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Abstract

In the literature various kinds of mappings between topological spaces have been defined. We introduce the notion of a new class of mappings called alpha-weakly continuous mappings and investigate its several properties and characterizations. Its connection with other existing concepts such as alpha-continuous and weakly alpha-continuous mappings are also investigated.

1. Introduction

The notion of $\alpha-open$ set (originally called $\alpha-sets$) in topological space was introduced by Njastad $[1965]$. Since then, it has been widely investigated in the literature. Throughout this paper, (X,τ) $(simply\ X)$ always mean topological space. A subset A of (X,τ) is called $\alpha-open$ $[Njastad;\ 1965]$ if $A\subseteq Int\left[Cl\left(Int(A)\right)\right]$. The complement of an $\alpha-open$ set is called an $\alpha-closed$ set. The intersection of all $\alpha-closed$ sets containing A is called the $\alpha-closure$ of A, denoted by $Cl_{\alpha}(A)$. A subset A is $\alpha-closed$ if and only if $A=Cl_{\alpha}(A)$. A point $x\in X$ is said to be an $\alpha-interior$ point of A if there exists an $\alpha-open$ set U containing x such that $U\subseteq A$. The set of all $\alpha-interior$ points of A is said to be $\alpha-interior$ of A $[Mashhour\ \&\ El\text{-}Deeb;\ 1983]$ and denoted by $Int_{\alpha}(A)$. We denote the family of all $\alpha-open$ sets of (X,τ) by τ^{α}. It is shown in $[Njastad;\ 1965]$ $(see\ also\ Ohba\ \&\ Umehara;\ 2000)$ that each of $\tau\subseteq\tau^{\alpha}$ and τ^{α} is a topology on X.

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2. Alpha-Weakly Continuous Mappings

Definition 1. A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be $\alpha-weakly$ continuous if for each $x\in X$ and each open sets V containing $f(x)$, there exists an $\alpha-open$ set U containing x such that $f(U)\subseteq Cl_{\alpha}(V)$.

Definition 2. $[Mashhour\ \&\ El\text{-}Deeb;\ 1983]$. A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be $\alpha-continuous$ if $f^{-1}(V)\in\tau^{\alpha}$ for every $V\in\sigma$, and, equivalently, if for each $x\in X$ and each open sets V of Y containing $f(x)$, there exists $U\in\tau^{\alpha}$ with $x\in U$ such that $f(U)\subseteq V$.

We remark that every $\alpha-continuous$ mapping is $\alpha-weakly$ continuous, but the converse is not true as the following example shows.

Example 3. Let X and Y be both the set of real numbers. Let τ be the usual topology for X and σ the cocountable topology for Y. Then the identity mapping $f:(X,\tau)\to(Y,\sigma)$ is $\alpha-weakly$ continuous and not $\alpha-continuous$. Firstly we show that f is not $\alpha-continuous$. Let $\sqrt{2}\in I =$ the set of irrational numbers. Then $I\in\sigma$ as $R-I$ is countable. Let $U\subseteq I$ such that $\sqrt{2}\in U$. Then $Int\left[Cl\left(Int(U)\right)\right]=\phi$. Hence $U\notin\tau^{\alpha}$. Thus f is not $\alpha-continuous$.
Now we show that f is $\alpha-weakly$ continuous. Let $x\in V$ and $V\in\sigma$. Then $R-V$ is countable. We show that $Cl_{\alpha}(V)=Y$. Let $y\in W\in\sigma^{\alpha}$. If W is countable, then $Int(W)=\phi$. Hence $Int\left[Cl\left(Int(W)\right)\right]=\phi$. This shows that $W\notin\sigma^{\alpha}$. Hence W is uncountable. Since $Y-V$ is countable. Hence $W\cap V\neq\phi$. It shows that $Cl_{\alpha}(V)=Y$. Now by definition it follows that f is $\alpha-weakly$ continuous.

Definition 4. $[Noiri;\ 1987]$. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be weakly $\alpha-continuous$ continuous if for each $x\in X$ and each open set V of Y containing $f(x)$, there exists $U\in\tau^{\alpha}$ containing x such that $f(U)\subseteq Cl(V)$.

Theorem 5. Every $\alpha-weakly$ continuous function is weakly $\alpha-continuous$.

Proof. We know that $\sigma\subseteq\sigma^{\alpha}$. Hence $Cl_{\alpha}(V)\subseteq Cl(V)$. The theorem follows.

Theorem 6. A mapping $f:(X,\tau)\to(Y,\sigma)$ is $\alpha-weakly$ continuous if and only if for every open set V in Y, $f^{-1}(V)\subseteq Int_{\alpha}\left[f^{-1}\left(Cl_{\alpha}(V)\right)\right]$.

Proof. Let $x \in X$ and V an open set containing $f(x)$. Then $x \in f^{-1}(V) \subseteq Int_{\alpha}\left[f^{-1}\left(Cl_{\alpha}(V)\right)\right]$. Put $U = Int_{\alpha}\left[f^{-1}\left(Cl_{\alpha}(V)\right)\right]$. Then U is $\alpha - open$ and $f(U) \subseteq Cl_{\alpha}(V)$. Conversely, let V be an open set of Y and $x \in f^{-1}(V)$. Then there exists an open set U in X such that $x \in U$ and $f(U) \subseteq Cl_{\alpha}(V)$. Therefore, we have $x \in U \subseteq f^{-1}\left[\left(Cl_{\alpha}(V)\right)\right]$ and hence $x \in Int_{\alpha}\left[f^{-1}\left(Cl_{\alpha}(V)\right)\right]$. This proves that $f^{-1}(V) \subseteq Int_{\alpha}\left[f^{-1}\left(Cl_{\alpha}(V)\right)\right]$.

Theorem 7. Let $f:(X,\tau) \to (Y,\sigma)$ be $\alpha - weakly$ continuous. If V is a clopen (both closed and open) subset of Y such that $f(x) \in V$, then $f^{-1}(V)$ is clopen in (X,τ^{α}).

Proof. Let $x \in X$ and V be a clopen subset of (Y,σ) such that $f(x) \in V$. Then there exists $U \in \tau^{\alpha}$ containing x such that $f(U) \subseteq Cl_{\alpha}(V)$. Since $\sigma \subseteq \sigma^{\alpha}$ implies every clopen subset of (Y,σ) is also a clopen subset of (Y,σ^{α}). Hence $x \in U$ and $f(U) \subseteq V$ and so $x \in U \subseteq f^{-1}(V)$. This shows that $f^{-1}(V) \in \tau^{\alpha}$. Since $Y - V$ is a clopen in (Y,σ), so $f^{-1}(Y-V) \in \tau^{\alpha}$. But $f^{-1}(Y-V) = X - f^{-1}(V)$. Therefore $f^{-1}(V)$ is closed in (X,τ^{α}). Hence $f^{-1}(V)$ is an $\alpha - clopen$ (both $\alpha - closed$ and $\alpha - open$) set in (X,τ^{α}).

Theorem 8. The following are equivalent for a mapping $f:(X,\tau) \to (Y,\sigma)$.

(1) f is $\alpha - weakly$ continuous.

(2) $f^{-1}(V) \subseteq Int_{\alpha}\left[f^{-1}\left(Cl_{\alpha}(V)\right)\right]$ for every open subset V of (Y,σ).

(3) $Cl_{\alpha}\left[f^{-1}\left(Int_{\alpha}(V)\right)\right] \subseteq f^{-1}(V)$ for every *closed* subset V of (Y,σ).

Proof. $(1) \Leftrightarrow (2)$: Follows from Theorem 6.

$(2) \Rightarrow (3)$: Let V be a closed subset of (Y,σ). Then $Y - V$ is an open set in (Y,σ).
So by hypothesis $f^{-1}(Y-V) \subseteq Int_{\alpha}\left[f^{-1}\left(Cl_{\alpha}(Y-V)\right)\right] = Int_{\alpha}\left[f^{-1}\left(Y - Int_{\alpha}(V)\right)\right]$
$= X - Cl_{\alpha}\left[f^{-1}\left(Int_{\alpha}(V)\right)\right]$. Thus $Cl_{\alpha}\left[f^{-1}\left(Int_{\alpha}(V)\right)\right] \subseteq f^{-1}(V)$.

$(3) \Rightarrow (1)$: Let $x \in X$ and let $f(x) \in V \in \sigma$. So $Y - V$ is a *closed* set in (Y,σ). So by hypothesis $Cl_{\alpha}\left[f^{-1}\left(Int_{\alpha}(Y-V)\right)\right] \subseteq f^{-1}(Y-V)$. Thus $x \notin Cl_{\alpha}\left[f^{-1}\left(Int_{\alpha}(Y-V)\right)\right]$. Hence there exists $U \in \tau^{\alpha}$ such that $x \in U$ and $U \cap f^{-1}\left(Int_{\alpha}(Y-V)\right) = \phi$ which implies that $f(U) \cap Int_{\alpha}(Y-V) = \phi$.

$\Rightarrow f(U) \subseteq Y - Int_{\alpha}(Y-V) \Rightarrow f(U) \subseteq Cl_{\alpha}(V)$. This shows that f is $\alpha - weakly$ continuous.

Theorem 9. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\mu)$ be any mappings and let $gof:(X,\tau)\to(Z,\mu)$ be the composition.

(1) If $g:(X,\tau^{\alpha})\to(Y,\sigma^{\alpha})$ is an open surjection and gof is $\alpha-weakly$ continuous, then g is $\alpha-weakly$ continuous.

(2) If $g:(X,\tau^{\alpha})\to(Y,\sigma^{\alpha})$ is continuous and g is $\alpha-weakly$ continuous, then gof is $\alpha-weakly$ continuous.

Proof. (1) Let $y\in Y$. Since f is a surjection, there exists $x\in X$ such that $f(x)=y$. Let $V\in\mu$ contain $(gof)(x)$. Since gof is $\alpha-weakly$ continuous, there exists, $U\in\tau^{\alpha}$ containing x such that $(gof)(U)\subseteq Cl_{\alpha}(V)$. By hypothesis, it follows that $W=f(U)\in\sigma^{\alpha}$ and contains $f(x)=y$. Thus $g(W)\subseteq Cl_{\alpha}(V)$. Hence g is $\alpha-weakly$ continuous.

(2) Let $x\in X$ and $W\in\mu$ such that $(g\circ f)(x)=g(f(x))\in W$. Let $y=f(x)$. Since g is $\alpha-weakly$ continuous. So there exists $V\in\sigma^{\alpha}$ such that $g(y)\in V$ and $g(V)\subseteq Cl_{\alpha}(W)$. Let $U=g^{-1}(V)$. Then U is open in (X,τ^{α}) as f is $\alpha-continuous$. Now $(g\circ f)(U)=g\left(f\left(f^{-1}(V)\right)\right)\subseteq g(V)$. Then $x\in U\in\tau^{\alpha}$ and $(g\circ f)(U)\subseteq Cl_{\alpha}(W)$. Hence $g\circ f$ is $\alpha-weakly$ continuous

Theorem 10. Let $f:X\to Y$ be a mapping and $g:X\to X\times Y$ be the graph mapping of f, given by $g(x)=(x,f(x))$ for every point $x\in X$. Then f is $\alpha-weakly$ continuous if and only if g is $\alpha-weakly$ continuous.

Proof. Necessity. Suppose that f is $\alpha-weakly$ continuous. Let $x\in X$ and $g(x)\in W\in\tau\times\sigma$. There exist $U_1\in\tau$ and $V\in\sigma$ such that $(x,f(x))\in U_1\times V\subseteq W$. Since f is $\alpha-weakly$ continuous, there exists $U_2\in\tau^{\alpha}$ containing x such that $f(U_2)\subseteq Cl_{\alpha}(V)$. Put $U=U_1\cap U_2$, then we have $x\in U\in\tau^{\alpha}$ and $g(U)\subseteq Cl_{\alpha}(W)$. This indicates that g is $\alpha-weakly$ continuous.
Sufficiency. Suppose that g is $\alpha-weakly$ continuous. Let $x\in X$ and V be any open set containing $f(x)$. Then $X\times V$ is an open set in $X\times Y$ containing $g(x)$. Since g is $\alpha-weakly$ continuous, there exists an $\alpha-open$ set U in X containing x such that $g(U)\subseteq Cl_{\alpha}(X\times V)$. It follows from Lemma 4 of $[Noiri;1978]$, that $Cl_{\alpha}(X\times V)\subseteq X\times Cl_{\alpha}(V)$. Since g is the graph mapping of f, we have $f(U)\subseteq Cl_{\alpha}(V)$. This shows that f is $\alpha-weakly$ continuous.

Let $\{(x_\lambda,\tau_\lambda):\lambda\in\Lambda\}$ and $\{(y_\lambda,\sigma_\lambda):\lambda\in\Lambda\}$ be any two families of spaces with the same index set Λ. Let $f_\lambda:(X_\lambda,\tau_\lambda)\to(Y_\lambda,\sigma_\lambda)$ be a function for each $\lambda\in\Lambda$. Let $f:(\prod X_\lambda,\prod\tau_\lambda)\to(\prod Y_\lambda,\prod\sigma_\lambda)$ denote the product function defined by $f(x_\lambda:\lambda\in\Lambda)=(f(x_\lambda):\lambda\in\Lambda)$ for every $(x_\lambda:\lambda\in\Lambda)\in\prod X_\lambda$. Moreover, let $p_\mu:\prod_{\lambda\in\Lambda}X_\lambda\to X_\mu$ and $q_\mu:\prod_{\lambda\in\Lambda}Y_\lambda\to Y_\mu$ be the natural projections. Then, we have the following result.

Theorem 11. The product function $f:(\prod X_\lambda,\prod\tau_\lambda)\to(\prod Y_\lambda,\prod\sigma_\lambda)$ is $\alpha-weakly$ continuous if and only if $f_\lambda:(X_\lambda,\tau_\lambda)\to(Y_\lambda,\sigma_\lambda)$ is $\alpha-weakly$ continuous for each $\lambda\in\Lambda$.

Proof. Necessity. Suppose that that f is $\alpha-weakly$ continuous. Let μ be an arbitrary fixed index of Λ. Since q_μ is continuous, by Theorem 9 $q_\mu\circ f=f_\mu\circ p_\mu$ is $\alpha-weakly$ continuous. Moreover, p_μ is an open continuous surjection and hence by Theorem 9 f_μ is $\alpha-weakly$ continuous.

Sufficiency. Suppose that f_λ is $\alpha-weakly$ continuous for each $\lambda\in\Lambda$. Let $x=(x_\lambda:\lambda\in\Lambda)\in\prod X_\lambda$ and $f(x)\in W\in\prod\sigma_\lambda$. There exists a basic open set $\prod V_\lambda$ such that $f(x)\in\prod V_\lambda\subseteq W$ and $\prod V_\lambda=\prod_{i=1}^{n}V_{\lambda_i}\times\prod_{\lambda\neq\lambda_i}^{n}Y_{\lambda_i}$, where $V_\lambda\in\sigma_\lambda$ for each $\lambda=\lambda_1,\lambda_2,...,\lambda_n$. Since f_λ is $\alpha-weakly$ continuous, for each $\lambda=\lambda_1,\lambda_2,...,\lambda_n$ there exists $U_\lambda\in\tau_\lambda^\alpha$ containing x_λ such that $f(U_\lambda)\subseteq Cl_\alpha(V_\lambda)$. Now, let us put $U=\prod_{i=1}^{n}U_{\lambda_i}\times\prod_{\lambda\neq\lambda_i}^{n}X_{\lambda_i}$, then we have $x\in U\in(\prod\tau_\lambda)^\alpha$ and $f(U)\subseteq Cl_\alpha(W)$. This indicates that f is $\alpha-weakly$ continuous.

Definition 12. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be weakly continuous if for each $x\in X$ and each $V\in\sigma$ containing $f(x)$, there exists $U\in\tau$ containing x such that $f(U)\subseteq Cl(V)$.

Every weakly continuous function is $\alpha-weakly$ continuous but the converse is not true by the following example.

Example 13. Let $X=Y=\{a,b,c\}$, $\tau=\{\phi,\{c\},X\}$ and $\sigma=\{\phi,\{a\},\{b\},\{a,b\},Y\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity function. Then f is $\alpha-weakly$ continuous but it is not weakly continuous.

Lemma 14. A function $f:(X,\tau)\to(Y,\sigma)$ is $\alpha-weakly$ continuous if and only if $f:(X,\tau^\alpha)\to(Y,\sigma)$ is weakly continuous.

Theorem 15. A function $f:(X,\tau)\to(\prod Y_\lambda,\prod\sigma_\lambda)$ is $\alpha-weakly$ continuous if and only if $q_\lambda\circ f:(X,\tau)\to(Y_\lambda,\sigma_\lambda)$ is $\alpha-weakly$ continuous for each $\lambda\in\Lambda$.

Proof. This follows immediately from Lemma 14 and the fact that a function $f:(X,\tau)\to(\prod Y_\lambda,\prod\sigma_\lambda)$ is weakly continuous if and only if $q_\lambda\circ f:(X,\tau)\to(Y_\lambda,\sigma_\lambda)$ is continuous for each $\lambda\in\Lambda$.

Theorem 16. If $f:(X,\tau)\to(Y,\sigma)$ is $\alpha-weakly$ continuous and (Y,σ) is regular, then f is continuous.

Proof. Let x be any point of X and V any open set of (Y,σ) containing $f(x)$. Since (Y,σ) is regular, there exists $W\in\sigma$ such that $f(x)\in W\subseteq Cl(W)\subseteq V$. Since f is $\alpha-weakly$ continuous, there exists $U\in\tau^\alpha$ containing x such that $f(U)\subseteq Cl_\alpha(W)\subseteq Cl(W)\subseteq V$. Therefore, f is $\alpha-continuous$ by Theorem 1 in [Mashhour et al; 1983] and hence it is continuou by the remark of [Mashhour et al; 1983].

Theorem 17. If $f:X\to Y$ is an $\alpha-weakly$ continuous mapping and Y is Hausdorff, then the graph $G(f)$ is an $\alpha-closed$ set of $X\times Y$.

Proof. Let $(x,y)\notin G(f)$. Then, we have $y\neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets W and V such that $f(x)\in W$ and $y\in V$. Since f is $\alpha-weakly$ continuous, there exists an $\alpha-open$ set U containing x such that $f(U)\subseteq Cl_\alpha(W)$. Since W and V are disjoint, we have $V\cap Cl_\alpha(W)=\phi$ and hence $V\cap f(U)=\phi$. This shows that $(U\times V)\cap G(f)=\phi$. It follows that $G(f)$ is $\alpha-closed$.

Definition 18. By an $\alpha-weakly$ continuous retraction, we mean an $\alpha-weakly$ continuous mapping $f:X\to A$, where $A\subseteq X$ and $f_A=\ f/A$ is the identity mapping on A.

Theorem 19. Let $A\subseteq X$ and $f:X\to Y$ be an $\alpha-weakly$ continuous retraction of X onto A. If X is a Hausdorff space, then A is an $\alpha-closed$ set in X .

Proof. Suppose that A is not an $\alpha-closed$ set in X . Then there exists a point $x\in Cl_\alpha(A)-A$. Since f is $\alpha-weakly$ continuous retraction, we have $f(x)\neq x$. Since X is Hausdorff, there exist disjoint open sets U and V such that $x\in U$ and $f(x)\in V$. Thus we get $U\cap Cl_\alpha(V)=\phi$. Now, let W be any $\alpha-open$ set in X containing x. Then $U\cap W$ is an $\alpha-open$ set containing x and hence

$(U \cap W) \cap A \neq \phi$ because $x \in Cl_\alpha(A)$. Let $y \in (U \cap W) \cap A$. Since $y \in A$, $f(y) = y \in U$ and hence $f(y) \notin Cl_\alpha(V)$. This gives that $f(W) \not\subset Cl_\alpha(V)$. This contradicts that f is $\alpha - weakly$ continuous. Hence A is $\alpha - closed$ in X .

Theorem 20. Let $f:(X,\tau) \to (Y,\sigma)$ be $\alpha - weakly$ continuous and A a subset of X such that if either $A \in PO(X,\tau)$ or $A \in SO(X,\tau)$. Then the restriction $f_A:(A,\tau_A) \to (Y,\sigma)$ is $\alpha - weakly$ continuous.

Proof. Since either $A \in PO(X,\tau)$ or $A \in SO(X,\tau)$. It follows from lemma 1.1 of $[Mashhour\ et\ al;1983]$ and $[\text{Reilly \& vamanamurthy;1984}]$ that $\tau^\alpha / A \subseteq (\tau/A)^\alpha$. Since f is $\alpha - weakly$ continuous, for each $x \in A$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau^\alpha$ containing x such that $f(U) \subseteq Cl_\alpha(V)$. Put $U_A = U \cap A$, then we have $x \in U_A \in (\tau/A)^\alpha$ and $(f/A)(U_A) \subseteq Cl_\alpha(V)$. This indicates that f/A is $\alpha - weakly$ continuous.

4. Alpha-Connected Spaces

Definition 21. A space X is said to be $\alpha - connected$ if X can not be written as the disjoint union of two non-empty $\alpha - open$ sets.
Every $\alpha - connected$ space is connected but the converse may not be true.

It is shown in Theorem 4 of $[Long$ et al;1973$]$ $(resp.\ Theorem\ 3\ of\ [Noiri;1974])$ that connectedness is invariant under almost continuous (resp. weakly continuous) surjections. It is also known that $S - cnnectedness$ is invariant under semi-continuous surjections. In $[Noiri\ \&\ Ahmad;1985]$ it is proved that the semi-weakly continuous image of an $S - connected$ space is connected. In $[Latif;1995]$ we prove that $S - connectedness$ is invariant semi-weakly semi-continuous mapping. However, we have the following.

Theorem 22. If X is an $\alpha - connected$ space and $f: X \to Y$ is $\alpha - weakly$ connected surjection, then Y is connected.

Proof. Suppose that Y is not connected. Then there exist two non-empty open sets V_1 and V_2 of Y such that $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = Y$. Hence, we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$, $f^{-1}(V_1) \cup f^{-1}(V_2) = X$ and $f^{-1}(V_1) \neq \phi \neq f^{-1}(V_2)$ because f is surjection. By Theorem 8, we have $f^{-1}(V_i) \subseteq Int_\alpha\left[f^{-1}(Cl_\alpha(V_i))\right]$, for $i = 1, 2$. Since each V_i is clopen and hence also $\alpha - clopen$. We obtain $f^{-1}(V_i) \subseteq Int_\alpha\left[f^{-1}(V_i)\right]$ and hence $f^{-1}(V_i)$ is $\alpha-open$ for $i=1,2$. This implies that X is not $\alpha-connected$. Therefore Y is connected.

Theorem 23. If X is an $\alpha-connected$ space and $f: X \to Y$ is $\alpha-continuous$ mapping with the closed graph, then f is constant.

Proof. Suppose that f is not constant. Then there exist two distinct points x_1, x_2 in X such that $f(x_1) \neq f(x_2)$. Since the graph $G(f)$ is closed and $(x_1, f(x_2)) \notin G(f)$, there exist open sets U and V containing x_1 and $f(x_2)$, respectively, such that $f(U) \cap V = \phi$. Since f is $\alpha-continuous$, U and $f^{-1}(V)$ are non-empty disjoint $\alpha-open$ sets. It follows that X is not $\alpha-connected$. Therefore f is constant.

Corollary 24. $[Thompson;1981]$. Let X be $\alpha-connected$. If $f: X \to Y$ is a continuous mapping with the closed graph, then f is constant.

Proof. Since every continuous mapping is $\alpha-continuous$, this is an immediate consequence of Theorem 23.

Definition 25. $[Maheshwari \& Thakur;1980]$. A function $f:(X,\tau) \to (Y,\sigma)$ is an $\alpha-irresolue$ mapping if and only if the inverse image of every $\alpha-open$ set in Y is an $\alpha-open$ set in X.

Theorem 26. If X is an $\alpha-connected$ space and $f: X \to Y$ is an $\alpha-irresolute$ mapping with the $\alpha-closed$ graph, then f is constant.

Proof. Suppose that f is not constant. Then there exist two distinct points x_1, x_2 in X such that $f(x_1) \neq f(x_2)$. Since the graph $G(f)$ is $\alpha-closed$ and $(x_1, f(x_2)) \notin G(f)$, there exist $\alpha-open$ sets U and V containing x_1 and $f(x_2)$, respectively, such that $f(U) \cap V = \phi$. Since f is $\alpha-irresolute$, U and $f^{-1}(V)$ are non-empty disjoint $\alpha-open$ sets. It follows that X is not $\alpha-connected$. Therefore f is constant.

5. Hausdorff and Urysohn Spaces

Definition 27. A space X is called a Urysohn space if for every pair of distinct points x and y in X, there exist open sets U and V in X such that $x \in U, y \in V$ and $Cl(U) \bigcap Cl(V) = \phi$.

Theorem 28. Let $f:(X,\tau)\to(Y,\sigma)$ be an $\alpha-weakly$ continuous injection and (Y,σ^{α}) be a Urysohn space. Then (X,τ^{α}) is a $T_2-space.$

Proof. For any distinct points $x_1,x_2\in X,$ we have $f(x_1)\neq f(x_2)$ because f is injection. Since Y is Urysohn, there exist open sets V_1 and V_2 in Y such that $f(x_1)\in V_1,$ $f(x_2)\in V_2$ and $Cl(V_1)\cap Cl(V_2)=\phi.$ We know that $\tau\subseteq\tau^{\alpha}$ which implies that $Cl_{\alpha}(V_i)\subseteq Cl(V_i)$ for $i=1,2.$ It follows that $Cl_{\alpha}(V_1)\cap Cl_{\alpha}(V_2)=\phi.$ Hence we have $Int_{\alpha}\left[f^{-1}(Cl_{\alpha}(V_1))\right]\cap Int_{\alpha}\left[f^{-1}(Cl_{\alpha}(V_2))\right]=\phi.$ Since f is $\alpha-weakly$ continuous, so by Theorem 8, we have $x_j\in f^{-1}(V_j)\subseteq Int_{\alpha}\left[f^{-1}(Cl_{\alpha}(V_j))\right]$ for $j=1,2.$ This implies that (X,τ^{α}) is $T_2.$

Theorem 29. Let $f:(X,\tau)\to(Y,\sigma)$ be an $\alpha-weakly$ continuous injection mapping and (Y,σ) be a $T_2-space.$ Then the graph $G(f)$ is an $\alpha-closed$ set of $X\times Y.$

Proof. Let $(x,y)\notin G(f).$ Then, we have $y\neq f(x).$ Since (Y,σ) is $T_2,$ there exist disjoint open sets S and T such that $f(x)\in S$ and $y\in T.$ Since f is $\alpha-weakly$ continuous, there exists an $\alpha-open$ set R containing x such that $f(R)\subseteq Cl_{\alpha}(S).$ Since S and T are disjoint, we have $T\cap Cl_{\alpha}(S)=\phi$ and hence $T\cap f(R)=\phi$. This shows that $(R\times T)\cap G(f)=\phi.$ It follows that $G(f)$ is $\alpha-closed$.

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References

1. Raja M. Latif, On semi-weakly semi-continuous mappings, Punjab University Journal of Mathematics, Vol. 28 (1995), 22 – 29.
2. P.E. Long and D.A. Carnahan, Comparing almost continuous functions, Proc. Amer. Math. Soc., 38 (1973), 413 – 418.
3. S.N. Maheshwari and S.S. Thakur, On $\alpha-irresolute$ mappings, Tamkang J. Math. 11 (1980), no, 2, 209 – 214.
4. A.S. Mashhour, I.A. Hasanein, and S.N. El-Deeb, $\alpha-continuous$ and $\alpha-open$ mappings, Acta Math. Hungar, 41 (1983), no. 3 – 4, 213 – 218.
5. O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961 – 970.
6. T. Noiri, On weakly continuous mappings, Proc. Amer. Math. Soc., 46 (1974), 120 – 124.

7. T. Noiri, A characterization of almost-regular spaces, Glasnik Matematicki, Vol. 13, No. 2 (1978), $335-338$.
8. T. Noiri and B. Ahmad, On semi-weakly continuous mappings, Kyungpook Math. J., 25 (1985), no. 2, $191-194$.
9. T. Noiri, Weakly $\alpha-continuous$ functions, Int. J. Math, Math, Sci. 10 (1987), no. 3, 483 -490.
10. T. Ohba and J. Umehara, A simple proof of τ^{α} being a topology, Mem. Fac. Sci. K *Kôchi* Univ. Ser. A Math. 21 (2000), $87-88$.
11. I.L. Reilly and M.K. Vamanamurthy, Connectedness and strong semi-continuity, Casopis Pest. Mat. 109 (1984), $261-265$.
12. T. Thompson, Characterizations of irreducible spaces. Kyungpook Math. J., 21 (1981), $191-194$.