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Abstract A double integral that naturally arises from bivariate Wishart distribution is considered. An attempt of evaluating the integral results in an infinite series of the product of several gamma functions. Since the density integral is an identity in so called correlation parameter, many identities in infinite series have been developed by repeatedly differentiating the density identity with respect to the parameter. The series representation of the integral has been evaluated in closed from for special cases. The results developed in the paper would be vital for studying properties especially product moments of bivariate Wishart distribution or bivariate chisqaure distribution. Some applications are outlined.

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1. Introduction

The probability density function of the elements of the Wishart matrix A for the bivariate case is given by

$$f_*(a_{11},a_{22},a_{12}) = \frac{\left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}}{2^m\sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)}\left(a_{11}a_{22}-a_{12}^2\right)^{(m-3)/2}$$
$$\times\exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2}-\frac{a_{22}}{2(1-\rho^2)\sigma_2^2}\right)\exp\left(\frac{\rho a_{12}}{2(1-\rho^2)\sigma_1\sigma_2}\right) \tag{1.1}$$

where $a_{11}>0, a_{22}>0, -\infty<a_{12}<\infty, m>2, -1<\rho<1$ (Anderson, 2003, 123). That is

$$\int_{a_{11}=0}^{\infty}\int_{a_{22}=0}^{\infty}\int_{a_{12}=-\infty}^{\infty}\left(a_{11}a_{22}-a_{12}^2\right)^{(m-3)/2}\exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2}-\frac{a_{22}}{2(1-\rho^2)\sigma_2^2}\right)\exp\left(\frac{\rho a_{12}}{2(1-\rho^2)\sigma_1\sigma_2}\right)da_{11}da_{22}da_{12} \tag{1.2}$$
$$=2^m\sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)(\sigma_1\sigma_2)^m\left(1-\rho^2\right)^{m/2}.$$

In this paper we evaluate, for special cases of integers a and b, the following integral

$$I(a,b;\rho)=\sum_{l=0}^{\infty}\left(\frac{\rho}{1-\rho^2}\right)^l\frac{1}{l!}\frac{\Gamma\left((l+1)/2\right)}{\Gamma\left((l+m)/2\right)}\int_0^{\infty}\int_0^{\infty}u_1^{\frac{m+l}{2}+a-1}u_2^{\frac{m+l}{2}+b-1}e^{-\frac{(u_1+u_2)}{2(1-\rho^2)}}du_1du_2, \tag{1.3}$$

which is a moment integral based on (1.2) .

Evidently, the evaluation of the integral results in an infinite series of the product of several gamma functions. Since the integral is an identity in ρ, many identities in infinite series have been developed by repeatedly differentiating an integral resulting from (1.1) with respect to ρ.

The series representation of the integral has been evaluated in closed from for special cases. A resulting integral involving Mittag-Leffler function is shown to have an elegant expression. The results developed in the paper would be vital for studying properties especially product moments of bivariate Wishart distribution. In what follows, for any positive integer k , we define

(i) $c^{(k)} = c(c-1)\cdots(c-k+1)$

(ii) $(c)_k = c(c+1)\cdots(c+k-1),\ (c)_0 = 1.$

2. The Density Identity

The equation in (1.2) will be called the density identity. The left hand side of the density identity has been transformed to have more elegant identities in the following theorem.

Theorem 2.1 For $m > 2$ and $-1 < \rho < 1$**,** we have

$$(i)\int_0^\infty\int_0^\infty\int_0^1 v^{-1/2}(1-v)^{(m-3)/2}\left(u_1u_2\right)^{m/2-1}\exp\left[-\frac{u_1+u_2-2\rho r\sqrt{u_1u_2}}{2-2\rho^2}\right]du_1du_2dr = 4\pi\,\Gamma(m-1)\left(1-\rho^2\right)^{m/2},$$

$$(ii)\int_0^\infty\int_0^\infty\sum_{l=0}^\infty\left(\frac{\rho}{1-\rho^2}\right)^l\frac{\Gamma\left((l+1)/2\right)}{l!\Gamma\left((l+m)/2\right)}\left(u_1u_2\right)^{(m+l-2)/2}\ e^{-(u_1+u_2)/\left(2-2\rho^2\right)}du_1du_2 = 2^m\sqrt{\pi}\ \Gamma(m/2)\left(1-\rho^2\right)^{m/2}.$$

Proof. Under the transformation $a_{11} = ms_1^2, a_{22} = ms_2^2, a_{12} = mrs_1s_2$ with Jacobian $J(a_{11}, a_{22,}a_{12} \to s_1^2, s_2^2, r) = m^3 s_1 s_2$, the integral in (1.1) yields

$$\int_{s_1^2=0}^\infty\int_{s_2^2=0}^\infty\int_{r=-1}^1 s_1^{m-2}\exp\left(-\frac{ms_1^2}{2(1-\rho^2)\sigma_1^2}\right)s_2^{m-2}\ \exp\left(-\frac{ms_2^2}{2(1-\rho^2)\sigma_2^2}\right)$$

$$\times\left(1-r^2\right)^{(m-3)/2}\exp\left(\frac{m\rho rs_1s_2}{2(1-\rho^2)\sigma_1\sigma_2}\right)ds_1^2ds_2^2dr$$

$$= (2/m)^m\sqrt{\pi}\ \Gamma(m/2)\Gamma((m-1)/2)\left(1-\rho^2\right)^{m/2}(\sigma_1\sigma_2)^m$$

By making the transformation $ms_1{}^2 = \sigma_1^2u_1,\ ms_2{}^2 = \sigma_2^2u_2,$ with Jacobian $J\left(s_1{}^2, s_2{}^2 \to u_1u_2\right) = \left(\sigma_1\sigma_2/m\right)^2$, we have

$$\int_{u_1=0}^\infty\int_{u_2=0}^\infty\int_{r=-1}^{r=1}\left(1-r^2\right)^{(m-3)/2}\left(u_1u_2\right)^{m/2-1}\exp\left[-\frac{1}{2}\left(\frac{u_1+u_2-2\rho r\sqrt{u_1u_2}}{1-\rho^2}\right)\right]du_1du_2dr = 4\pi\ \Gamma(m-1)\left(1-\rho^2\right)^{m/2}.$$

which can be simplified

$$\int_{u_1=0}^\infty\int_{u_2=0}^\infty\int_{v=0}^1\left(u_1u_2\right)^{m/2-1}e^{-(u_1+u_2)/\left(2-2\rho^2\right)}\ v^{-1/2}\left(1-v\right)^{(m-3)/2}\exp\left(\frac{\rho\sqrt{u_1u_2v}}{1-\rho^2}\right)dvdu_1du_2 = 4\pi\ \Gamma(m-1)\left(1-\rho^2\right)^{m/2}$$

which is equivalent to (i). Then integrating out v , we have

$$\int_{u_1=0}^{\infty}\int_{u_2=0}^{\infty}(u_1u_2)^{m/2-1}e^{-(u_1+u_2)/(2-2\rho^2)}\sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{\rho\sqrt{u_1u_2}}{1-\rho^2}\right)^l\frac{\Gamma((l+1)/2)\Gamma((m-1)/2)}{\Gamma((m+l)/2)}du_1du_2$$
$$=4\pi\ \Gamma(m-1)\left(1-\rho^2\right)^{m/2}.$$

Then by the use of the duplication formula of gamma function
$\Gamma(y)=2^{y-1}\Gamma\left(y/2\right)\Gamma\left((y+1)/2\right)/\sqrt{\pi}$ with $y=m-1$, we have

$$\int_0^{\infty}\int_0^{\infty}(u_1u_2)^{(m+l-2)/2}e^{-(u_1+u_2)/(2-2\rho^2)}\sum_{l=0}^{\infty}\left(\frac{\rho}{1-\rho^2}\right)^l\frac{\Gamma((l+1)/2)}{\Gamma((m+l)/2)}\left(\frac{\sqrt{\pi}}{2^{m-2}\Gamma(m/2)}\right)=4\pi\ \left(1-\rho^2\right)^{m/2}.$$

which is equivalent to (ii).

Theorem 2.2 For $m>\max(a,b)$, and $-1<\rho<1$, consider the integral $I(a,b;\rho)$ in (1.3). Then

$$I(a,b;\rho)=\left(2\left(1-\rho^2\right)\right)^{m+a+b}\sum_{l=0}^{\infty}\frac{(2\rho)^l}{l!}\Gamma\left(\frac{l+m}{2}+a\right)\Gamma\left(\frac{l+m}{2}+b\right)\frac{\Gamma\left((l+1)/2\right)}{\Gamma\left((l+m)/2\right)}$$

Proof. The proof is obvious by virtue of gamma functions.

Note that $I(a,b;\rho)=I(b,a;\rho)$. Since $I(a,b;\rho)$ can be written as

$$I(a,b;\rho)=\left(2\left(1-\rho^2\right)\right)^{m+a+b}\sum_{l=0}^{\infty}\rho^l\gamma_{k,m+2b}\ \frac{\Gamma\left(((l+m)/2+a)\right)}{\Gamma\left((l+m)/2\right)}$$
$$=\left(2\left(1-\rho^2\right)\right)^{m+a+b}\sum_{l=0}^{\infty}\rho^l\gamma_{l,m+2b}\left(\frac{l+m}{2}\right)_a,$$

where $\gamma_{l,m}=\dfrac{2^l}{l!}\Gamma\left((l+1)/2\right)\Gamma\left((l+m)/2\right),$ (2.1)

we have the following theorem to facilitate to get a closer form of $I(a,b;\rho)$.

3. The Evaluation of Some Infinite Series Involving Product of Several Gamma Functions

Theorem 3.1 Let $\gamma_{l,m}$ be defined by (2.1) and $L(m,\rho)=\sqrt{\pi}\Gamma(m/2)\left(1-\rho^2\right)^{-m/2}$. Then for $m>2, -1<\rho<1$, we have

$(i)\sum_{l=0}^{\infty}\rho^l\gamma_{l,m}=L(m,\rho)$

$(ii)\sum_{l=0}^{\infty}l\rho^l\gamma_{l,m}=m\rho^2\left(1-\rho^2\right)^{-1}L(m,\rho)=w_1(m,\rho)L(m,\rho)$

$(iii)\sum_{l=0}^{\infty}l^{(2)}\rho^l\gamma_{l,m}=\left(m(m+1)\rho^4+m\rho^2\right)\left(1-\rho^2\right)^{-2}L(m,\rho)=w_{(2)}(m,\rho)L(m,\rho)$

$(iv)\sum_{l=0}^{\infty}l^{(3)}\rho^l\gamma_{l,m}=w_{(3)}(m,\rho)L(m,\rho)$ where

$$w_3(m,\rho) = \left((m^3+3m^2+2m)\rho^6+(3m^2+6m)\rho^4\right)\left(1-\rho^2\right)^{-3}$$

$$= \left(m(m+1)(m+2)\rho^6+3m(m+2)\rho^4\right)\left(1-\rho^2\right)^{-3} L(m,\rho) = w_{(3)}(m)L(m,\rho)$$

$(v)\sum_{l=0}^{\infty} l^{(4)}\rho^l\ \gamma_{l,m} = w_{(4)}(m,\rho)L(m,\rho)$ where

$$w_{(4)}(m,\rho) = \left[(m^4+6m^3+11m^2+6m)\rho^8+(6m^3+30m^2+36m)\rho^6+(3m^2+6m)\rho^4\right]\left(1-\rho^2\right)^{-4}$$

$$= \left[m(m+1)(m+2)(m+3)\rho^8+6m(m+2)(m+3)\rho^6+3m(m+2)\rho^4\right]\left(1-\rho^2\right)^{-4}$$

$(vi)\sum_{l=0}^{\infty} l^2\rho^l\gamma_{l,m} = \left(m^2\rho^4+2m\rho^2\right)\left(1-\rho^2\right)^{-2} L(m,\rho) = w_2(m,\rho)L(m,\rho)$

$(vii)\sum_{l=0}^{\infty} l^3\rho^l\gamma_{l,m} = \left(m^3\rho^6+(6m^2+4m)\rho^4+4m\rho^2\right)\left(1-\rho^2\right)^{-3} L(m,\rho) = w_3(m,\rho)L(m,\rho)$

$(viii)\sum_{l=0}^{\infty} l^4\rho^l\ \gamma_{l,m} = w_4(m,\rho)L(m,\rho)$ where

$$w_4(m,\rho) = \left[m^4\rho^8+(12m^3+16m^2+8m)\rho^6+(28m^2+32m)\rho^4+8m\rho^2\right]\left(1-\rho^2\right)^{-4}$$

Proof. By evaluating (ii) of Theorem 2.1, we have

$$\sum_{l=0}^{\infty}\left(\frac{\rho}{1-\rho^2}\right)^l \frac{\Gamma\left((l+1)/2\right)\Gamma\left((l+m)/2\right)}{l!}\left(2-2\rho^2\right)^{m+l} = 2^m\sqrt{\pi}\ \Gamma(m/2)\left(1-\rho^2\right)^{m/2}$$

which can be simplified to be

$$\sum_{l=0}^{\infty}\frac{1}{l!}(2\rho)^l\ \Gamma\left((l+1)/2\right)\Gamma\left((l+m)/2\right) = \sqrt{\pi}\ \Gamma(m/2)\left(1-\rho^2\right)^{-m/2}$$

which is the same as (i). The identity in (i) can be rewritten as

$$\sum_{l=0}^{\infty}\rho^l\gamma_{l,m} = L(m,0)\left(1-\rho^2\right)^{-m/2}. \tag{3.1}$$

Differentiating both sides of the identity in (3.1) with respect to ρ we have

$$\sum_{l=0}^{\infty} l\rho^{l-1}\gamma_{l,m} = mL(m,0)\rho\left(1-\rho^2\right)^{-m/2-1} \tag{3.2}$$

which yields the identity in (ii). Differentiating the identity in (3.2) again, we have

$$\sum_{l=0}^{\infty} l(l-1)\rho^{l-2}\ \gamma_{l,m} = mL(m,0)\left[\rho\left\{(-m/2-1)\left(1-\rho^2\right)^{-m/2-2}(-2\rho)\right\}+\left(1-\rho^2\right)^{-m/2-1}\right]$$
$$= mL(m,0)\left[1+(m+1)\rho^2\right]\left(1-\rho^2\right)^{-m/2-2} \tag{3.3}$$

which yields the identity in (iii). Differentiating the identity in (3.3), we have

$$\sum_{l=0}^{\infty} l(l-1)(l-2)\rho^{l-3}\ \gamma_{l,m}$$
$$= mL(m,0)\Big[\left(1+(m+1)\rho^2\right)\left\{(-m/2-2)\left(1-\rho^2\right)^{-m/2-3}(-2\rho)\right\}+\left\{(m+1)2\rho\right\}\left(1-\rho^2\right)^{-m/2-2}\Big]$$
$$= mL(m,0)\left[(m^2+3m+2)\rho^3+(3m+6)\rho\right]\left(1-\rho^2\right)^{-m/2-3}$$

(3.4)

which yields the identity in (v). Differentiating the identity in (3.4), we have

$$\sum_{l=0}^{\infty} l(l-1)(l-2)(l-3)\rho^{l-4}\ \gamma_{l,m}$$
$$= mL(m,0)\Big[\left((m^2+3m+2)\rho^3+(3m+6)\rho\right)\left\{(-m/2-3)\left(1-\rho^2\right)^{-m/2-4}(2\rho)\right\}$$
$$+\left\{(m^2+3m+2)3\rho^2+(3m+6)\right\}\left(1-\rho^2\right)^{-m/2-3}\Big]$$
$$= mL(m,0)\left[(m^3+6m^2+11m+6)\rho^4+(6m^2+30m+36)\rho^2+(3m+6)\right]\left(1-\rho^2\right)^{-m/2-4}$$

which yields the identity in (v). The identity in (vi) can be proved as follows:

$$\sum_{l=0}^{\infty} l^2\rho^l\gamma_{l,m} = \sum_{l=0}^{\infty}\left(l^{(2)}+l\right)\rho^l\gamma_{l,m}$$
$$=\left(w_{(2)}(m,\rho)+w_{(1)}(m,\rho)\right)L(m,\rho)$$
$$=\left[w_{(2)}(m,\rho)\left(1-\rho^2\right)^2+w_{(1)}(m,\rho)\left(1-\rho^2\right)^2\right]L(m,\rho)\left(1-\rho^2\right)^{-2}$$
$$=\left[\left((m^2+m)\rho^4+m\rho^2\right)+m\rho^2\left(1-\rho^2\right)^2\right]L(m,\rho)\left(1-\rho^2\right)^{-2}.$$

The identity in (vii) can be proved as follows:

$$\sum_{l=0}^{\infty} l^3\rho^l\gamma_{l,m} = \sum_{l=0}^{\infty}\left(l^{(3)}+3l^{(2)}+l\right)\rho^l\gamma_{l,m}$$
$$=\left(w_{(3)}(m,\rho)+3w_{(2)}(m,\rho)+w_{(1)}(m,\rho)\right)L(m,\rho)$$
$$=\left[w_{(3)}\left(1-\rho^2\right)^3+3w_{(2)}(m,\rho)\left(1-\rho^2\right)^3+w_{(1)}(m,\rho)\left(1-\rho^2\right)^3\right]L(m,\rho)\left(1-\rho^2\right)^{-2} \text{ The}$$
$$=\Big[\left((m^3+3m^2+2m)\rho^6+(3m^2+6m)\rho^4\right)+3\left((m^2+m)\rho^4+m\rho^2\right)\left(1-\rho^2\right)$$
$$+m\rho^2\left(1-\rho^2\right)^2\Big]L(m,\rho)\left(1-\rho^2\right)^{-3}.$$

identity in (vii) can be proved as follows:

$$\sum_{l=0}^{\infty} l^4 \rho^l \gamma_{l,m} = \sum_{l=0}^{\infty}\left(l^{(4)} + 6l^{(3)} + 7l^{(2)} + l\right)\rho^l \gamma_{l,m}$$

$$= \left(w_{(4)}(m,\rho) + 6w_{(3)}(m,\rho) + 7w_{(2)}(m,\rho) + w_{(1)}(m,\rho)\right) L(m,\rho)$$

$$= \Big[w_{(4)}(m,\rho)\left(1-\rho^2\right)^4 + 6w_{(3)}(m,\rho)\left(1-\rho^2\right)^4 + 7w_{(2)}(m,\rho)\left(1-\rho^2\right)^4$$

$$+ w_{(1)}(m,\rho)\left(1-\rho^2\right)^4\Big] L(m,\rho)\left(1-\rho^2\right)^{-4}$$

$$= \Big[\left((m^4 + 6m^3 + 11m^2 + 6m)\rho^8 + (6m^3 + 30m^2 + 36m)\rho^6 + (3m^2 + 6m)\rho^4\right)$$

$$+ 6\left(m^3 + 3m^2 + 2m)\rho^6 + (3m^2 + 6m)\rho^4\right)\left(1-\rho^2\right)$$

$$+ 7\left((m^2 + m)\rho^4 + m\rho^2\right)\left(1-\rho^2\right)^2 + m\rho^2\left(1-\rho^2\right)^3\Big] L(m,\rho)\left(1-\rho^2\right)^{-4}$$

$$= w_4(m,\rho) L(m,\rho)$$

where $w_a(m,\rho)$, $(a = 1,2,3,4)$ are defined in the theorem.

Now consider

$$\psi(a,b;\rho) = \frac{\left(1-\rho^2\right)^{a+b+m/2}}{\sqrt{\pi}\,\Gamma(m/2+b)} W(a,b;\rho), \tag{3.5}$$

where

$$W(a,b;\rho) = 2^a \sum_{l=0}^{\infty} \rho^l \gamma_{l,m+2b} \left(\frac{l+m}{2}\right)_a \tag{3.6}$$

so that $I(a,b;\rho) = 2^{m+b}\left(1-\rho^2\right)^{m+a+b} W(a,b;\rho).$ (3.7)

Note that $W(0,0;\rho) = L(m,\rho)$, $\psi(a,b;\rho) = \psi(b,a;\rho)$ and $W(a,b;\rho) = W(b,a;\rho)$.

Theorem 3.2 For $m > \max(a,b)$, $-1 < \rho < 1$, $1 \le a \le 4, 1 \le b \le 4$ and $-4 \le b \le 4$, we have

$$(i)\ \psi(a,b;\rho) = 2^a \sum_{j=0}^{a} \binom{a}{j} (b + j + 1 - a)_{a-j} \left((m/2) + a - j\right)_j \rho^{2(a-j)},$$

$$(ii)\ W(a,b;\rho) = \frac{\sqrt{\pi}\, 2^a\, \Gamma(m/2+b)}{\left(1-\rho^2\right)^{a+b+m/2}} \sum_{j=0}^{a} \binom{a}{j} (b + j + 1 - a)_{a-j} \left((m/2) + a - j\right)_j \rho^{2(a-j)}.$$

Proof. By virtue of Theorem 3.1, it follows from (3.6) that

$$W(1,b;\rho) = \sum_{l=0}^{\infty} (l+m)\rho^l \gamma_{l,m+2b}$$

$$= \sum_{l=0}^{\infty} l\,\rho^l \gamma_{l,m+2b} + m\sum_{l=0}^{\infty} \rho^l \gamma_{l,m+2b}$$

$$= w_1(m+2b,\rho) L(m+2j,\rho) + mL(m+2b,\rho)$$

$$= (w_1(m+2b,\rho) + m) L(m+2b,\rho)$$

and then it follows from (3.5) that

$$\psi(1,b;\rho)=\frac{\left(1-\rho^2\right)^{1+b+m/2}}{\sqrt{\pi}\Gamma(m/2+b)}W\ (1,b;\rho)$$

$$=\frac{\left(1-\rho^2\right)^{1+b+m/2}}{\sqrt{\pi}\Gamma(m/2+b)}(w_1(m+2b,\rho)+m)L(m+2b,\rho)$$

$$=(w_1(m+2b,\rho)+m)\left(1-\rho^2\right)$$

$$=\left((m+2b)\rho^2\left(1-\rho^2\right)^{-1}+m\right)\left(1-\rho^2\right)$$

$$=m+2b\rho^2$$

By virtue of Theorem 3.1, it follows from (3.6) that

$$W\ (2,b;\rho)=\sum_{l=0}^{\infty}(l+m+2)(l+m)\rho^l\gamma_{l,m+2b}$$

$$=\sum_{l=0}^{\infty}\left(l^2+2(m+1)l+(m^2+2m)\right)\rho^l\gamma_{l,m+2b}$$

$$=\left(w_2(m+2b,\rho)+2(m+1)w_1(m+2b,\rho)+(m^2+2m)\right)L(m+2b,\rho)$$

and hence from (3.5)

$$\psi(2,b;\rho)=\left(w_2(m+2b,\rho)+2(m+1)w_1(m+2b,\rho)+(m^2+2m)\right)\left(1-\rho^2\right)^2$$

$$=\Big[\left((m+2b)^2\rho^4+2(m+2b)\rho^2\right)\left(1-\rho^2\right)^{-2}$$

$$+2(m+1)(m+2b)\rho^2\left(1-\rho^2\right)^{-1}+\left(m^2+2m\right)\Big]\left(1-\rho^2\right)^2$$

$$=4b(b-1)\rho^4+4b(m+2)\rho^2+m(m+2).$$

By virtue of Theorem 3.1, it follows from (3.6) that

$$W\ (3,b;\rho)=\sum_{l=0}^{\infty}(l+m+4)(l+m+2)(l+m)\rho^l\gamma_{l,m+2b}$$

$$=\sum_{l=0}^{\infty}\left(l^3+3(m+2)l^2+3(m^2+12m+8)+(m^3+6m^2+8m)\right)\rho^l\gamma_{l,m+2b}$$

$$=\Big(w_3(m+2b,\rho)+3(m+2)w_2(m+2b,\rho)+3(m^2+12m+8)w_1(m+2b,\rho)$$

$$+(m^3+6m^2+8m)\Big)L(m+2b,\rho)$$

and hence from (3.5)

$$\psi(3,b;\rho)=\Big(w_3(m+2b,\rho)+3(m+2)w_2(m+2b,\rho)+3(m^2+12m+8)w_1(m+2j,\rho)$$
$$+(m^3+6m^2+8m)\Big)\left(1-\rho^2\right)^3$$
$$=8b(b-1)(b-2)\rho^6+12b(b-1)(m+4)\rho^4$$
$$+6b(m+2)(m+4)\rho^2+m(m+2)(m+4).$$

where quantities $w_a(m+2b,\rho),(a=1,2,3)$ are defined in Theorem 3.1. By virtue of Theorem 3.1, it follows from (3.6) that

$$W\ (4,b;\rho)=\sum_{l=0}^{\infty}(l+m+6)(l+m+4)(l+m+2)(l+m)\rho^l\gamma_{l,m+2b}$$
$$=\sum_{l=0}^{\infty}\Big(l^4+4(m+3)l^3+2(3m^2+18m+22)l^2+4(m^3+9m^2+22m+12)l$$
$$+(m^4+12m^3+44m^2+48m)\Big)\rho^l\gamma_{l,m+2b}$$
$$=\Big(w_4(m+2b,\rho)+4(m+3)w_3(m+2b,\rho)+2(3m^2+18m+22)w_2(m+2j,\rho)$$
$$+4(m^3+9m^2+22m+12)w_1(m+2b,\rho)+(m^4+12m^3+44m^2+48m)\Big)L(m+2j,\rho)$$

and that hence from (3.5)

$$\psi(4,b;\rho)=\Big(w_4(m+2b,\rho)+4(m+3)w_3(m+2b,\rho)+2(3m^2+18m+22)w_2(m+2j,\rho)$$
$$+4(m^3+9m^2+22m+12)w_1(m+2b,\rho)+(m^4+12m^3+44m^2+48m)\Big)\left(1-\rho^2\right)^4$$
$$=16b(b-1)(b-2)(b-3)\rho^8+32b(b-1)(b-2)(m+6)\rho^6$$
$$+24b(b-1)(m+4)(m+6)\rho^4+8b(m+2)(m+4)(m+6)\rho^2$$
$$+m(m+2)(m+4)(m+6).$$

where the quantities $w_a(m+2b,\rho),\ a=1,2,3,4$ are defined in Theorem 3.1.

Hence the proof.

4. The Evaluation of the Integral with Some Applications

Theorem 4.1 For $m>2\max(a,b),\ -1<\rho<1,\ 1\le a\le 4,\ 1\le b\le 4$ and $-4\le b\le 4$ the integral in (1.2) can be evaluated to be

$$I(a,b;\rho)=\ 2^{a+b+m}\,\Gamma(m/2+b)\sqrt{\pi}(1-\rho^2)^{m/2}$$
$$\times\sum_{j=0}^{a}\binom{a}{j}\,(b+j+1-a)_{a-j}\,\left((m/2)+a-j\right)_j\,\rho^{2(a-j)}.$$

Proof. By plugging in part (ii) of Theorem 3.2 in (3.7) we have the theorem.

Note that $I(a,b;\rho)=I(b,a;\rho)$ and $I(a,b;\rho)=I(0,b;0)\ \psi(a,b;\rho)$ where $I(0,b;0)=2^b\,\Gamma(m/2+b)/\Gamma(m/2).$

A bivariate chisquare distribution is introduced in the following theorem whose proof is similar to that of Theorem 2.1. For other types of chisquare distribution, we refer to Kotz, Balakrishnan and Johnson (2000, 451).

Theorem 4.2 The random variables U and V are said to have a correlated bivariate chisquare distribution each with m degrees of freedom, if its probability density function is given by

$$(i)\, f(u,v) = \frac{(uv)^{m/2-1}\, e^{-(u+v)/(2-2\rho^2)}}{2^m\sqrt{\pi}\,\Gamma(m/2)(1-\rho^2)^{m/2}} \sum_{l=0}^{\infty}\left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^l \frac{\Gamma((l+1)/2)}{l!\,\Gamma((l+m)/2)},$$

$$\text{(ii)}\, f(u,v) = \frac{(1-\rho^2)^{-m/2}(uv)^{m/2-1}}{2^m\,\Gamma^2(m/2)} \exp\left(-\frac{u+v}{2(1-\rho^2)}\right) E\left[\exp\left(\frac{\rho\sqrt{uv}\sqrt{Y}}{1-\rho^2}\right)\right],$$

$m = N-1 > 2$, $-1<\rho<1$, $u>0, v>0$ and where Y has a beta distribution $B(a,b)$ with parameters $a = 1/2$ and $b = (m-1)/2$.

For integers a and b, the (a,b) th product moment of U and V is given by $\mu'(a,b;\rho) = E(U^aV^b)$ while the corrected product moment is given by $\mu(a,b;\rho) = E[(U-E(U))^a(V-E(V))^b]$ (Johnson, Kotz and Kemp, 1993, 46). The product momet of bivariate chisquare distribution is given in the following theorem.

Theorem 4.3 Let U and V have the bivariate chisquare distribution with pdf given by Theorem 4.2. Then for integers a and b, the (a,b) th product moment of U and V, denoted by $\mu'(a,b;\rho) = E(U^aV^b)$, is given by

$$\mu(a,b;\rho) = \frac{2^{a+b}\Gamma(m/2+b)}{\Gamma(m/2} \sum_{j=0}^{a}\binom{a}{j}(b+j+1-a)_{a-j}\left((m/2)+a-j\right)_j\, \rho^{2(a-j)}$$

where $m > 2\max(a,b)$, and $-1<\rho<1$.

Proof. It follows from Theorem 4.2 that

$$\mu'(a,b;\rho) = \frac{2^{a+b}(1-\rho^2)^{a+b}}{L(m,\rho)} \sum_{l=0}^{\infty}\rho^l \gamma_{l,m+2b}\left(\frac{l+m}{2}\right)_a$$

which, by (2.1) is equivalent to

$$\mu'(a,b;\rho) = = \frac{2^{a+b}(1-\rho^2)^{a+b}}{L(m,\rho)}\left(2\left(1-\rho^2\right)\right)^{-m+a+b} I(a,b;\rho).$$

That is $\mu'(a,b;\rho) = \dfrac{1}{2^m\Gamma(m/2)(1-\rho^2)^{m/2}\sqrt{\pi}} I(a,b;\rho)$ so that the theorem is obvious by Theorem 4.1.

The following corollary is obvious from the above theorem.

Corollary 4.1 Let U and V have the bivariate chisquare distribution with pdf given by Theorem 4.2. Then for integers a and b and $m > 2$, we have

$$(i)\ \mu(a,a;\rho) = 4^a(m/2)_a \sum_{j=0}^{a}\binom{a}{j}(j+1)_{a-j}\left((m/2)+a-j\right)_j\, \rho^{2(a-j)},$$

$(ii)\ \mu(a,-a;\rho) = \dfrac{2^a}{(m/2)^{(a)}} \sum_{j=0}^{a} \binom{a}{j} (j+1-2a)_{a-j} \left((m/2)+a-j\right)_j \rho^{2(a-j)}.$

The above two moments represent a- th moment of UV and U/V respectively.

The corrected product moments of the bivariate chisquare distribution can also be calculated by $\mu(a,b;\rho) = E[(U-E(U))^a(V-E(V))^b]$ with the help of $\mu'(a,b;\rho)$. The product moments of bivariate Wishart distribution can also be derived by the identities in Theorem 3.1.

Theorem 4.4 The product moment correlation between U and V is given by $\rho_{U,V} = \rho^2$.

Proof. By definition

$$\left[\left(E(U^2)-E^2(U)\right)\left(E(V^2)-E^2(V)\right)\right]^{1/2} \rho_{U,V} = E(UV)-E(U)E(V).$$

The theorem then follows by virtue of

$E(UV) = \mu'(1,1) = m(m+2\rho^2),$

$E(U) = \mu'(1,0) = m,$

$E(V) = \mu'(0,1),$

$E(U^2) = \mu'(2,0) = m(m+2),$

$E(V^2) = \mu'(0,2) = m(m+2).$

Note that if $\rho = 0$, the pdf in Theorem 2.2 becomes, as expected, the product of that of the two independent chisquare distributions each with m degrees of freedom.

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