Mahalanobis Moments of the Bivariate Chi-square Distribution

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Abstract Centered moments of the bivariate chi-square distribution have been derived by the use of raw product moments. Standardized moments have also been calculated for the distribution. In case of independence of the components of bivariate chi-square random vector, the results, as expected, are in agreement with the resulting situation of independence. The results are also in agreement with the case when the degrees of freedom converges to infinity.

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1. Introduction

The pdf of sum of squares and products for the bivariate normal distribution can be written as

\[\begin{align*}
f_1(a_{11}, a_{22}, a_{12}) &= \frac{1}{2^m \sqrt{\pi \Gamma(m/2)} \Gamma((m-1)/2)}
\left(a_{11} a_{22} - a_{12}^2 \right)^{(m-3)/2}
\times \exp\left(- \frac{a_{11}}{2(1-\rho^2)\sigma_1^2} - \frac{a_{22}}{2(1-\rho^2)\sigma_2^2}\right)
\exp\left(\frac{\rho a_{12}}{2(1-\rho^2)\sigma_1\sigma_2}\right)
\end{align*}\]

with \(a_{11} > 0, a_{22} > 0, -\infty < a_{12} < \infty, m > 2, -1 < \rho < 1\) (Anderson, 2003, 123).

Joarder (2005) calculated raw product moments of the above bivariate chi-square distribution. In this paper we derive centered moments of the bivariate chi-square distribution by the use of raw product moments. Standardized moments have also been calculated for the distribution. In case of independence of the components of bivariate chi-square random vector, the results, as expected, are in agreement with the resulting situation of independence. The results are also in agreement with the case when the degrees of freedom converges to infinity.
expected, are in agreement with the resulting situation. The results are also in agreement with
the case when the degrees of freedom converges to infinity. The kurtosis and skewness have
been analyzed by the standardized moments as well as by the graphs.

The product moments (also called as raw product moments or product moments around zero)
of order $a$ and $b$ for two random variables $X$ and $Y$ are defined by $\mu'(a,b) = E \left( X^a Y^b \right)$
while the centered product moments (sometimes called corrected product moments or central
mixed moments) are defined by

$$\mu(a,b) = E \left[ (X - \xi)^a (Y - \theta)^b \right]$$

(1.1)

where $\xi = E(X)$, $\theta = E(Y)$. See for example, Johnson, Kotz and Kemp (1993, 46) or Johnson,
Kotz and Balakrishnan (1997, 3). Evidently $\mu'(a,0) = E \left( X^a \right)$ is the $a$-th moment of $X$, and
$\mu'(0,b) = E \left( Y^b \right)$ is the $b$-th moment of $X$. In case $X$ and $Y$ are independent
$\mu'(a,b) = E \left( X^a \right) E \left( Y^b \right) = \mu'(a,0)\mu'(0,b)$. The correlation coefficient $\rho(-1 < \rho < 1)$ between
$X$ and $Y$ is denoted by

$$\rho_{X,Y} = \frac{\mu(1,1)}{\sqrt{\mu(2,0)\mu(0,2)}}.$$  

(1.2)

In a series of papers, Mardia (1970, 1974, 1975) defined and discussed the properties of
measures for kurtosis and skewness based on Mahalanobis distance. As it is difficult to derive
distribution of Mahalanobis distance for bivariate chisquare distribution, we derive
standardized moments (equivalently Mahalanobis moments) in terms of centered product
moments. It is worth mentioning that the second standardized moment accounts for kurtosis
while the third for skewness. In this paper standardized product moments are calculated for a
bivariate chisquare distribution introduced by Joarder (2005). Centered product moments of the
bivariate chisquare distribution that are deemed essential have been expressed in terms of raw
product moments.

2. Centered Product Moments of the Bivariate Chisquare Distribution

The following theorem is due to Joarder (2005).

**Theorem 2.1** The $(a,b)$ th moment of the distribution of $U$ and $V$, is given by

$$E(U^a V^b) = \frac{2^{a+b}(1-\rho^2)^{a+b}}{L(m, \rho)} \sum_{k=0}^{\infty} \frac{(2\rho)^k}{k!} \Gamma \left( \frac{k+m}{2} + a \right) \Gamma \left( \frac{k+m}{2} + b \right) \Gamma \left( (k+1)/2 \right) \Gamma \left( (k+m)/2 \right)$$
where \( L(m, \rho) = \sqrt{\pi} \Gamma(m/2)(1-\rho^2)^{-m/2} \), \( m > 2 \), and \(-1 < \rho < 1\).

It has been checked by Joarder (2056) that for the bivariate chisquare distribution
\[
E(UV) = E(UV^2) = \left[ 2 \pi \Gamma((m/2)+a)/\Gamma(m/2), a = 1, 2, \ldots \right].
\]
Some raw product moments of the bivariate chisquare distribution are given below.

\[
E(UV) = m \left( m + 2 \rho^2 \right),
\]
\[
E(UV^2) = m(m+2)(m+4\rho^2),
\]
\[
E(UV^3) = m(m+2)(m+4)(m+6\rho^2),
\]
\[
E(UV^4) = m(m+2)(m+4)(m+6)(m+8\rho^2),
\]
\[
E(UV^5) = m(m+2)(m+4)(m+6)(m+8)(m+10\rho^2),
\]
\[
E(UV^2) = m(m+2)\left[ 8\rho^4 + 8(m+2)\rho^2 + m(m+2) \right],
\]
\[
E(UV^3) = m(m+2)(m+4)\left[ 24\rho^6 + 12(m+2)\rho^2 + m(m+2) \right],
\]
\[
E(UV^4) = m(m+2)(m+4)(m+6)\left[ 48\rho^8 + 16(m+2)\rho^2 + m(m+2) \right],
\]
\[
E(UV^3) = m(m+2)(m+4)\left[ 48\rho^6 + 72(m+4)\rho^4 + 18(m+2)(m+4)\rho^2 + m(m+2)(m+4) \right].
\]

Central product moments of the bivariate chisquare distribution are expressed below in terms of raw product moments.

\[
E(UUV) = E(U^2) - m^2,
\]
\[
E(U^2) = E \left( U^2 \right) - m^2,
\]
\[
E(UV^2) = E(UV^2) - mE(U^2) + 2m^3,
\]
\[
E(UV^3) = E(UV^3) - 3mE(U^2) + 4m^2E(UV) - 4mE(U^3) + E(U^2)^2,
\]
\[
E(UV^4) = E(U^3) - 3mE(U^2) + 2m^3,
\]
\[
E(UV^5) = E(U^3) - 3mE(U^2) + 3m^2E(U^2) - mE(U^3) - 3m^4,
\]
\[
E(UV^6) = E(U^2) - 6mE(UV) - m^2E(U^3) + m^5 - 3m^4,
\]
\[
E(UV^7) = -5m^6 + 9m^4E(UV) - 18m^3E(U^2) + 6m^2E(U^3) + 9m^2E(U^2)^2 - 2m^3E(U^3) + 6m^2E(U^2)^2 + mE(U^4) - 6mE(U^3)^2 + E(U^4),
\]
\[
E(UV^8) = -3m^4 + 6mE(U^3) - 4mE(U^2)^2 + E(U^4),
\]
\[
E(UV^9) = -4m^3E(UV) + 6m^2E(U^3) - 4mE(U^2)^2 + E(U^4) - 6m^3E(U^2) + 4m^2E(U^3) - mE(U^4) + 4m^4.
\]
Important centered product moments of bivariate chi-square distribution are explicitly derived below with the help of the raw product moments.

\[ \mu(1,1) = 2m \rho^2, \]
\[ \mu(2,0) = 2m, \]
\[ \mu(2,2) = (8\rho^4 + 4)m^2 + (16\rho^4 + 32\rho^2)m, \]
\[ \mu(3,1) = 12\rho^2 m^2 + 48\rho^2 m, \]
\[ \mu(3,3) = (48\rho^6 + 72\rho^3)m^3 + (288\rho^6 + 1152\rho^4 + 576\rho^2 + 64)m^2 + (384\rho^6 + 2304\rho^4 + 1152\rho^2)m, \]
\[ \mu(4,0) = 12m^2 + 48m, \]
\[ \mu(4,2) = (96\rho^4 + 24)m^3 + (1344\rho^4 + 2176\rho^2 + 96)m^2 + (2304\rho^4 + 1536\rho^2)m. \]
\[ \mu(5,0) = 2080\rho^2 m^2 + 3840\rho^2 m, \]
\[ \mu(6,0) = 120m^3 + 2080m^2 + 3840m, \]

### 3. Mahalanobis Moments for the Bivariate Chi-square Distribution

Since \( \mu(2,0) = 2m, \mu(1,1) = 2m \rho^2, \mu(0,2) = 2m, \) the covariance matrix between \( U \) and \( V \) is given by

\[ \Omega = \begin{pmatrix} 2m & 2m \rho^2 \\ 2m \rho^2 & 2m \end{pmatrix} \]

while the correlation coefficient (see 1.1) between \( U \) and \( V \) is given by

\[ \rho_{U,V} = \frac{E(U - E(U)(V - E(V)))}{\sqrt{(E(U - E(U))^2)(E(V - E(V))^2)}} = \frac{\mu(1,1)}{\sqrt{\mu(2,0)\mu(0,2)}} = \rho^2. \]

The Mahalanobis moments or standardized moments of the bivariate chi-square distribution are given by

\[ \beta_i = E(W'\Omega^{-1}W), (i = 1,2,\cdots) \text{ where } W' = (U - m V - m). \]

Note that

\[ \beta_1 = E(trW'\Omega^{-1}W') = E(tr\Omega^{-1}WW') = tr[\Omega^{-1}E(WW')] = tr(\Omega^{-1}\Omega) = tr(I_2) = 2. \]

Obviously the first standardized moment for any distribution is the dimension of the random variable in question.
The coefficient of kurtosis and skewness (Kotz, Balakrishnan and Johnson, 2000, 77) for the bivariate chisquare distribution are given by \( \beta_2 = E \left( W^\prime \Omega W \right)^2 \) and \( \beta_3 = E \left( W^\prime \Omega^2 W \right)^3 \) respectively.

**Theorem 3.1** Let \( X \) and \( Y \) have a bivariate distribution with \( E(X^a Y^b) = E(X^b Y^a) \) and the correlation between them is \( \rho \). Then

\[
\begin{align*}
(i) \quad & \mu^2(0,2)(1-\rho^2)^2 E(Q^2) = 2\mu(4,0) + (4\rho^2 + 2)\mu(2,2) - 8\rho\mu(3,1) \\
(ii) \quad & \mu^3(0,2)(1-\rho^2)^3 E(Q^3) = 2\mu(6,0) - (8\rho^3 + 12\rho)\mu(3,3) - 12\rho^2\mu(5,1) + (24\rho^2 + 6)\mu(4,2).
\end{align*}
\]

**Theorem 3.2** For \( m > 0 \), the second and the third Mahalanobis moments of the bivariate chisquare distribution with pdf in Theorem 1.1 are given by

\[
\begin{align*}
(i) \quad & m(1-\rho^4)^2 E(Q^2) = 8m(1-\rho^4)^2 + 8(3 + 2\rho^2 - 11\rho^4 + 4\rho^8 + 2\rho^8), \\
(ii) \quad & m^3(0,2)(1-\rho^4)^3 E(Q^3) = 2m(6,0) - (8\rho^6 + 12\rho^2)\mu(3,3) - 12\rho^2\mu(5,1) + (24\rho^4 + 6)\mu(4,2).
\end{align*}
\]

**Proof.** Since the correlation between \( U \) and \( V \) is \( \rho_{U,V} = \rho^2 \), we replace \( \rho_{X,Y} = \rho \) by \( \rho_{U,V} = \rho^2 \) in the expressions in Theorem 3.1. Moreover since \( E(U^a V^b) = E(U^b V^a) \), it follows from Theorem 3.1 (i) that

\[
\mu^2(0,2)(1-\rho^4)^2 E(Q^2) = 2 + \mu(4,0) + (4\rho^2 + 2)\mu(2,2) - 8\rho^2\mu(3,1),
\]

Then part (i) in the following way:

\[
\begin{align*}
4m^2(1-\rho^4)^2 E(Q^2) & = 2(12m^2 + 48m) + (4\rho^4 + 2)(8\rho^4 m^2 + 16\rho^6 m + 32\rho^8 m + 4m^2) - 8\rho^4(12m^2 \rho^2 + 48m \rho^8) \\
& = 4(8\rho^8 - 16\rho^4 + 8)m^2 + 4(16\rho^8 + 32\rho^6 - 88\rho^4 + 16\rho^2 + 24)m.
\end{align*}
\]

The part (ii) follows from Theorem 3.1 (ii). The simplification seems to be a formidable task.

In case \( \rho = 0 \), the pdf of the joint probability distribution in Theorem 2.1, would be product of two independent chisquare random variables given by

\[
f(u,v) = \frac{(uv)^{n/2} e^{-(u+v)/2}}{2^n \Gamma(n/2)} , \quad u > 0, v > 0.
\]

**Corollary 3.1** For independent bivariate chisquare distribution with pdf given above we have

\[
\begin{align*}
(i) \quad & E(Q^2) = 8 + 24/m, \\
(ii) \quad & E(Q^3) = 48 + (592/m) + (960/m^2).
\end{align*}
\]

These moments coincide, as expected, to that of the bivariate normal distribution as \( m \) tends to infinity.
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References


