An Introduction to the Bivariate T-Distribution

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Abstract The bivariate *t*-distribution is a natural generalization of the bivariate normal distribution as a derived sampling distribution or as a scale mixture of itself with an inverted gamma distribution. For broad spectrum of researchers, the paper introduces the bivariate *t*-distribution as a probable model for population. Some characteristics especially different types of moments, distribution of sample correlation coefficient and simple linear regression with bivariate *t*-error are presented.

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***Key Words**:* bivariate *t*-distribution, product moments, standardized moments, correlation coefficient, linear regression

1. Introduction

It is well known that the bivariate *t*-distribution arises as a derived sampling distribution from the bivariate normal distribution and the chisquare distribution (Anderson, 2003, 289). In this paper we emphasize a scale mixture representation to calculate some product moments and standardized moments for the bivariate *t*-distribution. This compounds a nonnegative continuous distribution with the bivariate normal distribution. Though we have considered an 'inverted chisquare' distribution as the scaling or the compounding distribution, the generalization of the model to any other continuous distribution is evident. This section of the paper is a decent introduction of relevant materials for Section 2 to motivate those who are experts in statistical analysis by normal distributions but wish to check the robustness of their theories in a broader context of *t*-distributions. Wherever possible, matrix algebra has been avoided for broad spectrum of readers.

Since the use of the multivariate *t*-distribution is on the increase in business especially in stock returns, the paper will enlighten as well as stimulate research in business, econometrics and statistics. Interested readers may go through Joarder (1992), Kotz and Nadarajah (2004), Nadarajah and Kotz (2005) and Nadarajah and Kotz (2005a) and the references therein.

(i) The Univariate T-Distribution

Theorem 1.1 Let $Z \sim N(0,\tau^2)$ and $\nu \mathrm{T}^{-2} \underset{=}{d}\ W \sim \chi^2_\nu$ where the symbol $\underset{=}{d}$ means that both sides of it have the same distribution.

$(a)\ (\nu / W)^{1/2} Z \sim t_\nu$,

$$(b)\ h_{\mathrm{T}}(\tau)=\frac{2(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}\tau^{-(\nu+1)}e^{-\nu/(2\tau^2)},\ \ 0<\tau\,, \tag{1.1}$$

(c) the pdf of univariate *t*-dsitribution with ν degrees of freedom has the following representation:

$$f_1(z)=\int_0^\infty \frac{1}{\tau\sqrt{2\pi}}\ \exp\left(-\frac{z^2}{2\tau^2}\right)h_{\mathrm{T}}(\tau)d\tau. \tag{1.2}$$

Proof. The proof of the first two parts are well known.

By plugging (1.1) in (1.2) we have

$$f_1(z)=\int_0^\infty \frac{(\nu/2)^{\nu/2}\sqrt{2/\pi}}{\tau\Gamma(\nu/2)}\ \exp\left[-\left(\frac{z^2}{2\tau^2}+\frac{\nu}{2\tau^2}\right)\right]\tau^{-(\nu+1)}d\tau.$$

With the transformation $\nu/\tau^2=w$, we have

$$f_1(z)=\int_0^\infty \frac{(\nu/2)^{\nu/2}\sqrt{2/\pi}}{\Gamma(\nu/2)}\ \sqrt{w}\ \exp\left[-\left(\frac{z^2}{2\nu}+\frac{1}{2}\right)w\ \right]w^{(\nu+1)/2}\left(\frac{1}{2}w^{-3/2}\sqrt{\nu}\right)dw$$

$$=\int_0^\infty \frac{1}{2^{\nu/2}\Gamma(\nu/2)\sqrt{2\pi\nu}}\ \exp\left[-\left(\frac{z^2}{2\nu}+\frac{1}{2}\right)w\ \right]w^{(\nu-1)/2}dw$$

$$=\frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\Gamma(\nu/2)}\left(\frac{z^2}{\nu}+1\right)^{-(\nu+1)/2}$$

which is the probability density function (pdf) of t_ν and that proves part (c).

In what follows, for any nonnegative integer a, we denote the ascending and descending factorial by

$k_{\{a\}}=k(k+1)\cdots(k+a-2)(k+a-1),\ (k)_0=1,$ and

$k^{\{a\}}=k(k-1)\cdots(k-a+2)(k-a+1)$

respectively. Obviously, $k_{\{a\}}=(k+a-1)^{\{a\}}$ and $k^{\{a\}}=(k-a+1)_{\{a\}}$.

Corollary 1.1 Let T has the pdf given by (1.1). Then the a-th moment of T is given by

$$\gamma_a=E\left(\mathrm{T}^a\right)=\frac{(\nu/2)^{a/2}\Gamma(\nu/2-a/2)}{\Gamma(\nu/2)},\quad \nu>a.$$

Clearly

$$\gamma_2=\frac{\nu}{\nu-2},\ \ \nu>2,$$

$$\gamma_4=\frac{\nu^2}{(\nu-2)(\nu-4)},\ \ \nu>4,$$

$$\gamma_6=\frac{\nu^3}{(\nu-2)(\nu-4)(\nu-6)},\ \ \nu>6,$$

$$\gamma_{2a}=E\left(T^{2a}\right)=\frac{(\nu/2)^a}{(\nu/2-1)^{\{a\}}},\quad \nu>2a,$$

$$\gamma_{-2a} = E\left(\mathrm{T}^{-2a}\right) = (\nu/2)^{-a}(\nu/2)_{\{a\}},\ \ \nu > 2.$$

(ii) The Standard Bivariate T-Distribution

A distribution is said to have the standard bivariate *t*-distribution if it has the pdf given by

$$f_2(z_1, z_2) = \frac{1}{2\pi}\left[1 + \frac{1}{\nu}\left(z_1^2 + z_2^2\right)\right]^{-(\nu/2+1)}, \nu > 0. \tag{1.3}$$

Notice that though the components Z_1 and Z_2 in (1.3) are uncorrelated but they are not independent unless $\nu \to \infty$. Product moments of the above distribution are given by Corollary 3.3. Following the univariate *t*-distribution, the quantity ν may also be called the degrees of freedom though it is just a shape parameter here. Since the pdf in (1.3) is constant on the circle $z_1^2 + z_2^2 = r^2$, for any fixed r, the distribution is also called the circular *t*-distribution.

(iii) Correlated Bivariate T-Distribution

The following is the pdf of a correlated bivariate *t*-distribution

$$f_3(y_1, y_2) = \frac{1}{2\pi\sqrt{1-\rho^2}}\left(1 + \frac{y_1^2 + y_2^2 - 2\rho y_1 y_2}{\nu(1-\rho^2)}\right)^{-(\nu/2+1)} \tag{1.4}$$

which is a special case of the well known bivariate *t* distribution (Anderson, 2003, 289). The above distribution is permutation-symmetric as the components have common mean 0, common variance γ_2 and common correlation coefficient ρ where $\gamma_2(\nu - 2) = \nu,\ \nu > 2$ (See Tong, 1990, 202-203). The above distribution is explicitly considered for the first time in El-Bassiouni, Sultan and Moshref (2006). Product moments of the above distribution are given by Corollary 3.2.

Theorem 1.2 Let $Z = (Z_1, Z_2)' \sim N_2(0, \Sigma), \Sigma = (\sigma_{ik}), \sigma_{12} = \rho = \sigma_{21},\ (-1 < \rho < 1)$ independent of W where $\nu\mathrm{T}^{-2} \underline{\underline{d}}\ W \sim \chi_\nu^2$. Then

(*a*) $Y = (\nu/W)^{1/2} Z \sim T_2(0, \Sigma; \nu),$

(b) The pdf of a bivariate *t*-distribution with pdf in (1.4) can be written as

$$f_4(y) = (2\pi)^{-1}\int_0^\infty |\tau^2\Sigma|^{-1/2} \exp\left[y'(\tau^2\Sigma)^{-1} y\right] h_{\mathrm{T}}(\tau) d\tau \tag{1.5}$$

which is the scale mixture of bivariate normal and the 'inverted' chisquare distribution of T with pdf in (1.1).

Proof. (a) The pdf of Z_1, Z_2 and W is given by

$$f_5(z_1, z_2, w) = \frac{1}{2\pi}\exp\left(-\frac{z_1^2 + z_2^2 - 2\rho z_1 z_2}{2(1-\rho^2)}\right)\frac{w^{\nu/2-1}}{2^{\nu/2}\Gamma(\nu/2)}\exp(-w/2).$$

Letting $y_1 = (\nu/w)^{1/2} z_1,\ y_2 = (\nu/w)^{1/2} z_2,\ w = w$ with Jacobian w/ν, we have

$$f_6(y_1, y_2, w) = \frac{2^{-(\nu/2+1)} w^{\nu/2}}{\pi\nu\Gamma(\nu/2)\sqrt{1-\rho^2}}\exp\left[-\left(\frac{y_1^2 + y_2^2 - 2\rho y_1 y_2}{2\nu(1-\rho^2)} + \frac{1}{2}\right)w\right].$$

Integrating out w, we have the pdf of bivariate *t* distribution given by (1.4).

(b) The pdf in (1.5) can be written as

$$f_7(y_1, y_2)$$

$$= \int_0^\infty \frac{1}{2\pi\tau^2\sqrt{1-\rho^2}} \exp\left[-\frac{y_1^2+y_2^2-2\rho y_1 y_2}{2\tau^2(1-\rho^2)}\right] \frac{2(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \tau^{-(\nu+1)} e^{-\nu/(2\tau^2)} d\tau,$$

which can be simplified as

$$f_7(y_1, y_2)$$

$$= \frac{(\nu/2)^{\nu/2}}{\pi\Gamma(\nu/2)\sqrt{1-\rho^2}} \int_0^\infty \exp\left[-\left(\frac{y_1^2+y_2^2-2\rho y_1 y_2}{2(1-\rho^2)\tau^2}+\frac{\nu}{2\tau^2}\right)\right]\tau^{-(\nu+3)} d\tau.$$

With the transformation $\nu/\tau^2 = w$, we have

$$f_7(y_1, y_2)$$

$$= \frac{(\nu/2)^{\nu/2}}{\pi\Gamma(\nu/2)\sqrt{1-\rho^2}}$$

$$\times \int_0^\infty \exp\left[-\left(\frac{y_1^2+y_2^2-2\rho y_1 y_2}{2(1-\rho^2)}+\frac{\nu}{2}\right)w\right](\sqrt{\nu}w^{-1/2})^{-(\nu+3)}\left(\frac{1}{2}\sqrt{\nu}w^{-3/2}\right)dw$$

$$= \frac{1}{2^{\nu/2+1}\pi\nu\Gamma(\nu/2)\sqrt{1-\rho^2}} \int_0^\infty \exp\left[-\left(\frac{y_1^2+y_2^2-2\rho y_1 y_2}{2(1-\rho^2)\nu}+\frac{1}{2}\right)w\right]w^{\nu/2} dw$$

$$= \frac{\Gamma(\nu/2+1)}{2^{\nu/2+1}\pi\nu\Gamma(\nu/2)\sqrt{1-\rho^2}}\left[\left(\frac{y_1^2+y_2^2-2\rho y_1 y_2}{2(1-\rho^2)\nu}+\frac{1}{2}\right)\right]^{-(\nu/2+1)},$$

which simplifies to the pdf of the bivariate *t*-distribution given by (1.4).

(iii) A Location Scale Bivariate T-Distribution

The pdf of the bivariate normal distribution is given by

$$f_8(x) = (2\pi)^{-1}|\Sigma|^{-1/2}\exp\left(-(x-\theta)'\Sigma^{-1}(x-\theta)/2\right) \quad (1.6)$$

where $X' = (X_1, X_2)$, $\theta' = (\theta_1, \theta_2)$ is unknown vector of location parameters and Σ is the 2×2 unknown positive definite matrix of scale parameters. It is known that $E(X) = \theta$ and $Cov(X) = \Sigma$. The pdf in (1.6) can be rewritten as

$$f_8(x_1, x_2) = \frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2}\exp\left(\frac{-q(x_1,x_2)}{2}\right) \quad (1.7)$$

where

$$(1-\rho^2)q(x_1,x_2) = \left(\frac{x_1-\theta_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\theta_2}{\sigma_2}\right)^2 - \frac{2\rho(x_1-\theta_1)(x_2-\theta_2)}{\sigma_1\sigma_2}. \quad (1.8)$$

The pdf of the bivariate t - random vector is given by

$$f_9(x) = (2\pi)^{-1}|\Sigma|^{-1/2}\left(1+(x-\theta)'(\nu\Sigma)^{-1}(x-\theta)\right)^{-\nu/2-1} \quad (1.9)$$

where the scalar ν is assumed to be a known positive constant (Anderson, 2003, 289). The probability density function will be denoted by $T_2(\theta,\Sigma;\nu)$ in contrast to the bivariate normal by $N_2(\theta,\Sigma)$. It is worth noting that for the bivariate *t*-distribution with pdf in (1.9), $E(X) = \theta$

and $Cov(X) = \gamma_2\Sigma$ where $\gamma_2(\nu-2) = \nu$. The pdf (1.9) of the bivariate *t*-distribution can be written as a mixture of the bivariate normal distribution and the inverted chisquare distribution as follows:

$$f_9(x) = (2\pi)^{-1}\int_0^\infty |\tau^2\Sigma|^{-1/2}\left(1+(x-\theta)'(\nu\tau^2\Sigma)^{-1}(x-\theta)\right)^{-\nu/2-1} h_\Upsilon(\tau)d\tau \qquad (1.10)$$

which can be written as

$$f_9(x_1,x_2) = \int_0^\infty \frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2\tau^2}\exp\left[\frac{-q(x_1,x_2)}{2\tau^2}\right]h_\Upsilon(\tau)d\tau. \qquad (1.11)$$

where $q(x_1,x_2)$ is given by (1.8). The pdf in (1.11) can further be written as

$$f_9(x_1,x_2) = \frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2}\left(1+q(x_1,x_2)/\nu\right)^{-(\nu+2)/2}, \qquad (1.12)$$

where $q(x_1,x_2)$ is defined by (1.8) and is well known to be the pdf of the location-scale bivariate *t*-distribution (cf. Anderson, 2003, 123). Since the pdf in (1.12) is constant on the ellipse

$$\left(\frac{x_1-\theta_1}{\sigma_1}\right)^2+\left(\frac{x_2-\theta_2}{\sigma_2}\right)^2-\frac{2\rho(x_1-\theta_1)(x_2-\theta_2)}{\sigma_1\sigma_2}=c^2,$$

for any fixed c, the distribution is also called elliptical *t*-distribution. Note that even if $\rho = 0$ in the pdf in (1.12), the components X_1 and X_2 do not become independent unless $\nu\to\infty$.

The scale mixture in (1.10) can be represented by

$$(X \mid \Upsilon=\tau) \sim N_2(\theta,\tau^2\Sigma). \qquad (1.13)$$

It is interesting to note that the transformation

$$x_1 = \theta_1+\sigma_1 y_1,\; x_2=\theta_2+\sigma_2 y_2 \qquad (1.14)$$

in the location scale bivariate *t*-distribution with pdf in (1.12) yields the pdf of the correlated bivariate *t*-distribution as in (1.4). Also the following transformation

$$\begin{aligned} x_1 &= \theta_1+\sigma_1\sqrt{(1+\rho)/2}\; z_1+\sigma_1\sqrt{(1-\rho)/2}\; z_2, \\ x_2 &= \theta_2+\sigma_2\sqrt{(1+\rho)/2}\; z_1-\sigma_2\sqrt{(1-\rho)/2}\; z_2 \end{aligned} \qquad (1.15)$$

in the pdf in (1.12) yields that of the standard *t*-distribution given by (1.3) which is obvious by virtue of

$$\begin{aligned} &(1-\rho^2)q(x_1,x_2) \\ &=\left(\sqrt{(1+\rho)/2}\; z_1+\sqrt{(1-\rho)/2}\; z_2\right)^2 \\ &+\left(\sqrt{(1+\rho)/2}\; z_1-\sqrt{(1-\rho)/2}\; z_2\right)^2 \\ &-2\rho\left(\sqrt{(1+\rho)/2}\; z_1+\sqrt{(1-\rho)/2}\; z_2\right)\left(\sqrt{(1+\rho)/2}\; z_1-\sqrt{(1-\rho)/2}\; z_2\right) \\ &=(1-\rho^2)(z_1^2+z_2^2), \end{aligned}$$

and $J((x_1,x_2)\to(z_1,z_2)) = \mathrm{mod}\left(\frac{\partial x_1}{\partial z_1}\frac{\partial x_2}{\partial z_2}-\frac{\partial x_1}{\partial z_2}\frac{\partial x_2}{\partial z_1}\right)=\sigma_1\sigma_2\sqrt{1-\rho^2}$. Conversely, the transformation

$$z_1 = \frac{1}{\sqrt{2(1+\rho)}}\left(\frac{x_1-\theta_1}{\sigma_1}+\frac{x_2-\theta_2}{\sigma_2}\right),\ z_2 = \frac{1}{\sqrt{2(1-\rho)}}\left(\frac{x_1-\theta_1}{\sigma_1}-\frac{x_2-\theta_2}{\sigma_2}\right)$$

in (1.3) also yields the pdf in (1.12) since $z_z^2 + z_2^2 = q(x_1, x_2)$ and

$J((z_1,z_2)\to(x_1,x_2)) = \left(\sigma_1\sigma_2\sqrt{1-\rho^2}\right)^{-1}$.

2. Moments of the Bivariate T-Distribution

In the rest of the paper we concentrate mostly on the location scale bivariate *t*-distribution discussed above. It follows from (1.13) that the expected value and the covariance matrix of the distribution are given by

$E(X) = E[E(X \mid \mathrm{T})] = E(\theta) = \theta$ and

$$\begin{aligned} Cov(X) &= E[Cov(X \mid \mathrm{T})] + Cov[E(X \mid \mathrm{T})] \\ &= E\left(\mathrm{T}^2\Sigma\right) + Cov(\theta) \qquad (2.1) \\ &= \gamma_2\Sigma. \end{aligned}$$

The characteristic function of the bivariate *t*-distribution is given by

$$\phi_{X(t)} = E\left(e^{it'X}\right) = e^{it'\theta}\psi_\nu\left(\sqrt{\nu}\,|\Sigma^{1/2}t\,|\right), \qquad (2.2)$$

with

$$\psi_\nu\left(|t\,|\right) = \frac{|t\,|^{\nu/2}}{2^{\nu/2-1}\Gamma(\nu/2)}K_{\nu/2}(|t\,|), \qquad (2.3)$$

where $K_{\nu/2}(|t\,|)$ is the Macdonald function with order $\nu/2$ and argument $|t\,|$. The Macdonald function admits numerous integral and series representations (See Spainer and Oldham, 1987, Chapter 51). The characteristic function can be derived (Joarder and Alam, 1995) by

$$\begin{aligned} E\left(e^{it'X}\right) &= E\left[E\left(e^{it'X} \mid \mathrm{T}\right)\right] \\ &= E\left(e^{it'\theta}e^{-t'\mathrm{T}^2\Sigma t/2}\right) \\ &= e^{it'\theta}E[\exp(-\alpha\mathrm{T}^2)], \end{aligned}$$

where $\alpha = t'\Sigma t/2$. Since

$$E\left(e^{-\alpha\tau^2}\right) = \int_0^\infty e^{-\alpha\tau^2}h_{\mathrm{T}}(\tau)d\tau = \frac{(\alpha\nu)^{\nu/4}}{2^{\nu/4-1}\Gamma(\nu/2)}K_{\nu/2}\left(\sqrt{2\nu\alpha}\right),$$

where $K_\alpha(w) = K_{-\alpha}(w) = \frac{w^{-\alpha}}{2^{1-\alpha}}\int_0^\infty u^{\alpha-1}\exp\left(-u-\frac{w^2}{4u}\right)du$ (Lebedev, 1965), the characteristic function is given by (2.2).

Setting $t_2 = 0$ in the characteristic function (2.2), we immediately have $E\left(e^{it_2X_2}\right) = e^{it_2\theta_2}\psi_\nu\left(\sigma_1\sqrt{\nu}\,|t_1|\right)$. Hence the marginal probability density function of X_1 is given by

$$f_{10}(x_1) = \frac{\Gamma((\nu+1)/2)}{\sigma\sqrt{\nu\pi}\ \Gamma(\nu/2)}\left(1+\frac{1}{\nu\sigma_1^2}(x_1-\theta_1)^2\right)^{-\nu/2-1},\nu>0,$$

which is denoted by $X_1 \sim t\left(\theta_1, \sigma_1^2; \nu\right)$.

The raw product moment of any two variables X_1 and X_2 is defined by $E(X_1^a X_2^b)$ while the centered product moment of X_1 and X_2 is defined by

$$E[(X_1-\theta_1)^a (X_2-\theta_2)^b], \tag{2.4}$$

where $\theta_1 = E(X_1), E(X_2) = \theta_2$.

For the bivariate normal distribution it is advisable to calculate the above centered product moments first instead of the raw product moments. Let $\mu(a,b) = E[(X_1-\theta_1)^a (X_2-\theta_2)^b]$ be the centered product moments between the bivariate normal variables X_1 and X_2 with pdf in (1.7). Clearly, $\mu(1,0) = \mu(0,1) = 0$. It is well known from the univariate normal distribution that

$$\mu(2a+1,0) = 0 \text{ and}$$
$$\begin{aligned}\mu(2a,0) &= \sigma_1^2(2a-1)\mu(2a-2,0)\\ &= 1(3)(5)\cdots(2a-1)\sigma_1^{2a}.\end{aligned}$$

Because of the transformation in (1.14), we have

$$E(Y_1^a Y_2^b) = E\left[\left(\frac{X_1-\theta_1}{\sigma_1}\right)^a \left(\frac{X_2-\theta_2}{\sigma_2}\right)^b\right] = \sigma_1^{-a}\sigma_2^{-b}\mu\left(a,b\right), \tag{2.5}$$

which can be easily calculated for the bivariate normal distribution.

3. Centered Product Moments of Bivariate T-Distribution

The scale mixture representation of the bivariate *t*-distribution with pdf in (1.10) can be represented by

$$(X \mid \mathrm{T} = \tau) \sim N_2(\theta, \tau^2\Sigma) \text{ where } \Sigma = \begin{pmatrix} \mu(2,0) & \mu(1,1) \\ \mu(1,1) & \mu(0,2) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

The pdf's between (1.9) to (1.12) are equivalent. The following theorem is due to Joarder (2006) who proved it by applying Kendal and Stuart (1969, 91) to the scale mixture representation.

Theorem 3.1 Let X_1 and X_2 have the bivariate *t*-distribution with pdf in (1.12). Then the product moments between X_1 and X_2 are given by

$$\mu(a,b;\nu) = (a+b-1)\rho\sigma_1\sigma_2\mu(a-1,b-1)\gamma_2 + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,b-2)\gamma_4,$$

$$\mu(2a,2b;\nu) = \sigma_1^{2a}\sigma_2^{2b}\frac{(2a)!(2b)!}{2^{a+b}}\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}\gamma_{2a+2b},$$

$$\mu(2a+1,2b+1;\nu) = \sigma_1^{2a+1}\sigma_2^{2b+1}\frac{(2a+1)!(2b+1)!}{2^{a+b}}\rho\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}\gamma_{2a+2b+2},$$

$$\mu(2a,2b+1;\nu) = \mu(2a+1,2b;\nu) = 0$$

where γ_a is the a-th moment of T.

Corollary 3.1 Let X_1 and X_2 have the bivariate *t*-distribution with pdf in (1.12). Then the product moment correlation between X_1 and X_2 is given by $\rho_{X_1 X_2} = \rho$.

Proof. By definition
$$\left[\left(E(X_1-\theta_1)^2\right)\left(E(X_2-\theta_2)^2\right)\right]^{1/2}\rho_{X_1,X_2} = E(X_1-\theta_1)(X_2-\theta_2).$$
The corollary then follows by virtue of

$$E(X_1-\theta_1)^2 = \mu(2,0;\nu) = \gamma_2\sigma_1^2,$$
$$E(X_2-\theta_2)^2 = \mu(0,2;\nu) = \gamma_2\sigma_2^2,$$
$$E(X_1-\theta_1)(X_2-\theta_2) = \mu(1,1;\nu) = \gamma_2\rho\sigma_1\sigma_2.$$

We conclude this section by providing centered product moments of the bivariate correlated *t*-distribution and the standard *t*-distribution.

Corollary 3.2 The product moments of bivariate *t*-distribution with pdf in (1.4) are given by

$$\mu(a,b;\nu) = (a+b-1)\rho\mu(a-1,b-1)\gamma_2 + (a-1)(b-1)(1-\rho^2)\mu(a-2,b-2)\gamma_4,$$
$$\mu(2a,2b;\nu) = \frac{(2a)!(2b)!}{2^{a+b}}\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}\gamma_{2a+2b},$$
$$\mu(2a+1,2b+1;\nu) = \frac{(2a+1)!(2b+1)!}{2^{a+b}}\rho\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}\gamma_{2a+2b+2},$$
$$\mu(2a,2b+1;\nu) = \mu(2a+1,2b;\nu) = 0,$$
where γ_a is the a-th moment of T.

Corollary 3.3 The product moments of bivariate *t*-distribution with pdf in (1.3) are given by

$$\mu(a,b;\nu) = (a-1)(b-1)\mu(a-2,b-2)\gamma_4,$$
$$\mu(2a,2b;\nu) = \sigma_1^{2a}\sigma_2^{2b}\frac{(2a)!(2b)!}{2^{a+b}a!b!}\gamma_{2a+2b},$$
$$\mu(2a+1,2b+1;\nu) = 0,\ \mu(2a,2b+1;\nu) = \mu(2a+1,2b;\nu) = 0,$$
where γ_a is the a-th moment of T.

4. Standardized Moments of the Bivariate T-Distribution

Let the quantity $(X-\theta)'\Sigma^{-1}(X-\theta) = \|\Sigma^{-1/2}(X-\theta)\|^2 = Z'Z = R^2$ where R is the norm of the bivariate *t*-distribution. Since the covariance matrix for the bivariate *t*-distribution is given by
$$\Lambda = \begin{pmatrix}\mu(2,0;\nu) & \mu(1,1;\nu)\\ \mu(1,1;\nu) & \mu(0,2;\nu)\end{pmatrix} = \begin{pmatrix}\mu(2,0) & \mu(1,1)\\ \mu(1,1) & \mu(0,2)\end{pmatrix}\gamma_2 = \gamma_2\Sigma,$$
the quantity $(X-\theta)'\Sigma^{-1}(X-\theta) = \|\Sigma^{-1/2}(X-\theta)\|^2 = Z'Z = R^2$ is not the standardized distance. The standardized distance for the bivariate *t*-distribution is defined by $Q = (X-\theta)'\Lambda^{-1}(X-\theta) = (X-\theta)'(\gamma_2\Sigma)^{-1}(X-\theta) = \gamma_2^{-1}R^2$. The standardized moments are defined by $\beta_a = E(Q^a),\ a = 1,2,\ldots$ where β_2 and β_3 measure kurtosis and skewness

respectively (Kotz, Balakrishnan and Johnson, 2000, 77). The following theorem is obvious from Muirhead (1982, 37).

Theorem 4.1 Let $Q=(X-\theta)'\Lambda^{-1}(X-\theta)=(X-\theta)'(\gamma_2\Sigma)^{-1}(X-\theta)=\gamma_2^{-1}R^2$. Then

$(i)\ W=\gamma_2 Q/2 \sim F(2,\nu),\ \ \nu>2,$

$(ii)\ E(Q^a)=\left(\nu-2\right)^a\ \dfrac{\Gamma(a+1)\Gamma(\nu/2-a)}{\Gamma(\nu/2)},\ \ \nu>2a.$

Notice that as $\nu\to\infty$, the standardized moments, as expected, coincide with that of the bivariate normal distribution.

Corollary 4.1 Let $R^2=\gamma_2 Q$. Then

$$E(R^a)=\gamma_2^{a/2}E(Q^{a/2})=\left(\nu-2\right)^{a/2}\ \frac{\Gamma(a/2+1)\Gamma((\nu-a)/2)}{\Gamma(\nu/2)},\ \ \nu>a\,.$$

Corollary 4.2 The first three standardized moments of the bivariate *t*-distribution with pdf in (1.12) are given by

$$\beta_1=2,$$

$$\beta_2=8\frac{\nu-2}{\nu-4},\quad \nu>4,$$

$$\beta_3=48\frac{(\nu-2)^2}{(\nu-4)(\nu-6)},\quad \nu>6.$$

It can be checked that

$$\beta_2=8\frac{\gamma_4}{\gamma_2^2}\ \text{ and }\ \beta_3=8\frac{\gamma_6}{\gamma_2^3}$$

where the quantity $\dfrac{\gamma_4}{\gamma_2^2}$ is well known to be the coefficient of kurtosis of the mixing distribution of T with pdf in (1.1).

5. Shanon Entropy

The Shanon Entropy for any bivarite density function $f(x_1,x_2)$ is defined by

$H(f)=-E(\ln f(X_1,X_2)]$. It follows from (1.7) that

$$\ln f_8(x_1,x_2)=\ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)-\frac{q(x_1,x_2)}{2(1-\rho^2)}$$

so that

$$E(\ln f_8(X_1, X_2)) = -\ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) - \frac{E[q(X_1, X_2)]}{2(1-\rho^2)}$$
$$= -\ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) - 1.$$

Hence, the Shanon Entropy for the bivariate normal distribution with pdf in (1.7) is given by

$$H(f_8) = \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) + 1. \qquad (5.1)$$

Theorem 5.1 Let the bivariate *t*-distribution have the pdf in (1.12). Then the Shanon entropy for the bivariate *t*-distribution is given by

$$H(f_9) = \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) + E(\ln \mathrm{T}^2) + 1,$$

where $\nu\mathrm{T}^{-2} \sim \chi^2_\nu$.

Proof. By virtue of (1.13), it follows from (5.1) that

$$H(f_8) = E\left(\ln\left[2\pi(\sigma_1\mathrm{T})(\sigma_2\mathrm{T})\sqrt{1-\rho^2}\right]\right) + 1$$
$$= \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) + E(\ln \mathrm{T}^2) + 1.$$

Note that $E(\ln \mathrm{T}^2) = \ln\nu - E(\ln W),\ W \sim \chi^2_\nu$. For some other equivalent expressions see Nadarajah and Kotz (2005).

6. Distribution of the Correlation Coefficient for the Bivariate T-Model

The mean vector is $\bar{X}' = (\bar{X}_1, \bar{X}_2)$ so that the sums of squares and cross product matrix is given by $\sum_{j=1}^{N}(X_j - \bar{X})(X_j - \bar{X})' = A$. The symmetric bivariate matrix A can be written as $A = (a_{ik}), i = 1,2; k = 1,2$ where $a_{ii} = mS_i^2 = \sum_{j=1}^{N}(X_{ij} - \bar{X}_i)^2, \quad m = N-1, (i = 1,2)$ and $a_{12} = \sum_{j=1}^{N}(X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) = mRS_1S_2$. Fisher (1915) derived the distribution of A for $p = 2$ in order to study the distribution of the correlation coefficient from a normal sample.

Let $X' = (X_1, X_2)$ be bivariate t-random vector with probability density function in (1.9). Now consider a sample $X_1, X_2, \cdots X_N\ (N > 2)$ having the joint probability density function

$$f_5(x_1, x_2, \cdots, x_N) = \frac{\Gamma(\nu/2 + N)}{(\nu\pi)^N\,\Gamma(\nu/2} \ |\Sigma|^{-N/2}\left(1 + \sum_{j=1}^{N}(x_j - \theta)'(\nu\Sigma)^{-1}(x_j - \theta)\right)^{-\nu/2-1}, \qquad (6.1)$$

which is the bivariate *t*-model for the sample. Note that the observations in the sample are uncorrelated and not independent unless $\nu \to \infty$. The random symmetric positive definite matrix A is said to have a Wishart distribution based on bivariate t-population with $m = N - 1 > 2$ and $\Sigma(2\times 2) > 0$, written as $A \sim W(m, \Sigma; \nu)$ if its probability density function is given by

$$f_6(A) = C_\nu(m,2)\,|\Sigma|^{-m/2}\quad |A|^{(m-3)/2}\left(1+tr(\nu\Sigma)^{-1}A\right)^{-\nu/2-m}, \quad A>0, m>2 \qquad (6.2)$$

where $C_\nu(m,2) = \dfrac{\nu^{-m}\ \Gamma(\nu/2+m)}{\sqrt{\pi}\ \Gamma(\nu/2)\Gamma(m/2)\Gamma\left((m-1)/2\right)}$ (See Sutradhar and Ali, 1989, 160).

By the use of the duplication formula for gamma function given by

$\Gamma(z) = \pi^{-1/2}\ 2^{z-1}\ \Gamma((z+1)/2)\ \Gamma(z/2)$
(Anderson, 2003, 125) with $z = m-1$ we have
$C_\nu(m,2) = \dfrac{2^{m-2}\ \nu^{-m}\ \Gamma(m+\nu/2)}{\pi\ \Gamma(m-1)\Gamma(\nu/2)}$.

Lemma 6.1 Let $V = \sqrt{U_1U_2}$ be the geometric mean of two independent chisquare random variables $U_i \sim \chi^2_m\ (i=1,2)$**.** Then the moment generating function of V at ρ is given by
$M_V(\rho) = \sum_{k=0}^{\infty}\dfrac{(2\rho)^k}{k!}\dfrac{\Gamma^2((m+k)/2)}{\Gamma^2(m/2)}, \quad -1<\rho<1.$

Most of the results of this section are due to Joarder (2005).

Lemma 6.2 Let $I(\rho,m) = \int_0^\pi (\sin\theta)^{m-1}(1-\rho\sin\theta)^{-m}\,d\theta, \quad -1<\rho<1$**.** Then
$(i)\ I(\rho,m) = \dfrac{2^{m-1}}{\Gamma(m)}\ \sum_{k=0}^{\infty}\dfrac{(2\rho)^k}{k!}\ \Gamma^2\left((m+k)/2\right),$
$(ii)\ I(\rho,m) = \dfrac{2^{m-1}\ \Gamma^2(m/2)}{\Gamma(m)}\ M_V(\rho).$
where $M_V(\rho)$ is defined by Lemma 6.1.

Theorem 6.1 For $-1<\rho<1$ and $\nu>0$, let
$J(\rho,m,\nu) = \int_0^\infty\int_0^\infty (u_1u_2)^{m/2-1}\left(1+u_1+u_2-2\rho\sqrt{u_1u_2}\right)^{-\nu-m} du_1du_2$. Then
$(i)\ J(\rho,m,\nu) = \dfrac{\Gamma(\nu)}{\Gamma(m+\nu)}\sum_{k=0}^{\infty}\dfrac{(2\rho)^k}{k!}\Gamma^2\left(\dfrac{m+k}{2}\right),$
$(ii)\ J(\rho,m,\nu) = \dfrac{\Gamma(\nu)\Gamma^2(m/2)}{\Gamma(m+\nu)}\ M_V(\rho).$

Theorem 6.2 The probability density function of the correlation coefficient R based on the joint pdf in (6.1) is given by

$$h(r)=\frac{2^{m-2}\ \Gamma^2(m/2)\left(1-\rho^2\right)^{m/2}}{\pi\ \Gamma(m-1)}\quad\left(1-r^2\right)^{(m-3)/2}M_V\ (\rho r)$$
$$=\frac{2^{m-2}\left(1-\rho^2\right)^{m/2}}{\pi\ \Gamma(m-1)}\ \left(1-r^2\right)^{(m-3)/2}\sum_{k=0}^{\infty}\frac{(2\rho r)^k}{k\ !}\ \Gamma^2\left(\frac{m+k}{2}\right),\ -1<r<1$$

where $m>2,-1<\rho<1$ and $M_V(\rho r)$ is defined in Lemma 6.1 (cf. Johnson, Kotz and Balakrishanan, 1995, 548).

Proof. The pdf (6.2) of the elements of A based on bivariate *t*-model (6.1) can be written as

$$f_6(a_{11},a_{22},a_{12})=C_\nu(m,2)\ \left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}\left(a_{11}a_{22}-a_{12}^2\right)^{(m-3)/2}$$
$$\times\left(1+\frac{1}{\nu(1-\rho^2)}\left(\frac{a_{11}}{\sigma_1^2}+\frac{a_{22}}{\sigma_2^2}-\frac{2\rho a_{12}}{\sigma_1\sigma_2}\right)\right)^{-\nu/2-m}$$

where $a_{11}>0,a_{22}>0,-\infty<a_{12}<\infty,-1<\rho<1$, $m>2,\sigma_1>0,\sigma_2>0$. Under the transformation $a_{11}=ms_1^2,a_{22}=ms_2^2,a_{12}=mrs_1s_2$ with Jacobian $J(a_{11},a_{22},a_{12}\to r,s_1^2,s_2^2)=m^3s_1s_2$, followed by the transformation $ms_1^2=\sigma_1^2u_1$, $ms_2^2=\sigma_2^2u_2$ with Jacobian $J(s_1^2,s_2^2\to u_1,u_2)=m^{-2}\left(\sigma_1\sigma_2\right)^2$ and then integrating out u_1 and u_2 we have the probability density function of R as follows:

$$h(r)=C_\nu(m,2)\ \left(1-\rho^2\right)^{-m/2}\left(1-r^2\right)^{(m-3)/2}$$
$$\times\int_0^\infty\int_0^\infty(u_1u_2)^{m/2-1}\left[1+\frac{1}{\nu(1-\rho^2)}\left(u_1+u_2-2\rho r\sqrt{u_1u_2}\right)\right]^{-\nu/2-m}du_1du_2.$$

Then the transformation $u_1=y_1\left(1-\rho^2\right)$, $u_2=y_2\left(1-\rho^2\right)$ with Jacobian $J(u_1,u_2\to y_1,y_2)=\left(1-\rho^2\right)^2$ yields

$$h(r)=\ \nu^m\ C(m,\nu)\ \left(1-\rho^2\right)^{m/2}\left(1-r^2\right)^{(m-3)/2}J(\rho r,m,\nu/2)$$

where $J(\rho,m,\nu)$ is defined in Theorem 6.1 and the Theorem 6.2 follows.

7. Simple Linear Regression Model

The simple linear regression model is given by $y=\beta_0+\beta_1x+\varepsilon$. We assume

$$\left(\begin{pmatrix}x\\y\end{pmatrix}|\,\mathrm{T}=\tau\right)\sim N_2\left(\begin{pmatrix}\mu_1\\\mu_2\end{pmatrix},\tau^2\begin{pmatrix}\sigma_{xx}&\sigma_{xy}\\\sigma_{yx}&\sigma_{yy}\end{pmatrix}\right)$$

where $\sigma_{xx}=\sigma_x^2$, $\sigma_{yy}=\sigma_y^2=\sigma^2$, $\sigma_{12}=\rho\sigma_x\sigma_y$ where T has an invereted chisquare distribution with pdf in (1.1). Conditional on $\mathrm{T}=\tau$, $(\varepsilon\,|\,x)\sim NIID\left(0,\tau^2\sigma^2\right)$. Then for a given $\mathrm{T}=\tau$, we have

$$E(Y\ |\,x)=\mu_2+\rho\frac{\tau\sigma}{\tau\sigma_x}(x-\mu_1)=\left(\mu_2-\rho\frac{\sigma}{\sigma_x}\mu_1\right)+\rho\frac{\sigma}{\sigma_x}x$$

(cf. Hogg, McKean and Craig, 2005, 176). Since the above does not depend on $\mathrm{T}=\tau$, it is also true unconditionally. We write

$$\mu_{Y(x)} = E(Y \mid x) = \beta_0 + \beta_1 x$$

where $\beta_0 = \mu_2 - \frac{\rho\sigma}{\sigma_x}\mu_1$, $\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$ and $\beta_1 = \frac{\rho\sigma_y}{\sigma_x} = \frac{\rho\sigma_x\sigma_y}{\sigma_x^2} = \frac{\sigma_{xy}}{\sigma_{xx}} = \sigma_{xx}^{-1}\sigma_{xy}$.

Conditional on $\mathrm{T}=\tau$, $(Y \mid x) \sim N\left(E(Y \mid x), \left(1-\rho^2\right)\tau^2\sigma_2^2\right)$. Based on a sample of size N, the least square estimator of β_1 is given by $\hat{\beta}_1 = \frac{S_{xY}}{s_{xx}}$ where $S_{xY} = \sum(x-\bar{x})(Y-\bar{Y})$.

Conditional on $\mathrm{T}=\tau$, under $H_0: \beta_1 = 0$, $(Y \mid x) \sim N\left(\mu_2, \sigma^2\right)$ and hence

$\frac{TSS}{\sigma^2} = \sum\left(\frac{y-\bar{y}}{\sigma}\right)^2 \sim \chi^2(m)$, a chisquare distribution with m degrees of freedom.

$$\text{Given } \mathrm{T}=\tau,\ \hat{\beta}_1 = \frac{S_{xY}}{s_{xx}} = \sum\left(\frac{x-\bar{x}}{s_{xx}}\right)Y \sim N\left(\beta_1, \tau^2\sigma^2/s_{xx}\right) \qquad (7.1)$$

and under $H_0: \beta_1 = 0$, $\hat{\beta}_1 \sim N\left(0,\ \tau^2\sigma^2/s_{xx}\right)$. We also have

$$\frac{\hat{\beta}_1 - 0}{\sqrt{\tau^2\sigma^2/s_{xx}}} \sim N\left(0,\ 1\right), \quad \frac{\hat{\beta}_1^2}{\tau^2\sigma^2/s_{xx}} \sim \chi^2(1), \quad \frac{\hat{\beta}_1^2 s_{xx}}{\tau^2\sigma^2} = \frac{SSR}{\tau^2\sigma^2} \sim \chi^2(1)$$

since $SSR = \sum(\hat{y}-\bar{y})^2 = \hat{\beta}_1^2 s_{xx}$ and $SSE = \sum(y-\hat{y})^2$. Then given $\mathrm{T}=\tau$, we have

$$\frac{TSS}{\tau^2\sigma^2} = \frac{SSR}{\tau^2\sigma^2} + \frac{SSE}{\tau^2\sigma^2},$$

i.e. $\chi^2(m) = \chi^2(1) + SSE/(\tau^2\sigma^2)$. By additive Law of chisquares and by independence of SSR and SSE, it follows by Cochran's Theorem that

$\frac{SSE}{\tau^2\sigma^2} \sim \chi^2(m-1)$ and hence

$$\frac{\dfrac{\hat{\beta}_1 - 0}{\sqrt{\tau^2\sigma^2/s_{xx}}}}{\sqrt{\dfrac{SSE}{(m-1)\tau^2\sigma^2}}} = \frac{\hat{\beta}_1 - 0}{\sqrt{MSE/s_{xx}}}$$

which has a *t*-distribution with $(m-1)$ degrees f freedom. Alternatively, conditional on $\mathrm{T}=\tau$, it follows from (7.1) that

$$E(SSR) = s_{xx}E\left(\hat{\beta}_1^2\right) = s_{xx}\left[\left(E(\hat{\beta}_1)\right)^2 + V(\hat{\beta}_1)\right] = s_{xx}\left[\beta_1^2 + \frac{\tau^2\sigma^2}{s_{xx}}\right],$$

i.e. $E(SSR) = \tau^2\sigma^2 + \beta_1^2 s_{xx} = E(MSR)$,

$E(MSE) = \tau^2\sigma^2$.

We want to test $H_0 : \beta_1 = 0$ against the alternative that $H_0 : \beta_1 \neq 0$. If the null hypothesis is true, MSR and MSE are expected to be the same. If the alternative hypothesis is true, MSR is expected to be larger than MSE. That is the ratio MSR/MSE is expected to be large. Thus we argue that larger values of

$$F = \frac{MSR}{MSE}$$

supports the alternative hypothesis. The quantity F has a F-distribution with 1 and m degrees of freedom. Interested readers may go through Sutradhar (1988).

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