}(\tau)d\tau\right)(g^p \circ A^{1/2}u)(t). \end{aligned} \tag{2.19}$$

By combining (2.15)-(2.19), we have

$$\begin{aligned} \Psi'(t) \quad \leq \quad &||u_t||^2 + \frac{1}{2}\left[-1+(1+\eta)(1-l)^2\right]||A^{1/2}u||^2 \qquad (2.20) \\ &+(1+\frac{1}{\eta})\left(\int_0^t g^{2-p}(\tau)d\tau\right)(g^p \circ A^{1/2}u)(t). \end{aligned}$$

By choosing $\eta = l/(1-l)$, (2.14) is established.

Lemma 2.7 *Under the asumptions* (1.2)-(1.5), *the functional*

$$\chi(t) := - < u_t, \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau >$$

satisfies, along the solution of (1.1),

$$\chi'(t) \leq \delta\{1+2(1-l)^2\}||A^{1/2}u||^2+\{2\delta+\frac{3}{4\delta}\}\left[\int_0^t g^{2-p}(\tau)d\tau\right](g^p\circ A^{1/2}u)(t)$$
$$+\frac{g(0)}{4\delta}C_p(-(g'\circ A^{1/2}u)(t)+\{\delta-\int_0^t g(s)ds\}||u_t||^2, \quad \forall\delta>0. \qquad (2.21)$$

Proof. Direct computations, using (1.1), yield

$$\begin{aligned}\chi'(t) &= -< u_{tt},\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau >\\ &\quad -< u_t,\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau > -\left(\int_0^t g(s)ds\right)||u_t||^2\\ &= -< A^{1/2}u(t),\int_0^t g(t-\tau)A^{1/2}(u(t)-u(\tau))d\tau > \qquad (2.22)\\ &\quad -\langle\int_0^t g(t-\tau)A^{1/2}u(\tau)d\tau,\int_0^t g(t-s)(A^{1/2}u(t)-A^{1/2}u(s))\rangle\\ - &< u_t,\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau > -\left(\int_0^t g(s)ds\right)||u_t||^2\end{aligned}$$

Similarly to (2.15), we estimates the right-side terms of (2.22). So for $\delta>0$, we have :

The first term

$$-< A^{1/2}u(t),\int_0^t g(t-\tau)(A^{1/2}u(t)-A^{1/2}u(\tau))d\tau >$$
$$\leq \delta||A^{1/2}u||^2dx+\frac{1}{4\delta}(\int_0^t g^{2-p}(\tau)d\tau)(g^p\circ A^{1/2}u)(t). \qquad (2.23)$$

The second term

$$\begin{aligned}&< \int_0^t g(t-s)A^{1/2}u(s)ds,\int_0^t g(t-s)\left(A^{1/2}u(t)-A^{1/2}u(s)\right)ds >\\ &\leq \delta||\int_0^t g(t-s)A^{1/2}u(s)ds||^2+\tfrac{1}{4\delta}||\int_0^t g(t-s)\left(A^{1/2}u(t)-A^{1/2}u(s)\right)ds||^2\\ &\leq \delta||\int_0^t g(t-s)A^{1/2}\left(u(t)-Au(s)+u(t)\right)ds||^2dx\\ &\quad +\tfrac{1}{4\delta}||\int_0^t g(t-s)\left(A^{1/2}u(t)-A^{1/2}u(s)\right)ds||^2 \qquad (2.24)\\ &\leq \left(2\delta+\tfrac{1}{4\delta}\right)||\int_0^t g(t-s)\left(A^{1/2}u(t)-A^{1/2}u(s)\right)ds||^2+2\delta\left(1-l\right)^2||A^{1/2}u||^2\\ &\leq \left(2\delta+\tfrac{1}{4\delta}\right)\left[\int_0^t g^{2-p}(\tau)d\tau\right](g^p\circ A^{1/2}u)(t)+2\delta(1-l)^2||A^{1/2}u||^2.\end{aligned}$$

The third term

$$-<u_t,\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau>\leq\delta||u_t||^2+\frac{1}{4\delta}\left(\int_0^t -g^{'}(t-s)||u(t)-u(\tau)||d\tau\right)^2 \tag{2.25}$$

We then use, simarly to (2.11), Holder's inequality to estimate

$$\int_0^t -g^{'}(t-s)||u(t)-u(\tau)||d\tau \leq (\int_0^t -g^{'}(t-s)d\tau)^{1/2}\left[-g^{'}\circ u)(t)\right]^{1/2} \tag{2.26}$$
$$\leq \quad C_p\,(g(0))^{1/2}\left[-g^{'}\circ A^{1/2}u)(t)\right]^{1/2}.$$

Hence (2.25) and (2.26) give

$$-<u_t,\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau>\leq\delta||u_t||^2+\frac{g(0)}{4\delta}C_p(-(g'\circ A^{1/2}u)(t). \tag{2.27}$$

By combining (2.22)-(2.27), the assertion of the lemma is established.

Theorem 2.8 *Let* $(u_0,u_1)\in V\times H$ *be given. Assume that* (1.2)-(1.5) *hold. Then, for each* $t_0>0$, *there exist strictly positive constants* K *and* k *such that the solution of* (1.1) *satisfies, for all* $t\geq t_0$,

$$\begin{aligned}\mathcal{E}(t) &\leq Ke^{-kt}, \qquad p=1\\ \mathcal{E}(t) &\leq K(1+t)^{-1/(p-1)}, \quad p>1.\end{aligned} \tag{2.28}$$

Proof.

Since g is continuous, positive and $g(0)>0$ then for any $t_0>0$ we have

$$\int_0^t g(s)ds\geq\int_0^{t_0} g(s)ds=g_0>0, \qquad \forall t\geq t_0. \tag{2.29}$$

By using (2.8), (2.14), (2.21), and (2.29), we obtain

$$F'(t)\leq-\left[\varepsilon_2\{g_0-\delta\}-\varepsilon_1\right]||u_t||^2-\left[\frac{\varepsilon_1 l}{2}-\varepsilon_2\delta\{1+2(1-l)\}\right]||A^{1/2}u||^2 \tag{2.30}$$
$$-\xi\left(\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C_p-[\frac{\varepsilon_1}{l}+\varepsilon_2\{2\delta+\frac{3}{4\delta}\}]\int_0^t g^{2-p}(\tau)d\tau\right)(g^p\circ A^{1/2}u)(t).$$

At this point we choose δ so small that

$$g_0-\delta>\frac{1}{2}g_0, \qquad \frac{2}{l}\delta\{1+2(1-l)<\frac{1}{4}g_0.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{1}{4}g_0\varepsilon_2<\varepsilon_1<\frac{1}{2}g_0\varepsilon_2 \tag{2.31}$$

will make

$$\begin{aligned} k_1 &= \varepsilon_2\{g_0-\delta\}-\varepsilon_1>0 \\ k_2 &= \frac{\varepsilon_1 l}{2}-\varepsilon_2\delta\{1+2(1-l)\}>0. \end{aligned}$$

We then pick ε_1 and ε_2 so small that (2.9) and (2.31) remain valid and

$$\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C_p-[\frac{\varepsilon_1}{l}+\varepsilon_2\{2\delta+\frac{3}{4\delta}\}]\int_0^\infty g^{2-p}(\tau)d\tau>0.$$

Therefore, for all $t\geq t_0$. we have

$$F'(t)\leq-\beta\left[||u_t||^2+||A^{1/2}u||^2+(g^p\circ A^{1/2}u)(t)\right]. \tag{2.32}$$

Case 1. $p=1$: We combine (1.6), (2.9) and (2.32) to get

$$F'(t)\leq-\beta_1\mathcal{E}(t)\leq-\beta_1\alpha_1F(t) \qquad \forall t\geq t_0. \tag{2.33}$$

A simple integration of (2.33) leads to

$$F(t)\leq F(t_0)e^{\beta_1\alpha_1t_0}e^{-\beta\alpha_1t}, \qquad \forall t\geq t_0. \tag{2.34}$$

Thus (2.9), (2.34) yield

$$\mathcal{E}(t)\leq\alpha_2F(t_0)e^{\beta\alpha_1t_0}e^{-\beta\alpha_1t}=Ke^{-kt}, \qquad \forall t\geq t_0. \tag{2.35}$$

Case 2. $p>1$.

By using (1.4) and (1.5) we easily deduce that

$$\int_0^\infty g^{1-\theta}(\tau)d\tau<\infty, \quad \theta<2-p,$$

so lemma 2.2 yields

$$(g\circ A^{1/2}u)(t)\leq C\left\{(g^p\circ A^{1/2}u)(t)\right\}^{\theta/(p-1+\theta)}\left\{\left(\int_0^\infty g^{1-\theta}(\tau)d\tau\right)\mathcal{E}(0)\right\}^{(p-1)/(p-1+\theta)} \tag{2.36}$$

Therefore we get, for $\sigma>1$,

$$\begin{aligned} \mathcal{E}^\sigma(t) &\leq C\mathcal{E}^{\sigma-1}(0)\left(\int_\Omega u_t^2dx+||A^{1/2}u||^2\right)+C\left\{(g\circ A^{1/2}u)(t)\right\}^\sigma \\ &\leq C\mathcal{E}^{\sigma-1}(0)\left(\int_\Omega u_t^2dx+||A^{1/2}u||^2\right) \\ &\quad +C\left\{\mathcal{E}(0)\int_0^\infty g^{1-\theta}(\tau)d\tau\right\}^{\sigma(p-1)/(p-1+\theta)}\left\{(g^p\circ A^{1/2}u)(t)\right\}^{\sigma\theta/(p-1+\theta)}, \end{aligned} \tag{2.37}$$

where C is a generic positive constant. By choosing $\theta = \frac{1}{2}$ and $\sigma = 2p-1$ (hence $\sigma\theta/(p-1+\theta) = 1$), estimate (2.37) gives

$$\mathcal{E}^{\sigma}(t) \leq C \left\{ \int\limits_{\Omega} u_t^2 dx + ||A^{1/2}u||_2^2 + (g^p \circ A^{1/2}u)(t) \right\} \tag{2.38}$$

By combining (2.9), (2.32) and (2.38), we obtain

$$F'(t) \leq -\frac{\beta_2}{\Gamma}\mathcal{E}^{\sigma}(t) \leq -\frac{\beta_2}{\Gamma}(\alpha_1)^{\sigma} F^{\sigma}(t), \qquad \forall t \geq t_0, \tag{2.39}$$

for some constant $\beta_2 > 0$. A simple integration of (2.39) over (t_0, t) leads to

$$F(t) \leq C(1+t)^{-1/(\sigma-1)}, \qquad \forall t \geq t_0. \tag{2.40}$$

As a consequence of (2.40), we have

$$\int_0^{\infty} F(t)dt + \sup_{t\geq 0} tF(t) < \infty. \tag{2.41}$$

Therefore, by using Lemma 3.3, we have

$$\begin{aligned} g \circ A^{1/2}u &\leq C\left[\int_0^{\infty} \|A^{1/2}u(s)\|^2 ds + \sup_t t\|A^{1/2}u(t)\|^2\right]^{(p-1)/p} (g^p \circ A^{1/2}u)^{1/p} \\ &\leq C\left[\int_0^{\infty} F(s)ds + tF(t)\right]^{(p-1)/p} (g^p \circ \nabla u)^{1/p} \leq C(g^p \circ \nabla u)^{1/p}, \end{aligned}$$

which implies that

$$g^p \circ \nabla u \geq C(g \circ \nabla u)^p. \tag{2.42}$$

Consequently, a combination of (2.32) and (2.42) yields

$$F'(t) \leq -C\left[\int_{\Omega} u_t^2(t)dx + \|A^{1/2}u(t)\|^2 + (g \circ \nabla u)^p(t)\right], \ \forall\ t \geq t_0. \tag{2.43}$$

On the other hand, we have , similarly to (2.37),

$$\mathcal{E}^p(t) \leq C\left[\int_{\Omega} u_t^2(t)dx + \|A^{1/2}u(t)\|^2 + (g \circ \nabla u)^p(t)\right], \ \forall\ t \geq t_0. \tag{2.44}$$

Combining the last two inequalities and (2.9), we obtain

$$F'(t) \leq -CF^p(t), \quad t \geq t_0. \tag{2.45}$$

A simple integration of (2.45) over (t_0, t) gives

$$F(t) \leq K(1+t)^{-1/(p-1)}, \quad t \geq t_0.$$

This completes the proof.

Remark 2.1. Estimates (2.28) also hold for all $t \in [0, t_0]$ by virtue of continuity and boundedness of $\mathcal{E}$.

3 Applications.

1) **Wave Equation:**

$$\begin{cases} u_{tt} - \Delta u + \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau = 0, \text{ in } \Omega \times (0,\infty) \\ u(x,t) = 0, \ x \in \partial\Omega \ , t \geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is bounded with a smooth boundary $\partial\Omega$ and $g \geq 0$ satisfying (1.4) and (1.5).

Theorem 3.1. *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that* g *satisfies* (1.4) *and* (1.5)*. Then, for each* $t_0 > 0$*, there exist strictly positive constants* K *and* k *such that the solution of* (3.1) *satisfies, for all* $t \geq t_0$*, the decay estimates* (2.28).

Proof. It suffices to take

$$H = L^2(\Omega), \quad V = H_0^1(\Omega), \quad A = -\Delta$$

It is well known that

$$< -\Delta u, v >= \int_\Omega \nabla u . \nabla v dx, \qquad \forall u, v \in V$$

and, by Poincarė, we have

$$\int_\Omega u^2 dx \leq C_p \int_\Omega |\nabla u|^2 dx$$

The energy is

$$\mathcal{E}(t) := \frac{1}{2}\left(1 - \int\limits_0^t g(s)ds\right) ||\nabla u(t)||^2 + \frac{1}{2}||u_t||^2 + \frac{1}{2}(g \circ \nabla u)(t).$$

All conditions of Theorem 2.8 are satisfied. So (2.28) follow

Remak 3.1. Note that our result is proved without any condition on g'' and g'''. Unlike what was required in [6], we only assume (1.4) and (1.5).

2) **Petrovsky system**

$$\begin{cases} u_{tt} + \Delta^2 u - \int\limits_0^t g(t-\tau)\Delta^2 u(\tau)d\tau = 0, \quad \text{ in } \Omega \times (0,\infty) \\ u(x,t) = 0, \ \frac{\partial u}{\partial \nu} = 0, \ x \in \partial\Omega \ , \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega \end{cases} \tag{3.2}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is bounded with a smooth boundary $\partial\Omega$ and $g \geq 0$ satisfying (1.4) and (1.5).

Theorem 3.2. *Let* $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ *be given. Assume that* g *satisfies* (1.4) *and* (1.5)*. Then, for each* $t_0 > 0$*, there exist strictly positive constants* K *and* k *such that the solution of* (3.2) *satisfies, for all* $t \geq t_0$*, the decay estimates* (2.28).

Proof. It suffices to take

$$H = L^2(\Omega), \quad V = H_0^2(\Omega), \quad A = \Delta^2$$

consequently, we obtain

$$< \Delta^2 u, v >= \int_\Omega \Delta u \Delta v dx, \qquad \forall u, v \in V.$$

By using Poincarė's inequality and Green's formula, we have

$$\int_\Omega u^2 dx \leq C_p \int_\Omega |\Delta u|^2 dx$$

We define the energy by

$$\mathcal{E}(t) := \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) ||\Delta u(t)||^2 + \frac{1}{2}||u_t||^2 + \frac{1}{2}(g \circ \Delta u)(t).$$

All conditions of Theorem 2.8 are satisfied. So the decay estimates (2.28) follow.
3) **Higher-order Wave Equation:**

$$\begin{cases} u_{tt} + (-1)^m D^{2m} u - \int_0^t g(t-\tau)(-1)^m D^{2m} uu(\tau) d\tau = 0, \text{ in } (a,b) \times (0,\infty) \\ D^k u(a,t) = D^k u(b,t) = 0, \ t \geq 0, \ k = 0, 1, ..., m-1 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in (a,b). \end{cases} \tag{3.3}$$

We set

$$H_0^m(\Omega) = \{v \in H^m(\Omega) \quad / \quad v(x) = v'(x) = ... = v^{(m-1)}(x) = 0, \quad x = a, b\}$$

Theorem 3.3. *Let* $(u_0, u_1) \in H_0^m(\Omega) \times L^2(\Omega)$ *be given. Assume that* g *satisfies* (1.4) *and* (1.5). *Then, for each* $t_0 > 0$, *there exist strictly positive constants* K *and* k *such that the solution of* (3.3) *satisfies, for all* $t \geq t_0$, *the decay estimates* (2.28).
Proof. It suffices to take

$$H = L^2(\Omega), \quad V = H_0^m(\Omega), \quad A = (-1)^m D^{2m} u$$

By using "repeated" integration by parts, we easily see that

$$< Au, v >= \int_\Omega D^m u D^m v dx, \qquad \forall u, v \in V$$

and, by repeating Poincarė's inequality several times, we have

$$\int_\Omega u^2 dx \leq C_p \int_\Omega |D^m u|^2 dx$$

The energy is

$$\mathcal{E}(t) := \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) ||D^m u(t)||^2 + \frac{1}{2}||u_t||^2 + \frac{1}{2}(g \circ D^m u)(t)$$

All conditions of Theorem 2.8 are satisfied. So the decay estimates (2.28) follow.
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