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# Mahalanobis Moments of the Bivariate Chisqaure Distribution

**Anwar H. Joarder**

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

# Mahalanobis Moments of the Bivariate Chisqaure Distribution

Anwar H Joarder
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261 Saudi Arabia
Email: anwarj@kfupm.edu.sa

**Absract** Centered moments of the bivariate chisquare distribution have been derived by the use of raw product moments. Standardized moments have also been calculated for the distribution. In case of independence of the components of bivariate chisquare random vector, the results, as expected, are in agreement with the resulting situation of independence. The results are also in agreement with the case when the degrees of freedom converges to infinity.

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## 1. Introduction

The pdf of sum of squares and products for the bivariate normal distribution can be written as

$$f_1(a_{11},a_{22},a_{12})=\frac{\left(1-\rho^2\right)^{-m/2}(\sigma_1\sigma_2)^{-m}}{2^m\sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)}\left(a_{11}a_{22}-a_{12}^2\right)^{(m-3)/2}$$
$$\times\exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2}-\frac{a_{22}}{2(1-\rho^2)\sigma_2^2}\right)\exp\left(\frac{\rho a_{12}}{2(1-\rho^2)\sigma_1\sigma_2}\right)$$

$a_{11}>0, a_{22}>0, -\infty<a_{12}<\infty, m>2, -1<\rho<1$ (Anderson, 2003, 123).

By making some transformations, Joarder (2005) introduced the following chisquare distribution.

**Theorem 1.1** The random variables $U$ and $V$ are said to have a correlated bivariate chisquare distribution each with $m$ degrees of freedom, if its probability density function is given by

$$f_2(u,v)=\frac{(uv)^{m/2-1}\,e^{\frac{-(u+v)}{2(1-\rho^2)}}}{2^m\sqrt{\pi}\,\Gamma(m/2)\left(1-\rho^2\right)^{m/2}}\sum_{k=0}^{\infty}\left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^k\frac{\Gamma\left((k+1)/2\right)}{k!\Gamma\left((k+m)/2\right)},$$

$m=N-1>2,\ -1<\rho<1.$

Joarder (2005) calculated raw product moments of the above bivariate chisquare distribution. In this paper we derive centered moments of the bivariate chisquare distribution by the use of raw product moments. Standardized moments have also been calculated for the distribution. In case of independence of the components of bivariate chisquare random vector, the results, as expected, are in agreement with the resulting situation. The results are also in agreement with the case when the degrees of freedom converges to infinity. The kurtosis and skewness have been analyzed by the standardized moments as well as by the graphs.

The product moments (also called as raw product moments or product moments around zero) of order $a$ and $b$ for two random variables $X$ and $Y$ are defined by $\mu'(a,b) = E\left(X^a Y^b\right)$ while the centered product moments (sometimes called corrected product moments or central mixed moments) are defined by

$$\mu(a,b) = E\left[(X-\xi)^a (Y-\theta)^b\right] \tag{1.1}$$

where $\xi = E(X), \theta = E(Y)$. See for example, Johnson, Kotz and Kemp (1993, 46) or Johnson, Kotz and Balakrishnan (1997, 3). Evidently $\mu'(a,0) = E\left(X^a\right)$ is the $a$-th moment of $X$, and $\mu'(0,b) = E\left(Y^b\right)$ is the $b$-th moment of $X_2$. In case $X$ and $Y$ are independent $\mu'(a,b) = E\left(X^a\right)E\left(Y^b\right) = \mu'(a,0)\mu'(0,b)$. The correlation coefficient $\rho(-1<\rho<1)$ between $X$ and $Y$ is denoted by

$$\rho_{X,Y} = \frac{\mu(1,1)}{\sqrt{\mu(2,0)\mu(0,2)}}. \tag{1.2}$$

In a series of papers, Mardia (1970, 1974, 1975) defined and discussed the properties of measures for kurtosis and skewness based on Mahalanobis distance. As it is difficult to derive distribution of Mahalanobis distance for bivariate chisquare distribution , we derive standardized moments (equivalently Mahalanobis moments) in terms of centered product moments. It is worth mentioning that the second standardized moment accounts for kurtosis while the third for skewness. In this paper standardized product moments are calculated for a bivariate chisqaure distribution introduced by Joarder (2005). Centered product moments of the bivariate chisquare distribution that are deemed essential have been expressed in terms of raw product moments.

## 2. Centered Product Moments of the Bivariate Chaisquare Distribution

The following theorem is due to Joarder (2005).

**Theorem 2.1** The $(a,b)$ th moment of the distribution of $U$ and $V$, is given by

$$E(U^a V^b) = \frac{2^{a+b}\left(1-\rho^2\right)^{a+b}}{L(m,\rho)} \sum_{k=0}^{\infty} \frac{(2\rho)^k}{k!}\, \Gamma\left(\frac{k+m}{2}+a\right)\Gamma\left(\frac{k+m}{2}+b\right)\frac{\Gamma\left((k+1)/2\right)}{\Gamma\left((k+m)/2\right)}$$

where $L(m,\rho)=\sqrt{\pi}\;\Gamma(m/2)\left(1-\rho^2\right)^{-m/2}$, $m>2,$ and $-1<\rho<1$.

It has been checked by Joarder (2056) that for the bivariate chisquare distribution $E(U^aV^b)=E(U^bV^a)$ with $E(U)=E(V)=m$ . If $\rho=0$ , then $E(U^aV^b)=E(U^a)E(V^b)$, where $U$ is distributed as chisquare with $m$ degrees of freedom, and so is $V$ . Note that $E(U^a)=\; 2^a\;\Gamma((m/2)+a)/\Gamma(m/2), a=1,2,\cdots$. Some raw product moments of the bivariate chisquare distribution are given below.

$$E(UV)=m\left(m+2\rho^2\right),$$
$$E(UV^2)=m(m+2)\left(m+4\rho^2\right),$$
$$E(UV^3)=m(m+2)(m+4)\left(m+6\rho^2\right),$$
$$E(UV^4)=m(m+2)(m+4)(m+6)\left(m+8\rho^2\right),$$
$$E(UV^5)=m(m+2)(m+4)(m+6)(m+8)\left(m+10\rho^2\right),$$
$$E(U^2V^2)=m(m+2)\left[8\rho^4+8(m+2)\rho^2+m(m+2)\right],$$
$$E(U^2V^3)=m(m+2)(m+4)\left[24\rho^4+12(m+2)\rho^2+m(m+2)\right],$$
$$E(U^2V^4)=m(m+2)(m+4)(m+6)\left[48\rho^4+16(m+2)\rho^2+m(m+2)\right],$$
$$E(U^3V^3)=m(m+2)(m+4)\left[48\rho^6+72(m+4)\rho^4+18(m+2)(m+4)\rho^2+m(m+2)(m+4)\right].$$

Central product moments of the bivarite chisquare distribution are expressed below in terms of raw product moments.

$$\mu(1,1)=E(UV)-m^2,$$
$$\mu(2,0)=E\left(U^2\right)-m^2,$$
$$\mu(2,1)=-2mE(UV)+E(U^2V)-mE(U^2)+2m^3,$$
$$\mu(2,2)=-3m^4+2m^2E(U^2)+4m^2E(UV)-4mE(U^2V)+E(U^2V^2),$$
$$\mu(3,0)=E(U^3)-3mE(U^2)+2m^3,$$
$$\mu(3,1)=3m^2E(UV)-3mE\left(U^2V\right)+E\left(U^3V\right)+3m^2E\left(U^2\right)-mE\left(U^3\right)-3m^4,$$
$$\mu(3,2)=E(U^3V^2)-3mE(U^2V^2)-2mE(U^3V)+9m^2E(U^2V)-4m^3E(U^2)$$
$$-6m^3E(UV)+m^2E(U^3)+m^5-3m^4,$$
$$\mu(3,3)=-5m^6+9m^4E(UV)-18m^3E(U^2V)+6m^4E(U^2)+9m^2E(U^2V^2)-2m^3E(U^3)$$
$$+6m^2E(U^3V)\;-6mE\left(U^3V^2\right)+E(U^3V^3),$$
$$\mu(4,0)=-3m^4+6m^2E(U^2)-4mE(U^3)+E(U^4),$$
$$\mu(4,1)=-4m^3E(UV)+6m^2E(U^2V)-4mE(U^3V)+E(U^4V)-6m^3E(U^2)$$
$$+4m^2E(U^3)-mE(U^4)+4m^5,$$

$\mu(4,2) = -5m^6 + 7m^4 E(U^2) - 4m^3 E(U^3) + m^2 E(U^4) + 8m^4 E(UV) - 16m^3 E(UV^2)$
$\quad + 8m^2 E(U^3 V) - 2mE(U^4 V) + 6m^2 E(U^2 V^2) - 4mE(U^3 V^2) + E(U^4 V^2),$
$\mu(5,0) = 4m^5 - 10m^3 E(U^2) + 10m^2 E(U^3) - 5mE(U^4),$
$\mu(5,1) = 5m^4 E(UV) - 10m^3 E(U^2 V) + 10m^2 E(U^3 V) - 5mE(U^4 V) + E(U^5 V) - 5m^6$
$\quad + 10m^4 E(U^2) - 10m^3 E(U^3) + 5m^2 E(U^4) - mE(U^5),$
$\mu(6,0) = -5m^6 + 15m^4 E(U^2) - 20m^3 E(U^3) + 15m^2 E(U^4) - 6mE(U^5) + E(U^6),$

Important centered product moments of bivariate chisquare distribution are explicitly derived below with the help of the raw product moments.

$\mu(1,1) = 2m\rho^2,$
$\mu(2,0) = 2m,$
$\mu(2,2) = (8\rho^4 + 4)m^2 + (16\rho^4 + 32\rho^2)m,$
$\mu(3,1) = 12\rho^2 m^2 + 48\rho^2 m,$
$\mu(3,3) = (48\rho^6 + 72\rho^2)m^3 + (288\rho^6 + 1152\rho^4 + 576\rho^2 + 64)m^2 + (384\rho^6 + 2304\rho^4 + 1152\rho^2)m,$

$\mu(4,0) = 12m^2 + 48m,$
$\mu(4,2) = (96\rho^4 + 24)m^3 + (1344\rho^4 + 2176\rho^2 + 96)m^2 + (2304\rho^4 + 1536\rho^2)m.$
$\mu(5,1) = 120\rho^2 m^3 + 2080\rho^2 m^2 + 3840\rho^2 m,$
$\mu(6,0) = 120m^3 + 2080m^2 + 3840m,$

## 3. Mahalanobis Moments for the Bivariate Chisquare Distribution

Since $\mu(2,0) = 2m$, $\mu(1,1) = 2m\rho^2$, $\mu(0,2) = 2m$, the covariance matrix between $U$ and $V$ is given by

$$\Omega = \begin{pmatrix} 2m & 2m\rho^2 \\ 2m\rho^2 & 2m \end{pmatrix}$$

while the correlation coefficient (see 1.1) between $U$ and $V$ is given by

$\rho_{U,V} = \dfrac{E(U - E(U))(V - E(V))}{\sqrt{(E(U - E(U))^2 (E(V - E(V))^2}} = \dfrac{\mu(1,1)}{\sqrt{\mu(2,0)\mu(0,2)}} = \rho^2.$ The Mahalanobis moments or standardized moments of the bivariate chisquare distribution are given by $\beta_i = E\left(W'\Omega^{-1}W\right)^i, (i = 1, 2, \cdots)$ where $W' = (U - m \ \ V - m)$. Note that

$$\beta_1 = E(trW'\Omega^{-1}W) = E(tr\Omega^{-1}WW') = tr[\Omega^{-1}E(WW')] = tr(\Omega^{-1}\Omega) = tr(I_2) = 2.$$

Obviously the first standardized moment for any distribution is the dimension of the random variable in question.

The coefficient of kurtosis and skewness (Kotz, Balakrishnan and Johnson, 2000, 77) for the bivariate chisquare distribution are given by $\beta_2 = E\left(W'\Omega^{-1}W\right)^2$ and $\beta_3 = E\left(W'\Omega^{-1}W\right)^3$ respectively.

**Theorem 3.1** Let $X$ and $Y$ have a bivariate distribution with $E(X^aY^b) = E(X^bY^a)$ and the correlation between them is $\rho$. Then

$$(i)\mu^2(0,2)(1-\rho^2)^2E(Q^2) = 2\mu(4,0) + (4\rho^2+2)\mu(2,2) - 8\rho\mu(3,1)$$
$$(ii)\mu^3(0,2)(1-\rho^2)^3E(Q^3) = 2\mu(6,0) - (8\rho^3+12\rho)\mu(3,3) - 12\rho\mu(5,1) + (24\rho^2+6)\mu(4,2).$$

**Theorem 3.2** For $m > 0$, the second and the third Mahalanobis moments of the bivariate chisquare distribution with pdf in Theorem 1.1 are given by

$$(i)m(1-\rho^4)^2E(Q^2) = 8m(1-\rho^4)^2 + 8(3+2\rho^2-11\rho^4+4\rho^6+2\rho^8),$$
$$(ii)\mu^3(0,2)(1-\rho^4)^3E(Q^3) = 2\mu(6,0) - (8\rho^6+12\rho^2)\mu(3,3) - 12\rho^2\mu(5,1) + (24\rho^4+6)\mu(4,2).$$

**Proof.** Since the correlation between $U$ and $V$ is $\rho_{U,V} = \rho^2$, we replace $\rho_{X,Y} = \rho$ by $\rho_{U,V} = \rho^2$ in the expressions in Theorem 3.1. Moreover since $E(U^aV^b) = E(U^bV^a)$, it follows from Theorem 3.1 (i) that

$$\mu^2(0,2)(1-\rho^4)^2E(Q^2) = 2+\mu(4,0) + (4\rho^4+2)\mu(2,2) - 8\rho^2\mu(3,1),$$

Then part (i) in the following way:

$$\begin{aligned}&4m^2(1-\rho^4)^2E(Q^2)\\&= 2(12m^2+48m) + (4\rho^4+2)(8\rho^4m^2+16\rho^4m+32\rho^2m+4m^2) - 8\rho^2(12m^2\rho^2+48m\rho^2)\\&= 4(8\rho^8-16\rho^4+8)m^2 + 4(16\rho^8+32\rho^6-88\rho^4+16\rho^2+24)m.\end{aligned}$$

The part (ii) follows from Theorem 3,1 (ii). The simplification seems to be a formidable task.

In case $\rho = 0$, the pdf of the joint probability distribution in Theorem 2.1, would be product of two independent chisquare random variables given by

$$f(u,v) = \frac{(uv)^{m/2-1}e^{-(u+v)/2}}{2^m\,\Gamma^2(m/2)},\ u > 0, v > 0.$$

**Corollary 3.1** For independent bivariate chisquare distribution with pdf given above we have

$$(i)E(Q^2) = 8 + 24/m,$$
$$(ii)E(Q^3) = 48 + (592/m) + (960/m^2).$$

These moments coincide, as expected, to that of the bivariate normal distribution as $m$ tends to infinity.

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