Moments of the Bivariate T-Distribution

Anwar H. Joarder

Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia
Email: anwarj@ kfupm.edu.sa

Abstract The probability density function of the bivariate *t*-distribution can be represented by a scale mixture representation of the bivariate normal distribution with an 'inverted' chisquare distribution. The product moments of the bivariate *t*-distribution are derived by exploiting the scale mixture. Some standardized moments of the bivariate *t*-distribution are also derived.

AMS Mathematics Subject Classifications: 60E10, 60-02

***Key Words**:* Bivariate normal distribution, bivariate *t*-distribution, product moments, standardized moments

1. Introduction

The bivariate *t*-distribution arises as a derived sampling distribution from the bivariate normal distribution and the chisquare distribution (Anderson, 2003, 289). In this paper we emphasize a scale mixture representation to calculate some product moments and standardized moments of the bivariate *t*-distribution.

Because of the increase in the use of the multivariate *t*-distribution in business, especially, in stock returns, the paper will stimulate research in business, econometrics and statistics. Interested readers may go through Lange, Little and Taylor (1989), Billah and Saleh (2000), Kibria and Saleh (2000) and Kotz and Nadarajah (2004).

(i) The Univariate T-Distribution

Theorem 1.1 Let $Z \sim N(0,\tau^2)$ and $\nu \mathrm{T}^{-2} \underset{=}{d}\ W \sim \chi^2_\nu$ where the symbol $\underset{=}{d}$ means that both sides of it have the same distribution. Then

(a) $(\nu/W)^{1/2} Z \sim t_\nu$,

$$(b)\ h_{\mathrm{T}}(\tau) = \frac{2(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \tau^{-(\nu+1)} e^{-\nu/(2\tau^2)},\ \ 0<\tau\ , \qquad (1.1)$$

(c) the pdf of the univariate *t*-dsitribution with ν degrees of freedom has the following representation:

$$f_1(z) = \int_0^\infty \frac{1}{\tau\sqrt{2\pi}}\ \exp\left(-\frac{z^2}{2\tau^2}\right) h_{\mathrm{T}}(\tau) d\tau. \qquad (1.2)$$

Proof. The proof of the first two parts are well known while the third part follows by transformation $\nu/\tau^2 = w$ with Jacobian $J(\tau \to w) = w^{-3/2}\sqrt{\nu}/2.$

The representation in (1.2) can be expressed by $(Z \mid \mathrm{T}=\tau) \sim N(0,\tau^2)$. Thus
$E(Z)=E[E(Z \mid \mathrm{T})]=E(0)=0$ and

$$
\begin{aligned}
Cov(Z) &= E[Var(Z \mid \mathrm{T})]+Var[E(Z \mid \mathrm{T})] \\
&= E(\mathrm{T}^2)+Var(0) \\
&= \gamma_2
\end{aligned}
$$

where $\gamma_2=\nu/(\nu-2),\ \nu>2$. In what follows, for any nonnegative integer a, we assume $k_{\{a\}}=k(k+1)\cdots(k+a-1),\ k_{\{0\}}=1;\ k^{\{a\}}=k(k-1)\cdots(k-a+1),\ k^{\{0\}}=1.$

Corollary 1.1 The a-th moment of T is given by

$$\gamma_a=E\left(\mathrm{T}^a\right)=\frac{(\nu/2)^{a/2}\Gamma(\nu/2-a/2)}{\Gamma(\nu/2)},\quad \nu>a.$$

Clearly $\gamma_2=\nu/(\nu-2),\ \nu>2$ and

$$\gamma_{2a}=E\left(\mathrm{T}^{2a}\right)=\frac{(\nu/2)^a}{(\nu/2-1)^{\{a\}}},\quad \nu>2a,$$

$$\gamma_{-2a}=E\left(\mathrm{T}^{-2a}\right)=(\nu/2)^{-a}(\nu/2)_{\{a\}},\ \ \nu>2.$$

(ii) A Location Scale Bivariate T-Distribution

Let $X'=(X_1,X_2)$ be the bivariate t-random vector with probability density function

$$f_2(x)=(2\pi)^{-1}|\Sigma|^{-1/2}\left(1+(x-\theta)'(\nu\Sigma)^{-1}(x-\theta)\right)^{-\nu/2-1} \tag{1.3}$$

where $\theta'=(\theta_1,\theta_2)$ is an unknown vector of location parameters and Σ is the 2×2 unknown positive definite matrix of scale parameters while the scalar ν is assumed to be a known positive constant (Anderson, 2003, 289). The distribution will be denoted by $T_2(\theta,\Sigma;\nu)$ in contrast to the bivariate normal distribution denoted by $N_2(\theta,\Sigma)$. The pdf of the bivariate *t*-distribution in (1.3) can be written as a mixture of the bivariate normal distribution and an inverted chisquare distribution as follows:

$$f_3(x)=(2\pi)^{-1}\int_0^\infty |\tau^2\Sigma|^{-1/2}\exp\left(\frac{-1}{2}(x-\theta)'(\nu\tau^2\Sigma)^{-1}(x-\theta)\right)h_{\mathrm{T}}(\tau)d\tau \tag{1.4}$$

which can be represented by
i.e. $(X \mid \mathrm{T}=\tau)\sim N_2(\theta,\tau^2\Sigma).$ (1.5)

The pdf in (1.4) can be written as

$$f_4(x_1,x_2)$$

$$=\int_0^\infty \frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2\tau^2}\exp\left[\frac{-1}{2\tau^2(1-\rho^2)}\left\{\left(\frac{x_1-\theta_1}{\sigma_1}\right)^2+\left(\frac{x_2-\theta_2}{\sigma_2}\right)^2-\frac{2\rho(x_1-\theta_1)(x_2-\theta_2)}{\sigma_1\sigma_2}\right\}\right]h(\tau)d\tau. \tag{1.6}$$

which can further be written as

$$f_5(x_1,x_2)=\frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2}\left(1+\frac{q(x_1,x_2)}{\nu}\right)^{-(\nu+2)/2}, \tag{1.7}$$

where

$$(1-\rho^2)q(x_1,x_2)=\left(\frac{x_1-\theta_1}{\sigma_1}\right)^2+\left(\frac{x_2-\theta_2}{\sigma_2}\right)^2-\frac{2\rho(x_1-\theta_1)(x_2-\theta_2)}{\sigma_1\sigma_2}. \tag{1.8}$$

The above is the explicit form of the pdf of the bivariate *t*-distribution (cf. Anderson, 2003, 123). As $\nu\to\infty$, the pdf of the bivariate *t*-distribution converges to the bivariate normal distribution with pdf

$$f_6(x_1,x_2)=\frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2}\ \exp\left(\frac{-q(x_1,x_2)}{2}\right), \tag{1.9}$$

where $q(x_1,x_2)$ is defined by (1.8). The following is a special case of the bivariate *t*-distribution with pdf (1.7):

$$f_7(y_1,y_2)=\frac{(1-\rho^2)^{-1/2}}{2\pi}\ \left(1+\frac{y_1^2+y_2^2-2\rho y_1 y_2}{(1-\rho^2)\nu}\right)^{-(\nu+2)/2}. \tag{1.10}$$

2. The Moments of the Bivariate T-Distribution

In the rest of the paper we mostly concentrate on the location scale representation of the bivariate *t*-distribution with equivalent probability density functions between (1.4) to (1.7). It follows from (1.5) that the expected value and the covariance matrix of the distribution are given by

$$E(X)=E[E(X\mid \mathrm{T})]=E(\theta)=\theta \text{ and}$$

$$\begin{aligned}Cov(X)&=E[Cov(X\mid \mathrm{T})]+Cov[E(X\mid \mathrm{T})]\\&=E\left(\mathrm{T}^2\Sigma\right)+Cov(\theta)\\&=\gamma_2\Sigma,\end{aligned}$$

where

$$\Sigma=\begin{pmatrix}\mu(2,0) & \mu(1,1)\\ \mu(1,1) & \mu(0,2)\end{pmatrix}=\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_2\sigma_2 & \sigma_2^2\end{pmatrix}.$$

The raw product moments of X_1 and X_2 of order a and b are defined $\mu'(a,b;m)=E(X_1^aX_2^b)$ while the centered product moments of X_1 and X_2 are defined by

$$\mu(a,b;m)=E[(X_1-\theta_1)^a(X_2-\theta_2)^b], \tag{2.1}$$

where $\theta_1=E(X_1),\ \theta_2=E(X_2)$.

For the bivariate normal distribution it is advisable to calculate the centered product moments instead of the raw product moments $\mu'(a,b)=E(X_1^aX_2^b)$. Clearly, $\mu(1,0)=\mu(0,1)=0$. It is well known from the univariate normal distribution that

$$\mu(2a+1,0)=0 \text{ and}$$

$$\begin{aligned}\mu(2a,0)&=\sigma_1^2(2a-1)\mu(2a-2,0)\\&=1(3)(5)\cdots(2a-1)\sigma_1^{2a}.\end{aligned}$$

3. Centered Product Moments of the Bivariate T-Distribution

The following theorem is due to Kendal and Stuart (1969, 91).

Theorem 3.1 The centered product moments $\mu(a,b) = E[(X_1-\theta_1)^a (X_2-\theta_2)^b]$ of the bivariate normal distribution with pdf (1.9) are given by

$$\mu(a,b) = \sigma_1^a \sigma_2^b \lambda(a,b) \text{ where}$$
$$\lambda(a,b) = (a+b-1)\rho\lambda(a-1,b-1) + (a-1)(b-1)(1-\rho^2)\lambda(a-2,b-2),$$
$$\lambda(2a,2b) = \frac{(2a)!(2b)!}{2^{a+b}} \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!},$$
$$\lambda(2a+1,2b+1) = \frac{(2a+1)!(2b+1)!}{2^{a+b}} \rho \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!},$$
$$\lambda(2a,2b+1) = \lambda(2a+1,2b) = 0.$$

The above can be rewritten as

$$\mu(a,b) = (a+b-1)\rho\sigma_1\sigma_2\mu(a-1,b-1) + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,b-2),$$
$$\mu(2a,2b) = \sigma_1^{2a}\sigma_2^{2b} \frac{(2a)!(2b)!}{2^{a+b}} \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!},$$
$$\mu(2a+1,2b+1) = \sigma_1^{2a+1}\sigma_2^{2b+1} \frac{(2a+1)!(2b+1)!}{2^{a+b}} \rho \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!},$$
$$\mu(2a,2b+1) = \mu(2a+1,2b) = 0.$$

Theorem 3.2 The centered product moments of the bivariate *t*-distribution with pdf in (1.7) are given by

$$\mu(a,b;\nu) = (a+b-1)\rho\sigma_1\sigma_2\mu(a-1,b-1)\gamma_2 + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,b-2)\gamma_4,$$
$$\mu(2a,2b;\nu) = \sigma_1^{2a}\sigma_2^{2b} \frac{(2a)!(2b)!}{2^{a+b}} \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!} \gamma_{2a+2b},$$
$$\mu(2a+1,2b+1;\nu) = \sigma_1^{2a+1}\sigma_2^{2b+1} \frac{(2a+1)!(2b+1)!}{2^{a+b}} \rho \sum_{j=0}^{\min(a,b)} \frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!} \gamma_{2a+2b+2},$$
$$\mu(2a,2b+1;\nu) = \mu(2a+1,2b;\nu) = 0$$

where γ_a is the a-th moment of T.

Proof. By the scale mixture representation in (1.5), we have

$$\begin{aligned}\mu(a,b;m) &= E[E[(X_1-\theta_1)^a (X_2-\theta_2)^b \mid \mathrm{T}]] \\ &= E[(a+b-1)\rho(\sigma_1\mathrm{T})(\sigma_2\mathrm{T})\mu(a-1,b-1) \\ &\quad + (a-1)(b-1)(1-\rho^2)\upsilon^2(\sigma_1^2\mathrm{T}^2)(\sigma_2^2\mathrm{T}^2)\mu(a-2,b-2)] \\ &= (a+b-1)\rho\sigma_1\sigma_2\mu(a-1,b-1)E(\mathrm{T}^2) \\ &\quad + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,b-2)E(\mathrm{T}^4).\end{aligned}$$

Similarly we have

$$\mu(2a,2b;\nu)=\sigma_1^{2a}\sigma_2^{2b}\,\frac{(2a)!(2b)!}{2^{a+b}}\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}E\,(\mathrm{T}^{2a+2b})$$

$$\mu(2a+1,2b+1;\nu)=\sigma_1^{2a+1}\sigma_2^{2b+1}\,\frac{(2a+1)!(2b+1)!}{2^{a+b}}\rho\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}E\,(\mathrm{T}^{2a+2b+2})$$

$$\mu(2a,2b+1;\nu)=\mu(2a,2b+1)E\,(\mathrm{T}^{2a+2b+1})=0,$$

$$\mu(2a+1,2b;\nu)=\mu(2a+1,2b)E\,(\mathrm{T}^{2a+2b+1})=0.$$

Some special cases are given below:

$$\mu(a,1;\nu)=(a-1)\rho\sigma_1\sigma_2\mu(a-1,0)\gamma_2$$

$$\mu(1,b;\nu)=(b-1)\rho\sigma_1\sigma_2\mu(0,b-1)\gamma_2$$

$$\mu(a,2;\nu)=(a+1)\rho\sigma_1\sigma_2\mu(a-1,1)\gamma_2+(a-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,0)\gamma_4$$

$$\mu(2,b;\nu)=(b+1)\rho\sigma_1\sigma_2\mu(1,b-1)\gamma_2+(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(0,b-2)\gamma_4$$

$$\mu(a,3;\nu)=(a+2)\rho\sigma_1\sigma_2\mu(a-1,2)\gamma_2+2(a-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,1)\gamma_4$$

$$\mu(2a,0;\nu)=\frac{(2a)!}{2^a}\sigma_1^{2a}\ \gamma_{2a}$$

$$\mu(0,2b;\nu)=\frac{(2b)!}{2^b}\sigma_1^{2b}\ \gamma_{2b}$$

$$\mu(2a,2;\nu)=\sigma_1^{2a}\sigma_2^2\,\frac{(2a)!(2)}{2^{a+1}a!}(1+2\rho)\gamma_{2a+2b}$$

$$\mu(2a+1,1;\nu)=\sigma_1^{2a+1}\sigma_2\,\frac{(2a+1)!}{2^{a+b}}\rho\ \gamma_{2a+2}$$

$$\mu(1,2b+1;\nu)=\sigma_1\sigma_2^{2b+1}\,\frac{(2b+1)!}{2^b}\rho\ \gamma_{2b+2}$$

$$\mu(2a+1,3;\nu)=\sigma_1^{2a+1}\sigma_2^3\,\frac{(2a+1)!}{2^a a!(a-1)!}\rho[3(a-1)!+2\rho^2 a!]\ \gamma_{2a+4},\ a\geq 1.$$

$$\mu(2a+1,5;\nu)=\sigma_1^{2a+1}\sigma_2^5\,\frac{(2a+1)!5!}{2^{a+2}}\rho\left[\frac{1}{a!(2)}+\frac{4\rho^2}{(a-1)!(6)}+\frac{16\rho^4}{120(a-2)!}\right]\gamma_{2a+6},\ a\geq 2.$$

Some special moments needed for calculating standardized moments (See Section 4) are given below:

$$\mu(1,0;\nu)=0,$$

$$\mu(1,1;\nu)=\sigma_1\sigma_2\rho\gamma_2,$$

$$\mu(1,2;\nu)=0,$$

$$\mu(1,3;\nu)=3\sigma_1\sigma_2^3\rho\gamma_4,$$

$$\mu(1,4;\nu)=0,$$

$$\mu(1,5;\nu)=15\sigma_1\sigma_2^5\rho\gamma_6,$$

$$\mu(2,0;\nu)=\sigma_1^2\gamma_2,$$

$$\mu(2,1;\nu)=0,$$

$$\mu(2,2;\nu)=\sigma_1^2\sigma_2^2(1+2\rho^2)\gamma_4,$$
$$\mu(2,3;\nu)=0,$$
$$\mu(2,4;\nu)=3\sigma_1^2\sigma_2^4(1+4\rho^2)\gamma_6,$$

$$\mu(3,0;\nu)=\mu(0,3;\nu)=0,$$
$$\mu(3,1;\nu)=3\sigma_1^3\sigma_2\rho\gamma_4,$$
$$\mu(3,2;\nu)=0,$$
$$\mu(3,3;\nu)=3\sigma_1^3\sigma_2^3\rho(3+2\rho^2)\gamma_6,$$

$$\mu(4,0;\nu)=3\sigma_1^4\ \gamma_4,$$
$$\mu(4,1;\nu)=0,$$
$$\mu(4,2;\nu)=3\sigma_1^4\sigma_2^2(1+4\rho^2)\gamma_6,$$

$$\mu(5,0;\nu)=0,$$
$$\mu(5,1;\nu)=15\sigma_1^5\sigma_2\rho\gamma_6,$$

$$\mu(6,0;\nu)=90\sigma_1^6\gamma_6.$$

By putting $\sigma_1=\sigma_2=1$, we will have the moments $E(Y_1^aY_2^b)$ for the correlated bivarite *t*-distribution with pdf in (1.10).

Corollary 3.1 The product moment correlation between X_1 and X_2 is given by $\rho_{X_1X_2}=\rho$.

Proof. By definition

$$\left[\left(E(X_1-\theta_1)^2\right)\left(E(X_2-\theta_2)^2\right)\right]^{1/2}\rho_{X_1,X_2}=E(X_1-\theta_1)(X_2-\theta_2).$$

The corollary then follows by virtue of

$$E(X_1-\theta_1)^2=\mu(2,0;\nu)=\sigma_1^2\gamma_2,\ \ \nu>2,$$
$$E(X_2-\theta_2)^2=\mu(0,2;\nu)=\sigma_2^2\gamma_2,\ \ \nu>2,$$
$$E(X_1-\theta_1)(X_2-\theta_2)=\mu(1,1;\nu)=\sigma_1\sigma_2\rho\gamma_2,\ \ \nu>2.$$

Note that if $\rho=0$ in the pdf in (1.7), the components X_1 and X_2 do not become independent unless $\nu\to\infty$.

Corollary 3.2 The product moments of the standard bivariate *t*-distribution (i.e., $\theta_1=\theta_2=0,\sigma_1=\sigma_2=1,\rho=0)$ are given by

$$\mu(a,b;\nu)=(a-1)(b-1)\mu(a-2,b-2)\gamma_4,\ \ \nu>4,$$
$$\mu(2a,2b;\nu)=\sigma_1^{2a}\sigma_2^{2b}\frac{(2a)!(2b)!}{2^{a+b}a!b!}\gamma_{2a+2b},\ \ \nu>2a+2b,$$
$$\mu(2a+1,2b+1;\nu)=0,$$
$$\mu(2a,2b+1;\nu)=\mu(2a+1,2b;\nu)=0,$$

where γ_a is the a-th moment of T.

Corollary 3.3 The product moments of the correlated bivariate *t*-distribution $(\theta_1=\theta_2=0,\sigma_1=\sigma_2=1,-1<\rho<1)$ with pdf in (1.10) are given by

$$\mu(a,b;\nu)=(a+b-1)\rho\mu(a-1,b-1)\gamma_2+(a-1)(b-1)(1-\rho^2)\mu(a-2,b-2)\gamma_4,\ \ \nu>4$$
$$\mu(2a,2b;\nu)=\frac{(2a)!(2b)!}{2^{a+b}}\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}\gamma_{2a+2b},\ m>2a+2b$$
$$\mu(2a+1,2b+1;\nu)=\frac{(2a+1)!(2b+1)!}{2^{a+b}}\rho\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}\gamma_{2a+2b+2},\ \ \nu>2a+2b+2$$
$$\mu(2a,2b+1;\nu)=\mu(2a+1,2b;\nu)=0$$

where γ_a is the a-th moment of T.

4. The Standardized Moments of the Bivariate T-Distribution

Let $\mu(a,b)=E[(X_1-\theta_1)^a(X_2-\theta_2)^b]$ and $\mu(a,b;\nu)=E[(X_1-\theta_1)^a(X_2-\theta_2)^b]$ be the centered product moments of the bivarite normal distribution with pdf in (1.9) and the bivariate *t*-distribution with pdf in (1.7) respectively. Since the covariance matrix for the bivariate *t*-distribution is given by

$$\Lambda=\begin{pmatrix}\mu(2,0;\nu) & \mu(1,1;\nu)\\ \mu(1,1;\nu) & \mu(0,2;\nu)\end{pmatrix}=\begin{pmatrix}\mu(2,0) & \mu(1,1)\\ \mu(1,1) & \mu(0,2)\end{pmatrix}\gamma_2=\gamma_2\Sigma,$$

the quantity

$(X-\theta)'\Sigma^{-1}(X-\theta)=\|\Sigma^{-1/2}(X-\theta)\|^2=Z'Z=R^2$ is not the standardized distance. The standardized distance for the bivariate *t*-distribution is defined by

$$Q=(X-\theta)'(\gamma_2\Sigma)^{-1}(X-\theta) \tag{4.1}$$

Let $\beta_a=E(Q^a),\ (a=1,2,\cdots)$ be the a-th standardized moment of Q (Joarder, 2006). Some properties of standardized moments are discussed below.

Theorem 4.1 Let $Q=(X-\theta)'(\gamma_2\Sigma)^{-1}(X-\theta)$ be the standardized distance. Then

$$E(Q^a)=(\nu-2)^a\ \frac{\Gamma(a+1)\Gamma(\nu/2-a)}{\Gamma(\nu/2)},\ \ \nu>2a.$$

Proof. The quantity Q can be written as $Q=(X-\theta)'(\gamma_2\Sigma)^{-1}(X-\theta)=\gamma_2^{-1}Z'Z=\gamma_2^{-1}R^2$ where $\Sigma^{-1/2}(X-\theta)=Z$. Then from (1.3), the pdf of Z is

$$f_8(z)=\frac{1}{2\pi}\left(1+\frac{z'z}{\nu}\right)^{-\nu/2-1}. \tag{4.2}$$

By making the polar transformation

$z_1=r\cos\theta, z_2=r\sin\theta\ \ r\in(0,\infty),\theta\in(0,2\pi)$ in (4.2) with Jacobian r, it follows that the probability density functions of R and Θ are given by

$g_1(r)=r(1+r^2/\nu)^{-(\nu/2+1)},\ r\in(0,\infty)$ and

$g_2(\theta)=(2\pi)^{-1}\ \ ,\theta\in(0,2\pi)$

respectively. Then it can be checked easily that $R^2/2 \sim F(2,\nu)$, and consequently $W = \gamma_2 Q/2 \sim F(2,\nu)$, $\nu > 2$ (cf. Muirhead, 1982, 37).

Corollary 4.1 The first three standardized moments of the bivariate *t*-distribution are given by

$$E(Q) = 2,$$
$$E(Q^2) = 8\frac{\nu-2}{\nu-4},\ \nu > 4,$$
$$E(Q^3) = \frac{48(\nu-2)^2}{(\nu-4)(\nu-6)},\ \nu > 6.$$

Proof. The a-th moment of Q is given by

$$\begin{aligned} E(Q^a) &= 2^a \gamma_2^{-a} E(W^a) \\ &= 2^a \gamma_2^{-a} (\nu/2)^a \frac{\Gamma(a+1)\Gamma(\nu/2-a)}{\Gamma(\nu/2)} \\ &= (\nu-2)^a \frac{\Gamma(a+1)\Gamma(\nu/2-a)}{\Gamma(\nu/2)},\ \nu > 2a. \end{aligned}$$

Notice that as $\nu \to \infty$, the standardized moments of the bivariate *t*-distribution, as expected, coincide with that of the bivariate normal distribution.

Following Johnson, Kotz and Kemp (1993) we write moment ratios of the distribution of T by

$$\alpha_k(\mathrm{T}) = \mu_k (\mu_2)^{-k/2},\ k = 1,2,\cdots \tag{4.3}$$

Note that $\alpha_4(\mathrm{T})$ is an index of kurtosis while $\alpha_3(\mathrm{T})$ is an index of skewness for the univariate random variable T. In the following theorem we calculate the standardized moments of the bivariate *t*-dsitribution from Theorem 2.1 of Joarder (2006) by the use of the centered product moments of Section 3 of the present paper.

Theorem 4.2 The the first three standardized moments of the bivariate *t*-distribution are given by

$$\beta_1 = 2,\ \beta_2 = 8\frac{\gamma_4}{\gamma_2^2},\ \beta_3 = 48\frac{\gamma_6}{\gamma_2^3},$$

where the moment ratio $\alpha_k(\mathrm{T})$ is defined in (4.3).

Proof. Since the correlation coefficient between X_1 and X_2 is ρ (See Corollary 3.1), by the use of moments from Theorem 3.2 of the present paper in Theorem 2.1 of Joarder (2006), we have

$$\begin{aligned} (i)\ & (\sigma_1^2\gamma_2)(\sigma_2^2\gamma_2)(1-\rho^2)E(Q) \\ &= 2[(\sigma_1^2\gamma_2)(\sigma_2^2\gamma_2) - 2\rho(\sigma_1\sqrt{\gamma_2})(\sigma_2\sqrt{\gamma_2})(\sigma_1\sigma_2\gamma_2\rho)] \\ &= 2\sigma_1^2\sigma_2^2\gamma_2^2(1-\rho^2), \end{aligned}$$

$$(ii)\left[(\sigma_1^2\gamma_2)(\sigma_2^2\gamma_2)(1-\rho^2)\right]^2 E(Q^2)$$
$$= 3(\sigma_1^4\gamma_4)(\sigma_2^2\gamma_2)^2 + 4\rho^2(\sigma_1^2\gamma_2)(\sigma_2^2\gamma_2)[\sigma_1^2\sigma_2^2(1+2\rho^2)\gamma_4]$$
$$+3(\sigma_2^4\gamma_4)(\sigma_1^2\gamma_2)^2 - 4\rho(\sigma_1^2\gamma_2)^{1/2}(\sigma_2^2\gamma_2)^{3/2}(3\sigma_1^3\sigma_2\rho\gamma_4)$$
$$+2(\sigma_1^2\gamma_2)(\sigma_2^2\gamma_2)[\sigma_1^2\sigma_2^2(1+2\rho^2)\gamma_4], -4\rho(\sigma_1^2\gamma_2)^{3/2}(\sigma_2^2\gamma_2)^{1/2}(3\sigma_1\sigma_2^3\rho\gamma_4),$$

so that

$$(1-\rho^2)^2 E(Q^2)\gamma_2^4 = \left(3+4\rho^2(1+2\rho^2)+3-4\rho(3\rho)+2(1+2\rho^2)-4\rho(3\rho)\right)\gamma_2^2\gamma_4$$
$$= \left(6+(4\rho^2+2)(1+2\rho^2)-24\rho^2\right)\gamma_2^2\gamma_4$$
$$= \left(8(1-\rho^2)^2\right)\gamma_2^2\gamma_4,$$

$$(iii)\left[(\sigma_1^2\gamma_2)(\sigma_2^2\gamma_2)(1-\rho^2)\right]^3 E(Q^3)$$
$$= (90\sigma_1^6\gamma_6)(\sigma_2^2\gamma_2)^3 - 8\rho^3(\sigma_1^2\gamma_2)^{3/2}(\sigma_2^2\gamma_2)^{3/2}[3\sigma_1^3\sigma_2^3\rho(3+2\rho^2)\gamma_6]$$
$$+(\sigma_1^2\gamma_2)^3(90\sigma_2^6\gamma_6) - 6\rho(\sigma_1^2\gamma_2)^{1/2}(\sigma_2^2\gamma_2)^{5/2}[15\sigma_1^5\sigma_2\rho\gamma_6]$$
$$+12\rho^2(\sigma_1^2\gamma_2)(\sigma_2^2\gamma_2)^2[3\sigma_1^4\sigma_2^2(1+4\rho^2)\gamma_6] + 3(\sigma_1^2\gamma_2)(\sigma_2^2\gamma_2)^2[3\sigma_1^4\sigma_2^2(1+4\rho^2)\gamma_6]$$
$$+3(\sigma_1^2\gamma_2)^2(\sigma_2^2\gamma_2)[3\sigma_1^2\sigma_2^4(1+4\rho^2)\gamma_6] + 12\rho^2(\sigma_2^2\gamma_2)^2(\sigma_2^2\gamma_2)[3\sigma_1^2\sigma_2^4(1+4\rho^2)\gamma_6]$$
$$-6\rho(\sigma_1^2\gamma_2)^{5/2}(\sigma_2^2\gamma_2)^{1/2}(15\sigma_1\sigma_2^5\rho\gamma_6) - 12\rho(\sigma_1^2\gamma_2)^{3/2}(\sigma_2^2\gamma_2)^{3/2}[3\sigma_1^3\sigma_2^3\rho(3+2\rho^2)\gamma_6]$$

so that

$$(1-\rho^2)^3\gamma_2^6 E(Q^3)$$
$$= \left(90-8\rho^3[3\rho(3+2\rho^2)]+(90)-6\rho[15\rho]+12\rho^2[3(1+4\rho^2)]\right.$$
$$\left.+3[3(1+4\rho^2)]+3[3(1+4\rho^2)]+12\rho^2[3(1+4\rho^2)]-6\rho(15\rho)-12\rho[3\rho(3+2\rho^2)]\right)\gamma_2^3\gamma_6$$
$$= \left(48(1-\rho^2)^3\right)\gamma_2^3\gamma_6.$$

Hence, the second and the third standardized moments of the bivariate *t*-distribution are given by

$$\beta_2 = 8\frac{\gamma_4}{\gamma_2^2} = 8\alpha_4(\mathrm{T}),\ \beta_3 = 48\frac{\gamma_6}{\gamma_2^3} = 48\alpha_6(\mathrm{T}),$$

which can be simplified by Corollary 1.1 as the following:

$$\beta_2 = 8\frac{\nu-2}{\nu-4},\ \nu > 4;\ \beta_3 = 48\frac{(\nu-2)^2}{(\nu-4)(\nu-6)},\ \nu > 6.$$

Note that the coefficient of kurtosis β_2 of the bivariate t-distribution is 8 times that of the invereted chisquare distribution with pdf in (1.1).

Corollary 4.1 Let T be an inverted chisqaure variable with pdf in (1.1), and $\mu_a(\mathrm{T}) = E(\mathrm{T}-E(\mathrm{T}))^a$. Then, as $\nu \to \infty$, we have

(i) $\mu_4(\mathrm{T}) \to \left(\mu_2(\mathrm{T})\right)^2$,

(ii) $\mu_6(\mathrm{T}) \to (\mu_2(\mathrm{T}))^3$.

Proof. As $\nu \to \infty$, the bivariate *t*-distribution converges to the bivariate normal distribution in which case it is known that $\beta_2 \to 8, \beta_3 \to 48$ (Joarder, 2006). It is obvious from Theorem 4.2 that this happens if (i) and (ii) are true.

5. Shanon Entropy

The Shanon Entropy for any bivarite density function $f\ (x_1, x_2)$ is defined by $H(f\ ) = -E\,(\ln f\ (X_1, X_2)]$. It follows from (1.9) that

$$\ln f_6(x_1, x_2) = \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) - \frac{q(x_1, x_2)}{2(1-\rho^2)}$$

so that

$$\begin{aligned} E\,(\ln f_6(X_1, X_2)) &= -\ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) - \frac{E\,[q(X_1, X_2)]}{2(1-\rho^2)} \\ &= -\ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) - 1. \end{aligned}$$

Hence, the Shanon Entropy for the bivariate normal distribution with pdf in (1.9) is given by

$$H(f_6) = \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) + 1. \tag{5.1}$$

Theorem 5.1 Let the bivariate *t*-distribution have the pdf in (1.7). Then the Shanon entropy for the bivariate *t*-distribution has the following equivalent representations:

$$(i)\ H(f_5) = \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) + E\,(\ln \mathrm{T}^2) + 1$$

$$(ii)\ H(f_5) = \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) + \frac{\nu+2}{2}\ E\left[\ln\left(1 + q(X_1, X_2)/\nu\right)\right],$$

$$(iii)\ H(f_5) = \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) + \ln\left[\frac{\sqrt{\pi}\,\Gamma(\nu/2+1)}{\Gamma((\nu+1)/2)}\right] + \frac{\nu+2}{2}\left[\Psi(\nu/2+1) - \Psi(\nu/2)\right],$$

where $\nu \mathrm{T}^{-2} \sim \chi^2_\nu$ and $\Psi(t) = d\ \ln\Gamma(t)/dt$ denotes the digamma function. (Nadarajah and Kotz, 2005).

Proof. By virtue of (1.5), it follows from (5.1) that

$$\begin{aligned} H(f_5) &= E\left(\ln\left[2\pi(\sigma_1\mathrm{T})(\sigma_2\mathrm{T})\sqrt{1-\rho^2}\right]\right) + 1 \\ &= \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) + E\,(\ln \mathrm{T}^2) + 1, \end{aligned}$$

which is part (ii) of the theorem. It follows from (1.7) that

$$E[\ln f_5(X_1, X_2)] = -\ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right) - \frac{\nu+2}{2}\, E\left[\ln\left(1+q(X_1,X_2)/\nu\right)\right],$$

which yields (ii) in the theorem. Note that $E(\ln \mathrm{T}^2) = \ln\nu - E(\ln W)$, $W \sim \chi^2_\nu$. For part (iii), see Nadarajah and Kotz (2005).

Acknowledgements

The author acknowledges the excellent research support provided by King Fahd University of Petroleum and Minerals, Saudi Arabia. The author is also grateful to Prof Saralees Nadarajah for providing me with preprints of some papers on *t*-distributions.

References

1. Anderson, T.W. (2003). An *Introduction to Multivariate Statistical Analysis*. John Wiley and Sons. New York.

2. Billah, M.B. and Saleh, A.K.M.E. (2000). Performance of the large sample tests in the actual forecasting of the pretest estimators for regression model with *t*-error. *Journal of Applied Statistical Science*, 9(3), 237-251.

3. Joarder, A.H. (2006). Standradized Moments of Bivariate Distributions. Technical Report. Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals, Saudi Arabia.

4. Johnson, N.L.; Kotz and Kemp, A. (1993). *Univariate Discrete Distribution.* John Wiley and Sons.

5. Kendal, M.G. and Stuart, A. (1969). *The Advanced Theory of Statistics*, Charles Griffin and Co. Ltd, London.

6. Kibria, B.M.G. and Saleh, A.K.M.E. (2000). Conflict among the shrinkage estimators induced by W, LR and LM tests under a Student's *t* regression model. *Journal of the Korean Statsitical Society*, 33(4), 411-433.

7. Kotz, S. and Nadarajah, S. (2004). *Multivariate t-Distributions and Their Applications.* Cambridge University Press.

8. Lange, K.L.; Little, R.J.A. and Taylor, J.M.G. (1989) Robust statistical modeling using the *t* distribution. *Journal of American Statistical Association*, 84, 881-896.

9. Muirhead, R.J. (1982). *Aspects of Multivariate Statistical Theory*. John Wiley and Sons.

10. Nadaraja, S. adnd Kotz, S. (2005). Mathematical properties of multivariate *t*-dstributomn. *Acta Appliacandae Mathematicae.* To appear.

File: MSA\ MomentsBivariateTDpa.doc