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Anwar H Joarder

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

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Anwar H Joarder
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261 Saudi Arabia
Email: anwarj@kfupm.edu.sa

Abstract A correlated bivariate chisquare distribution is introduced. Some moments of the product and quotient of two correlated chisquare random variables have been derived. Some identities involving product of two gamma functions have been developed to facilitate the calculation of moments. An application is demonstrated while deriving the distribution of correlation coefficient based on a bivariate *t*-distribution.

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1. Introduction

Fisher (1915) derived the distribution of sum of squares and sum of products in order to study the distribution of correlation coefficient from a normal sample. Let $X_1, X_2, \cdots, X_N \ (N > 2)$ be a two-dimensional independent random vectors where $X_j = (X_{1j}, X_{2j})'$, $j = 1, 2, \cdots, N$ is distributed as bivariate normal distribution denoted by $N_2(\theta, \Sigma)$ with $\theta = (\theta_1, \theta_2)'$ and $\Sigma = (\sigma_{ik})$, $i = 1, 2;\ k = 1, 2$. The sums of squares and sum of products are given by $a_{ii} = \sum_{j=1}^{N} (X_{ij} - \bar{X}_i)^2 = mS_i^2,\ m = N - 1, (i = 1, 2)$ and $a_{12} = \sum_{j=1}^{N} (X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) = mrs_1s_2$ respectively. The distribution of a_{11}, a_{22} and a_{12} was derived by Fisher (1915) and may be called the bivariate Wishart distribution after Wishart (1928) who obtained the distribution of p-variate Wishart matrix as the joint distribution of sample variances and covariances from multivariate normal population. Obviously a_{11}/σ_{11} has a chisquare distribution with m degrees of freedom.

In this paper we introduce a bivariate chisquare distribution that follow from the bivariate Wishart distribution. There are a number of bivariate chisquare and gamma distributions excellently reviewed by Kotz, Balakrishnan and Johnson (2000). Some moments of integer order on the product and quotient of two correlated chisquare random variables have been derived. The details of the painstaking algebraic calculations are omitted as they are very long. In case the correlation coefficient vanishes, the moments, as expected, coincides with that of the bivariate independent chisquares random variables.

An application of the geometric mean of the product of two chisquare random variables is outlined while deriving the distribution of correlation coefficient assuming that the parent population has a bivariate *t*-distribution. The bivariate *t*-distribution offers a more viable alternative with respect to business data especially stock return data because its tails are fatter. In what follows, for any nonnegative integer k , we will use the notation

$(c)_k = c(c+1)\cdots(c+k-1)$, $(c)_0 = 1$, $(c)^k = c(c-1)\cdots(c-k+1)$

2. A Bivariate Chisquare Distribution

The pdf of the bivariate Wishart distribution originally derived by Fisher (1915) can be written as

$$f_1(a_{11},a_{22},a_{12}) = \frac{\left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}}{2^m\sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)}\left(a_{11}a_{22}-a_{12}^2\right)^{(m-3)/2}$$
$$\times\exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2}\right)\exp\left(-\frac{a_{22}}{2(1-\rho^2)\sigma_2^2}\right)\exp\left(\frac{\rho a_{12}}{2(1-\rho^2)\sigma_1\sigma_2}\right) \quad (2.1)$$

where $a_{11} > 0, a_{22} > 0, -\infty < a_{12} < \infty, m > 2,$ $\sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2,$ $\sigma_{12} = \rho\sigma_1\sigma_2$, $\sigma_1 > 0, \sigma_2 > 0$ and ρ , $(-1 < \rho < 1)$, is the product moment correlation coefficient between $X_{1j}, j = 1,2,\cdots,N$ and $X_{2j}, j = 1,2,\cdots,N$ (Anderson, 2003, 123).

With the notations in Section 1, let $U_1 = mS_1^2/\sigma_1^2$ and $U_2 = mS^2/\sigma_2^2$. Since the correlation coefficient between $X_{1j}, j = 1,2,\cdots,N$ and $X_{2j}, j = 1,2,\cdots,N$ is ρ, the correlation between U_1 and U_2 is ρ^2.

Theorem 2.1 The random variables U_1 and U_2 are said to have a correlated bivariate chisquare distribution with m degrees of freedom if the joint probability density function is given by

$$(i) f(u,v) = \frac{(uv)^{m/2-1} e^{\frac{-(u+v)}{2(1-\rho^2)}}}{2^m\sqrt{\pi}\,\Gamma(m/2)\left(1-\rho^2\right)^{m/2}}\sum_{l=0}^{\infty}\left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^k\frac{\Gamma\left((l+1)/2\right)}{l!\Gamma\left((l+m)/2\right)},$$

$$\text{(ii)}\ f(u,v) = \frac{\left(1-\rho^2\right)^{-m/2}\left(uv\right)^{m/2-1}}{2^m\,\Gamma^2(m/2)}\exp\left(\frac{-(u+v)}{2(1-\rho^2)}\right) E\left[\exp\left(\frac{\rho\sqrt{uv}\sqrt{Y}}{1-\rho^2}\right)\right]$$

$m = N-1 > 2$, $-1 < \rho < 1$ and where Y has a beta distribution $B(a,b)$ with parameters $a = 1/2$ and $b = (m-1)/2$.

Proof. The theorem follows from (2.1) by the transformation $a_{11} = ms_1^2, a_{22} = ms_2^2,$ $a_{12} = mrs_1s_2$ with Jacobian $m^3s_1s_2$, followed by further transformation with $u_1 = ms_1^2/\sigma_1^2$ and $u_2 = ms_2^2/\sigma_2^2$ with Jacobian $J(s_1^2, s_2^2 \to u,v) = m^{-2}\sigma_1^2\sigma_2^2$ and integration over r .

Several forms of the pdf can be obtained with the application of the duplication formula of gamma function given by $\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\Gamma(z+1/2)/\sqrt{\pi}$.

In case $\rho = 0$, the pdf in Theorem 2.1 becomes the product of two independent chisquare random variables each with m degrees of freedom.

Theorem 2.2 The product moment $\mu'(i,j;\rho) = E\left(U_1^i U_2^j\right)$ is given by

$$E\left(U_1^i U_2^j\right) = \frac{2^{i+j}\left(1-\rho^2\right)^{i+j}}{L(m,\rho)} \sum_{k=0}^{\infty} \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+m}{2}+i\right) \Gamma\left(\frac{k+m}{2}+j\right) \Gamma\left(\frac{k+1}{2}\right) \left(\Gamma\left(\frac{k+m}{2}\right)\right)^{-1}$$

where $L(m,\rho) = \sqrt{\pi}\;\Gamma(m/2)\left(1-\rho^2\right)^{-m/2}, m > 2\max(i,j), -1 < \rho < 1$.

Note that $\mu'(i,j;\rho) = \mu'(j,i;\rho) = E\left(U_1^i U_2^j\right)$.

3. Moments of the Product and Quotient of Two Correlated Chisquare Random Variables

Let $b_{k,m} = \dfrac{2^k}{k!}\Gamma\left(\dfrac{k+1}{2}\right)\Gamma\left(\dfrac{k+m}{2}\right), m > 1$ (3.1)

so that the product moment in Theorem 2.2 can be written as

$$E\left(U_1^i U_2^j\right) = \frac{2^{i+j}\left(1-\rho^2\right)^{i+j}}{L(m,\rho)} \sum_{k=0}^{\infty} \frac{(2\rho)^k}{k!} \left(\frac{k+m}{2}\right)_i \Gamma\left(\frac{k+m+2j}{2}\right) \Gamma\left(\frac{k+1}{2}\right)$$

$$= \frac{2^{i+j}\left(1-\rho^2\right)^{i+j}}{L(m,\rho)} \sum_{k=0}^{\infty} \rho^k \gamma_{k,m+2j} \left(\frac{k+m}{2}\right)_i .$$

To facilitate the calculation of moments, first we have the following theorem.

Theorem 3.1 Let $\gamma_{k,m}$ be defined by (3.1). Then for $L(m,\rho)$, $m > 2$, $-1 < \rho < 1$ defined in Theorem 2.2, we have

$$(i)\ \sum_{k=0}^{\infty} \rho^k \gamma_{k,m} = L(m,\rho)$$

$$(ii)\ \sum_{k=0}^{\infty} k\,\rho^k \gamma_{k,m} = m\rho^2\left(1-\rho^2\right)^{-1} L(m,\rho) = w_1(m,\rho)L(m,\rho)$$

$$(iii)\ \sum_{k=0}^{\infty} k^{(2)} \rho^k \gamma_{k,m} = w_{(2)}(m,\rho)L(m,\rho),\ \ w_{(2)}(m,\rho) = \left(m(m+1)\rho^4 + m\rho^2\right)\left(1-\rho^2\right)^{-2}$$

$$(iv)\ \sum_{k=0}^{\infty} k^{(3)} \rho^k \gamma_{k,m} = w_{(3)}(m,\rho)L(m,\rho),$$

$$w_{(3)}(m,\rho) = \left((m^3+3m^2+2m)\rho^6 + (3m^2+6m)\rho^4\right)\left(1-\rho^2\right)^{-3}$$

$$= \left((m)_3\rho^6 + 3m(m+2)\rho^4\right)\left(1-\rho^2\right)^{-3}$$

$(v)\ \sum_{k=0}^{\infty} k^{(4)} \rho^k \ \gamma_{k,m} = w_{(4)}(m,\rho) L(m,\rho),$

$$w_{(4)}(m,\rho) = \left[(m^4 + 6m^3 + 11m^2 + 6m)\rho^8 + (6m^3 + 30m^2 + 36m)\rho^6 + (3m^2 + 6m)\rho^4\right]\left(1-\rho^2\right)^{-4}$$

$$= \left[(m)_4 \rho^8 + 6m(m+2)(m+3)\rho^6 + 3m(m+2)\rho^4\right]\left(1-\rho^2\right)^{-4}$$

$(vi)\ \sum_{k=0}^{\infty} k^2 \rho^k \gamma_{k,m} = w_2(m,\rho) L(m,\rho),\ \ w_2(m,\rho) = \left(m^2\rho^4 + 2m\rho^2\right)\left(1-\rho^2\right)^{-2}$

$(vii)\ \sum_{k=0}^{\infty} k^3 \rho^k \gamma_{k,m} = w_3(m,\rho) L(m,\rho),\ \ w_3(m,\rho) = \left(m^3\rho^6 + (6m^2 + 4m)\rho^4 + 4m\rho^2\right)\left(1-\rho^2\right)^{-3}$

$(viii)\ \sum_{k=0}^{\infty} k^4 \rho^k \ \gamma_{k,m} = w_4(m,\rho) L(m,\rho),$

$$w_4(m,\rho) = \left[m^4\rho^8 + (12m^3 + 16m^2 + 8m)\rho^6 + (28m^2 + 32m)\rho^4 + 8m\rho^2\right]\left(1-\rho^2\right)^{-4}$$

Proof. Since $\mu'(0,0,0;\rho) = \frac{1}{L(m,\rho)} \sum_{k=0}^{\infty} \rho^k \ \gamma_{k,m}$, the proof of (i) is obvious by virtue of $\mu'(0,0,0;\rho) = 1$ where $\mu'(l_1, l_2, l; \rho)$ is defined in Theorem 2.2. The identity in (i) can be rewritten as $\sum_{k=0}^{\infty} \rho^k \gamma_{k,m} = L(m,0)\left(1-\rho^2\right)^{-m/2}$. By repeatedly differentiating with respect to ρ and by the use of Stirling number of the second kind, the other identities follow.

Note that $\frac{L(m+c,\rho)}{L(m,\rho)} = \frac{\Gamma\left((m+c)/2\right)}{\Gamma\left(m/2\right)}\ (1-\rho^2)^{-c/2}$. The following corollary follows from Theorem 2.2.

Corollary 3.1 For $m > 2, -1 < \rho < 1$, the i th moment of $V = U_1 U_2$ is given by

$$E\left(V^i\right) = \frac{4^i\left(1-\rho^2\right)^{2i}}{L(m,\rho)} \sum_{k=0}^{\infty} \frac{(2\rho)^k}{k!} \Gamma^2\left(\frac{k+m}{2} + i\right) \Gamma\left(\frac{k+1}{2}\right)\left(\Gamma\left(\frac{k+m}{2}\right)\right)^{-1}$$

where $L(m,\rho)$ is defined in Theorem 2.2. In case j is a nonnegative integer, then

$$E\left(V^i\right) = \frac{4^i\left(1-\rho^2\right)^{2i}}{L(m,\rho)} \sum_{k=0}^{\infty} \rho^k \gamma_{k,m+2i} \ \left(\frac{k+m}{2}\right)_i .$$

The moments of V are evaluated from Corollary 3.1 by applying Theorem 3.1 though they involve formidable algebraic manipulations.

Corollary 3.2 For $m > 2, -1 < \rho < 1$**,** the first four raw moments of V is given by

$(i)\ E\left(V\right) = m\left(m + 2\rho^2\right),$

$(ii)\ E\left(V^2\right) = m(m+2)\left[8\rho^4 + 8(m+2)\rho^2 + m(m+2)\right],$

$(iii)\ E\left(V^3\right)=16(m/2)_3\left[24\rho^6+36(m+4)\rho^4+9(m+2)(m+4)\rho^2+4(m/2)_3\right],$

$(iv)\ E\left(V^4\right)=256\left(\frac{m}{2}\right)_4\left[24\rho^8+48(m+6)\rho^6+72\left(\frac{m+4}{2}\right)_2\rho^4+16\left(\frac{m+2}{2}\right)_3\rho^2+\left(\frac{m}{2}\right)_4\right].$

In case $\rho=0$, then V will be the product of two independent chisqaure random variables each with m degrees of freedom and evidently the resulting moments are in agreement with that situation. Since the geometric mean of the chisquare variables are important in many applications, we define $G=\left(U_1U_2\right)^{1/2}$, the geometric mean of U_1 and U_2, and provide its moments below.

Corollary 3.3 For $m>2, -1<\rho<1$, the first four raw moments of G is given by

$(i)\ E(G)=\frac{2\left(1-\rho^2\right)^{m/2+1}}{\sqrt{\pi}\ \Gamma(m/2)}\sum_{k=0}^{\infty}\rho^k\gamma_{k,m+1}\frac{\Gamma\left((k+m+1)/2\right)}{\Gamma\left((k+m)/2\right)},$

$(ii)\ E\left(G^2\right)=m\left(m+2\rho^2\right),$

$(iii)\ E\left(G^3\right)=\frac{8\left(1-\rho^2\right)^{m/2+3}}{\sqrt{\pi}\ \Gamma(m/2)}\sum_{k=0}^{\infty}\rho^k\ \gamma_{k,m+3}\frac{\Gamma\left((k+m+3)/2\right)}{\Gamma\left((k+m)/2\right)},$

$(iv)\ E\left(G^4\right)=m(m+2)\left[8\rho^4+8(m+2)\rho^2+m(m+2)\right].$

Corollary 3.4 For $m>2, -1<\rho<1$, the moment generating function of G at h is given by

$$M_G(h)=\frac{1}{L(m,\rho)}\sum_{j=0}^{\infty}\frac{(4h)^j\left(1-\rho^2\right)^{2j}}{j\,!}\sum_{k=0}^{\infty}\rho^k\gamma_{k,m+j}\frac{\Gamma\left((k+m+j)/2\right)}{\Gamma\left((k+m)/2\right)}$$

where $L(m,\rho)$ is defined in Theorem 2.2**.** The following corollary follows from Corollary 3.4 by putting $\rho=0$.

Corollary 3.5 Let $G=\sqrt{U_1U_2}$ be the geometric mean of two independent chisquare random variables U_1 and U_2 each with m degrees of freedom. Then the j the moment and moment generating function of G are given by

$E\left(G^j\right)=\frac{2^j\ \Gamma^2((m+j)/2)}{\Gamma^2(m/2)}$ and $M_G(h)=\sum_{j=0}^{\infty}\frac{(2h)^j}{j\,!}\frac{\Gamma^2((m+j)/2)}{\Gamma^2(m/2)}$

respectively.

Let $W=U_1/U_2$, the ratio of two correlated chisqaure variables U_1 and U_2 that have probability density function in Theorem 2.1. Then the following corollary follows from Theorem 2.2.

Corollary 3.6 For $m>2j, -1<\rho<1$, the j th moment of W is given by

$$E\left(W^{j}\right)=\frac{1}{L(m,\rho)}\sum_{k=0}^{\infty}\frac{(2\rho)^{k}}{k!}\Gamma\left(\frac{k+m}{2}+j\right)\Gamma\left(\frac{k+m}{2}-j\right)\Gamma\left(\frac{k+1}{2}\right)\left(\Gamma\left(\frac{k+m}{2}\right)\right)^{-1}$$

where $L(m,\rho)$ is defined in Theorem 2.2. In case j is a nonnegative integer we have

$$E\left(W^{j}\right)=\frac{1}{L(m,\rho)}\sum_{k=0}^{\infty}\rho^{k}\gamma_{k,m-2j}\left(\frac{k+m}{2}\right)_{j},\quad m>2j,-1<\rho<1.$$

The moments of W are calculated from Corollary 3.6 by applying Theorem 3.1.

Corollary 3.7 For $m>8,-1<\rho<1$, the first four moments of W are given below:

$(i)\ E(W)=\dfrac{m-2\rho^{2}}{m-2},\quad m>2,$

$(ii)\ E\left(W^{2}\right)=\dfrac{1}{(m-2)(m-4)}\left(24\rho^{4}-8(m+2)\rho^{2}+m(m+2)\right),\ m>4,$

$$(iii)\ E\left(W^{3}\right)=\frac{2^{-3}\Gamma(m/2-3)}{\Gamma(m/2)}$$
$$\times\left[-480\rho^{6}+144(m+4)\rho^{4}-18(m+2)(m+4)\rho^{2}+m(m+2)(m+4)\right],\ m>6,$$

$$(iv)\ E\left(W^{4}\right)=\frac{2^{-4}\Gamma(m/2-4)}{\Gamma(m/2)}\Big[13440\rho^{8}-3840(m+6)\rho^{6}+480(m+4)(m+6)\rho^{4}$$
$$-256(m/2+1)_{3}\ \rho^{2}+8(m/2)_{3}\Big],\ m>8.$$

In case $\rho=0$, then W will be the ratio of two independent chisquare variables each with the same degrees of freedom, and the above moments will be simply $E\left(W^{j}\right)=(m/2)_{j}/(m/2-j)_{j},\ \ m>2j$ which are evidently in agreement with the situation.

Corollary 3.8 For $m>2j,-1<\rho<1$, the moment generating function of W at h is given by

$$M_{W}(h)=\frac{1}{L(m,\rho)}\sum_{j=0}^{\infty}\frac{h^{j}}{j!}\sum_{k=0}^{\infty}\rho^{k}\gamma_{k,m-2j}\left(\frac{k+m}{2}\right)_{j}$$

where $L(m,\rho)$ is defined in Theorem 2.2**.**

4. An Application to the Distribution of Correlation Coefficient Based on Bivariate T-Population

The random symmetric positive definite matrix $A=(a_{ik}),\ i=1,2;k=1,2$ is said to have a bivariate Wishart distribution based on bivariate t-population with $m=N-1>2$ and $\Sigma=(\sigma_{ik})>0,\ i=1,2;k=1,2$, written as $A\sim W_{2}(m,\Sigma;\nu)$ if its probability density function is given by

$$f_{2}(A)=C(m,\nu)\,|\Sigma|^{-m/2}\ \ |A|^{(m-3)/2}\left(1+tr(\nu\Sigma)^{-1}A\right)^{-\nu/2-m},\quad A>0,m>2$$

where $C(m,\nu)=\dfrac{2^{m-2}\ \nu^{-m}\ \Gamma(m+\nu/2)}{\pi\ \Gamma(m-1)\Gamma(\nu/2)}$ (See Sutradhar and Ali, 1989, 160) where $a_{11}>0, a_{22}>0, -\infty<a_{12}<\infty, -1<\rho<1,\ m>2, \sigma_1>0, \sigma_2>0$.

We want to derive the sampling distribution of the product moment correlation coefficient R. First we have the following two lemmas.

Lemma 4.1 For $m>2$, let $I(\rho,m)=\int_0^{\pi}(\sin\theta)^{m-1}(1-\rho\sin\theta)^{-m}\,d\theta,\quad -1<\rho<1$. Then

$(i)\ I(\rho,m)=\dfrac{2^{m-1}}{\Gamma(m)}\sum_{k=0}^{\infty}\dfrac{(2\rho)^k}{k\,!}\ \Gamma^2\big((m+k)/2\big),$

$(ii)\ I(\rho,m)=\dfrac{2^{m-1}\ \Gamma^2(m/2)}{\Gamma(m)}\ M_G(\rho)$

where $M_G(\rho)$ is defined in Corollary 3.5.

Proof. Since $|\rho\sin\theta|<1$, we have $(1-\rho\sin\theta)^{-m}\ =\sum_{k=0}^{\infty}\dfrac{\rho^k}{k\,!}\dfrac{\Gamma(m+k)}{\Gamma(m)}\ (\sin\theta)^k$ so that

$$I(\rho,m)=\sum_{k=0}^{\infty}\frac{\rho^k}{k\,!}\frac{\Gamma(m+k)}{\Gamma(m)}\int_0^{\pi}(\sin\theta)^{m+k-1}d\theta$$

$$=\sum_{k=0}^{\infty}\frac{\rho^k}{k\,!}\frac{\Gamma(m+k)}{\Gamma(m)}\frac{\sqrt{\pi}\ \Gamma((m+k)/2)}{\Gamma((m+k+1)/2)}$$

since $\int_0^{\pi}(\sin\theta)^m\,d\theta=\dfrac{\sqrt{\pi}\Gamma((m+1)/2)}{\Gamma(m/2+1)}$. Next, replacing $\Gamma(m+k)$ by the duplication formula of gamma function $\Gamma(z)=\dfrac{2^{z-1}}{\sqrt{\pi}}\Gamma\left(\dfrac{z+1}{2}\right)\Gamma\left(\dfrac{z}{2}\right)$ with $z=m+k$, we have (i), which can also be written as (ii) by virtue of Corollary 3.5.

Lemma 4.2 For $m>2$, let $J(\rho,m,\nu)=\int_0^{\infty}\int_0^{\infty}(u_1u_2)^{m/2-1}\left(1+u_1+u_2-2\rho\sqrt{u_1u_2}\right)^{-\nu-m}du_1du_2$.

Then

$(i)\ J(\rho,m,\nu)=\dfrac{\Gamma(\nu)}{\Gamma(m+\nu)}\sum_{k=0}^{\infty}\dfrac{(2\rho)^k}{k\,!}\Gamma^2\left(\dfrac{m+k}{2}\right),$

$(ii)\ J(\rho,m,\nu)=\dfrac{\Gamma(\nu)\Gamma^2(m/2)}{\Gamma(m+\nu)}\ M_G(\rho).$

Proof. The integral can be written as

$$J(\rho,m,\nu)=4\int_0^{\infty}\int_0^{\infty}(y_1y_2)^{m-1}\left(1+y_1^2+y_2^2-2\rho y_1y_2\right)^{-\nu-m}dy_1dy_2.$$

Further transformation $y_1 = w\cos\theta,\ y_2 = w\sin\theta$ with Jacobian $J(y_1, y_2 \to w, \theta) = w$ yields

$$J(\rho, m, \nu) = 2^{-m+3} \int_{\theta=0}^{\pi/2} \int_{w=0}^{\infty} \left(\sin 2\theta\right)^{m-1} w^{2m-1} \left(1 + w^2 - \rho w^2 \sin 2\theta\right)^{-\nu-m} \, dw d\theta .$$

Next, the transformations $w^2 = u,\ 2\theta = \alpha$ yield

$$J(\rho, m, \nu) = \frac{1}{2^{m-1}} \int_{\alpha=0}^{\pi} (\sin\alpha)^{m-1} \int_{u=0}^{\infty} u^{m-1} \left[1 + (1 - \rho\sin\alpha) u\right]^{-\nu-m} du d\alpha. \qquad (4.1)$$

Then by virtue of $\int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{\Gamma(m)\Gamma(n)}{a^n b^m \Gamma(m+n)}$, the last integral of (4.1) becomes $\frac{\Gamma(m)\Gamma(\nu/2)}{\Gamma(m+\nu/2)} (1 - \rho\sin\theta)^{-m}$ so that

$$J(\rho, m, \nu) = \frac{1}{2^{m-1}} \frac{\Gamma(m)\Gamma(\nu)}{\Gamma(m+\nu)} \int_{\alpha=0}^{\pi} (\sin\alpha)^{m-1} \, (1 - \rho r\sin\alpha)^{-m} d\alpha.$$

and hence the lemma follows by Lemma 4.1.

It is known that the distribution of correlation coefficient is robust under violation of normality at least in the bivariate elliptical class of distributions (Fang and Anderson, 1990 or Ali and Joarder, 1991). However a direct derivation would provide more insight for those who have been recently using *t*-distributions as parent populations. See for example Joarder and Ahmed (1996), Billah and Saleh (2000), Kibria (2004), Kotz and Nadarajh (2004) and the references therein.

Theorem 4.1 The probability density function of the correlation coefficient R based on a bivariate *t*-population is given by

$$h(r) = \frac{2^{m-2}\ \Gamma^2(m/2)\left(1-\rho^2\right)^{m/2}}{\pi\ \Gamma(m-1)} \left(1 - r^2\right)^{(m-3)/2} M_G(\rho r)$$

$$h(r) = \frac{2^{m-2}\left(1-\rho^2\right)^{m/2}}{\pi\ \Gamma(m-1)} \left(1 - r^2\right)^{(m-3)/2} \sum_{k=0}^{\infty} \frac{(2\rho r)^k}{k!}\ \Gamma^2\left(\frac{m+k}{2}\right),\ -1 < r < 1$$

(cf. Muirhead, 1982, 154) where $m > 2, -1 < \rho < 1$ and $M_G(\rho r)$ is defined in Corollary 3.4.

Proof. The pdf of the elements of A based on bivariate *t*-distribution can be written as

$$f_3(a_{11}, a_{22}, a_{12}) = C(m, \nu)\ \left(1-\rho^2\right)^{-m/2} (\sigma_1\sigma_2)^{-m} \left(a_{11}a_{22} - a_{12}^2\right)^{(m-3)/2}$$
$$\times\left(1 + \frac{1}{\nu(1-\rho^2)}\left(\frac{a_{11}}{\sigma_1^2} + \frac{a_{22}}{\sigma_2^2} - \frac{2\rho a_{12}}{\sigma_1\sigma_2}\right)\right)^{-\nu/2-m} .$$

Under the transformation $a_{11} = ms_1^2, a_{22} = ms_2^2, a_{12} = mrs_1 s_2$ with Jacobian $J(a_{11}, a_{22}, a_{12} \rightarrow r, s_1^2, s_2^2) = m^3 s_1 s_2$, the pdf of S_1^2, S_2^2 and R is given by

$$f_4\left(s_1^2, s_1^2, r\right) = m^m\ C(m,\nu)(\sigma_1\sigma_2)^{-m}\ \left(1-\rho^2\right)^{-m/2}\left(1-r^2\right)^{(m-3)/2}(s_1 s_2)^{m-2}$$
$$\times\left(1+\frac{1}{\nu(1-\rho^2)}\left\{\frac{ms_1^2}{\sigma_1^2}+\frac{ms_1^2}{\sigma_1^2}-2\rho r\frac{ms_1 s_2}{\sigma_1\sigma_2}\right\}\right)^{-\nu/2-m}$$

Then keeping r intact, the transformation $ms_1^2 = u_1\sigma_1^2$, $ms_2^2 = u_2\sigma_2^2$ with Jacobian $J(s_1^2, s_2^2 \rightarrow u_1, u_2) = m^{-2}\left(\sigma_1\sigma_2\right)^2$, we have the pdf of U_1, U_2 and R given below.

$$f_5\left(u_1, u_2, r\right) = C(m,\nu)\ \left(1-\rho^2\right)^{-m/2}\left(1-r^2\right)^{(m-3)/2}(u_1 u_2)^{m/2-1}$$
$$\times\left[1+\frac{1}{\nu(1-\rho^2)}\left(u_1+u_2-2\rho r\sqrt{u_1 u_2}\right)\right]^{-\nu/2-m}$$

By integrating out u_1 and u_2 followed by a transformation $u_1 = \nu\left(1-\rho^2\right)y_1$ and $u_2 = \nu\left(1-\rho^2\right)y_2$ with Jacobian $J(u_1, u_2 \rightarrow y_1, y_2) = \nu^2\left(1-\rho^2\right)^2$, we have

$$h(r) = \nu^m\ C(m,\nu)\ \left(1-\rho^2\right)^{m/2}\left(1-r^2\right)^{(m-3)/2}$$
$$\times 4\int_0^\infty\int_0^\infty (y_1 y_2)^{m/2-1}\left(1+y_1+y_2-2\rho r\sqrt{y_1 y_2}\right)^{-\nu/2-m} dy_1 dy_2.$$

Then the theorem follows by Lemma 4.2.

The above results indicate the robustness of the distribution or of tests on correlation coefficient. Thus the assumption of bivariate normality under which all the tests on correlation are developed can be relaxed to a broader class of bivariate *t*-distribution. Interested readers may go through Muddapur (1988) for some exciting exact tests on correlation coefficient.

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