Characterizations of α−I−Open Sets and α−I−Continuity

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Abstract

Given a space $(X, \tau, I)$ consisting of a nonempty set $X$ with topology $\tau$ and an ideal $I$ of subsets of $X$ which has a heredity and finite additivity properties. The main objective of this paper is to introduce the notion of $\alpha-I$-open set and discuss its numerous topological properties.

1. Introduction

In the beginning of ninetieth, Jankovic and Hamlett [Jankovic et al., 1992] have defined the concept of $I$-open set via the local function which was given by Vaidyanathaswamy [Vaidyanathaswamy, 1954]. The latter concept was also established utilizing the concept of an ideal whose topic in general topological spaces was treated in the classical text by [Kuratowski, 1933] in 1933. In 1992, Abd El-Monsef et al. [Abd El-Monsef et al., 1992] studied a number of properties of $I$-open sets as well as $I$-closed sets and $I$-continuous functions and investigated several of their properties. We devote this paper to generalize the $I$-openness by presenting the class of $\alpha-I$-open sets. The class of an $\alpha-I$-continuous function have been also established and studied. The connections between these new concepts with the corresponding types are discussed. Some of their characterizations and other numerous properties are studied.

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2. Preliminaries

Throughout the present paper, \((X, \tau)\) and \((Y, \sigma)\) are denoted by the topological spaces on which no separation axioms are assumed unless the needed ones will be explicitly stated. In \((X, \tau)\), the usual closure and the interior of any \(W \subseteq X\) will be denoted by \(\text{Cl}(W)\) and \(\text{Int}(W)\), respectively. The notion of an ideal topological space \((X, \tau, I)\) means a topological space \((X, \tau)\) with an ideal \(I\) on \(X\) which is a nonempty subclass of a power set \(P(X)\) having the heredity and finite additivity properties i.e. \(I \subseteq P(X)\) is an ideal on \(X\) if \(W \in I\) and \(S \subseteq W\), then \(S \in I\) and also if \(W \in I\) and \(S \in I\), then \(W \cup S \in I\) [Kuratowski, 1933]. Sometimes an ideal is called a dual filter [Jankovic et al., 1990]. For any \(W \subseteq X\) in \((X, \tau, I)\), \(W^*(I, \tau)\) or simply \(W^*(I)\) means the local function of \(W\) with respect to \(I\) and \(\tau\) and is defined by \(W^*(I, \tau) = \{x \in X : W \cap N \in I, N \in \tau(x)\}\) [Vaidyanathaswamy, 1954], where \(\tau(x)\) denotes the open-neighbourhood of \(x\). The simplest ideals are \(\{\phi\}\) and \(P(X)\) which satisfy \(\{\phi\} \subseteq I \subseteq P(X)\), for any ideal \(I\) on \(X\). Also, between the useful ideals is the collection of all nowhere dense subsets of \((X, \tau)\) which is an ideal and denoted by \(I_n\). i.e., \(I_n = \{W \subseteq X : \text{Int}[\text{Cl}(W)] = \phi\}\). For \((X, \tau, I)\), we say that the topology \(\tau\) is compatible with an ideal \(I\), denoted by \(\tau \sim I\), if the following holds: for every \(W \subseteq X\) and every \(x \in W\) there exists \(N \in \tau(x)\) such that \(N \cap W \in I\), then \(W \in I\) [Njastad, 1966]. \(W \subseteq X\) is said to be \(I\)−open [Jankovic et al., 1992] (resp. semi-open [Levine, 1963], \(\alpha\)−open [Njastad, 1965], \(\beta\)−open [Abd El-Monsef et al., 1983]) set if \(W \subseteq \text{Int}(W^*)\) (resp. \(W \subseteq \text{Cl}[\text{Int}(W)]\)), \(W \subseteq \text{Int}[\text{Cl}(\text{Int}(W))]\), \(W \subseteq \text{Cl}[\text{Int}(\text{Cl}(W))]\) while their complements are the
corresponding types of closeness. The classes of all previous types of openness in $(X, \tau)$ are denoted by $IO(X, \tau)$, $SO(X, \tau)$, $\tau''$ and $BO(X, \tau)$, respectively.

3. $\alpha-I$–Open Sets

**Definition 1.** In an ideal topological space $(X, \tau, I)$, $W \subseteq X$ is said to be an $\alpha-I$–open set if $W \subseteq Int \left[ Cl \left( Int(W^*) \right) \right]$, The complement set $(X-W)$ is called $\alpha-I$–closed. The collection of all $\alpha-I$–open (resp. $I$–open) sets of $(X, \tau)$ will be denoted by $\alpha IO(X, \tau)$ (resp. $IO(X, \tau)$).

**Remark 2.** $IO(X, \tau) \subseteq \alpha IO(X, \tau)$.

**Proposition 3.** Arbitrary union of $\alpha-I$–open sets is also $\alpha-I$–open.

**Proof.** Let $(X, \tau, I)$ be any ideal topological space and $W_i \in \alpha IO(X, \tau)$ for $i \in \nabla$, this means that for each $i \in \nabla$, $W_i \subseteq Int \left[ Cl \left( Int(W^*_i) \right) \right]$ and so

$$\bigcup_{i \in \nabla} W_i \subseteq \bigcup_{i \in \nabla} Int \left[ Cl \left( Int(W^*_i) \right) \right] \subseteq Int \left[ \bigcup_{i \in \nabla} Cl \left( Int(W^*_i) \right) \right] \subseteq Int \left[ Cl \left( \bigcup_{i \in \nabla} Int(W^*_i) \right) \right] \subseteq Int \left[ Cl \left( \bigcup_{i \in \nabla} W^*_i \right) \right].$$

Hence $\bigcup_{i \in \nabla} W_i \in \alpha IO(X, \tau)$.

**Lemma 4.** [Jankovic et al., 1990]. Let $(X, \tau)$ be a topological space with an ideal $I_n$ on $X$. Then $X$ coincides with its local function (i.e., $X = X^*$).

**Definition 5.** Let $X$ be a nonempty set and let $\tau \subseteq P(X)$. We say that $\tau$ is a supratopology on $X$ if $\phi, X \in \tau$ and $\tau$ is closed under arbitrary union.

**Theorem 6.** For an ideal topological space $(X, \tau, I_n)$, the class $\alpha IO(X, \tau)$ forms a supratopology.

**Proof.** Follows by the fact $\phi^* = \{ x \in X : \phi \cap N_x \notin I_n \} = \phi$ and both of Lemma 4 and Proposition 3.

The above Theorem leads us to present the following question: Does $\alpha IO(X, \tau)$ form a topology? The answer is no for the next remark.
Remark 7. A finite intersection of $\alpha-I$–open sets need not in general $\alpha-I$–open, as Example 8 shows.

Example 8. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a, b\}, \{a, b, c\}\}$ with $I = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. We deduce that the two sets $W_1 = \{a, b\}$ and $W_2 = \{b, c\}$ are $\alpha-I$–open while their intersection is not.

The preceding remark turns our attention to establish the next result.

Proposition 9. In an ideal topological space $(X, \tau, I)$, if $U \in \tau$ and $W \in \alpha IO(X, \tau)$, then $U \cap W$ is $\alpha-I$–open.

Proof. By hypothesis and the fact that $U \cap Cl(W) \subseteq Cl(U \cap W)$, we have, $U \cap Int(W) \subseteq (U \cap W)$ and $(U \cap W^*) \subseteq (U \cap W)^*$, it follows that $U \cap W \subseteq U \cap Int\left[ Cl\left( Int\left( W^*\right)\right)\right] \subseteq Int\left[ U \cap Cl\left( Int\left( W^*\right)\right)\right] \subseteq Int\left[ Cl\left( U \cap Int\left( W^*\right)\right)\right]$ $\subseteq Int\left[ Cl\left( U \cap W^*\right)\right] \subseteq Cl\left( Int\left( U \cap W^*\right)\right)$. Hence the result.

Theorem 10. The intersection of an $\alpha-I$–open set with an $\alpha$–open set is a $\beta$–open set.

Proof. Let $(X, \tau, I)$ be an ideal topological space, $W \in \alpha IO(X, \tau)$ and $G \in \tau^\alpha$. Then $W \cap G \subseteq Int\left[ Cl\left( Int\left( W^*\right)\right)\right] \cap Cl\left( Int\left( G\right)\right) \subseteq Cl\left[ Int\left( W^*\right) \cap Int\left( Cl\left( Int\left( G\right)\right)\right)\right]$ $= Cl\left[ Int\left( W^*\right) \cap Cl\left( Int\left( G\right)\right)\right] \subseteq Cl\left[ Int\left( Cl\left( W \cap G\right)\right)\right]$ $\subseteq Cl\left[ Cl\left( Cl\left( W \cap G\right)\right)\right] = Cl\left[ Cl\left( W \cap G\right)\right]$, which completes the proof.

Proposition 11. In an ideal topological space $(X, \tau, I)$, let $U \in \tau$ and $W \in \alpha IO(X, \tau)$. Then $U \cap W \in \alpha IO(U, \tau | U)$.

Proof. $U \cap W \subseteq U \cap Int\left[ Cl\left( Int\left( W^*\right)\right)\right] \subseteq Int\left[ Cl\left( U \cap Int\left( W^*\right)\right)\cap U\right]$
\[\text{Proposition 12.}\] Let \((X, \tau, I)\) be an ideal topological space and let \(W \subseteq U \subseteq X\), \(U \in \tau\) and \(W \in \alpha IO(U, \tau | U)\). Then \(W \in \alpha IO(X, \tau)\).

**Proof.** We notice that \(W \subseteq \text{Int}_U \left[ \text{Cl}_U \left( \text{Int}_U (W^*) \right) \right] \subseteq \text{Int} \left[ \text{Cl}_U \left( \text{Int}_U (W^*) \right) \right]\). Since \(\text{Int}_U (W^*)\) is an open set in \(U\), there exists \(V \in \tau\) such that \(\text{Int}_U (W^*) = U \cap V \in \tau\). Hence \(W \subseteq \text{Int} \left[ \text{Cl}_U \left( U \cap V \right) \right] = \text{Int} \left[ \text{Cl}(U \cap V) \right] \subseteq \text{Int} \left[ \text{Cl}(U \cap V) \right] = \text{Int} \left[ \text{Cl} \left( \text{Int}(W^*) \right) \right] \subseteq \text{Int} \left[ \text{Cl}(\text{Int}(W^*)) \right] \subseteq \text{Int} \left[ \text{Cl}(\text{Int}(W^*)) \right]\). Thus \(W \in \alpha IO(X, \tau)\).

Jankovic and Hamlett [Jankovic et al., 1992] have denoted by \(I*J\) the extension of an ideal \(I\) via other one \(J\) which are given on the same nonempty set \(X\) with respect to \((X, \tau)\) and defined as: \(I*J = \{A \subseteq X : A^*(I) \in J\}\). They showed also that \(I*J\) is an ideal on \(X\) and \(I \subseteq I*J\).

**Lemma 13.** [Jankovic et al., 1992]. Let \((X, \tau, I)\) be an ideal topological space with \(I_n \subseteq I\) and \(\tau \sim I\). Then for any other ideal \(J\) on \(X\), \(\tau \sim (J*I)\).

For any \((X, \tau, I)\), the authors in [Jankovic et al., 1992] denote \((I*I_n)\) by \(\tilde{I}\) which is a compatible extension of \(I\) and it is also defined as \(\tilde{I} = \{A \subseteq X : \text{Int}(A^*) = \emptyset\}\).

**Theorem 14.** (1) Given \((X, \tau, I)\), any \(W \subseteq X\) is \(\alpha - I - \text{open}\). Then \(W \subseteq W^*(\tilde{I})\).

(2) Given \((X, \tau, I)\), \(W \subseteq X\) such that \(W \subseteq \text{Int} \left[ W^*(\tilde{I}) \right]\). Then \(W\) is \(\alpha - I - \text{open}\).
**Proof.** (1) Let \( x \not\in W^*(\bar{I}) \). Then there exists an open neighbourhood \( N \) of \( x \) such that \( \text{Int}(N \cap W)^*(I) = \emptyset \). Since \( N \cap W^*(I) \subseteq (N \cap W)^*(I) \) [Kuratowski, 1966], then \( N \cap \text{Int}[W^*(I)] \subseteq \text{Int}(N \cap W)^*(I) = \emptyset \). This means that \( x \not\in \text{Cl} \left[ \text{Int}(W^*) \right] \).

Hence \( x \not\in \text{Int} \left[ \text{Cl} \left( \text{Int}(W^*) \right) \right] \). Since \( W \) is \( \alpha-I \)-open. So we get \( x \not\in W \).

(2) Since \( I_o \subseteq \bar{I} \), then for any \( W \subseteq X \), \( W^*(\bar{I}) \subseteq W^*(I_o) = \text{Cl} \left[ \text{Int}(\text{Cl}(W)) \right] \) [Vaidyanathaswamy, 1960]. If we put \( W^* \) instead of \( W \), the last inclusion gives \( (W^*(\bar{I}))^*(\bar{I}) \subseteq \text{Cl} \left[ \text{Int}(\text{Cl}(W^*(\bar{I}))) \right] = \text{Cl} \left[ \text{Int}(W^*(\bar{I})) \right] \), for \( W^*(\bar{I}) \) is closed \([Jankovic et al., 1990, Theorem(2.3)]\). By \( \bar{I} \sim \tau \), we get \( W^*(\bar{I}) = (W^*(\bar{I}))^*(\bar{I}) \). Therefore, \( W^*(\bar{I}) \subseteq \text{Cl} \left[ \text{Int}(W^*(\bar{I})) \right] \subseteq W^*(\bar{I}) \). Hence \( W \subseteq \text{Int}[W^*(\bar{I})] \subseteq \text{Int}[\text{Cl}(\text{Int}(W^*(\bar{I})))] \subseteq \text{Int}[\text{Cl}(\text{Int}(W^*(I)))] \), for \( I \subseteq \bar{I} \).

Therefore, \( W \in \alpha IO(X, \tau) \) and the proof is complete.

4. On \( \alpha-I \)-Continuity

**Definition 15.** A function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is called \( \alpha-I \)-continuous if for every \( V \in \sigma \), \( f^{-1}(V) \in \alpha IO(X, \tau) \).

The following Theorem gives some characterizations of \( \alpha-I \)-continuity.

**Theorem 16.** For a function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \), the following are equivalent:

(i) \( f \) is \( \alpha-I \)-continuous.

(ii) The inverse image of each closed set in \((Y, \sigma)\) is \( \alpha-I \)-closed.

(iii) For each \( x \in X \) and each \( V \in \sigma \) containing \( f(x) \), there exists \( W \in \alpha IO(X, \tau) \) containing \( x \) such that \( f(W) \subseteq V \).

**Proof.** Routine.
Definition 17. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called $\beta$-continuous \cite{Abd El-Monsef et al., 1983} (resp. $\beta$-irresolute \cite{Mahmoud et al., 1989}) if the inverse image of each open set (resp. $\beta$-open set) in $(Y, \sigma)$ is $\beta$-open set (resp. $\beta$-open set) in $(X, \sigma)$.

Proposition 18. (1) Let $f: (X, \tau, I) \rightarrow (Y, \sigma, J)$ be $\alpha-I$-continuous and $g: (Y, \sigma, J) \rightarrow (Z, \mu)$ be continuous. Then $gof$ is $\alpha-I$-continuous.

(2) Let $f: (X, \tau, I) \rightarrow (Y, \sigma, J)$ be $\beta$-irresolute and $g: (Y, \sigma, J) \rightarrow (Z, \mu)$ be $\alpha-I$-continuous. Then $gof$ is $\beta$-continuous.

Proof. (1) Let $W \in \mu$. Then $(gof)^{-1}(W) = (f^{-1}g^{-1})(W) = f^{-1}(g^{-1}(W))$. Since $g^{-1}(W)$ is open as $g$ is continuous. Now since $f$ is $\alpha-I$-continuous. So $f^{-1}(g^{-1}(W))$ is $\alpha-I$-open. Hence $gof$ is $\alpha-I$-continuous.

(2) Let $W \in \mu$. Then $(gof)^{-1}(W) = (f^{-1}g^{-1})(W) = f^{-1}(g^{-1}(W))$. Since $g$ is $\alpha-I$-continuous. So $g^{-1}(W)$ is $\alpha-I$-open and hence $g^{-1}(W)$ is $\beta$-open. Now since $f$ is $\beta$-irresolute. So $f^{-1}(g^{-1}(W))$ is $\beta$-open. Hence $gof$ is $\beta$-continuous.

Theorem 19. The restriction of an $\alpha-I$-continuous function to an open set is also $\alpha-I$-continuous.

Proof. Let $f: (X, \tau, I) \rightarrow (Y, \sigma)$ be an $\alpha-I$-continuous function and $U \in \tau$. To show that $f|U$ is $\alpha-I$-continuous, let $V \in \sigma$, then $f^{-1}(V)\in \alpha IO(X, \tau)$. By Proposition 11, $(f|U)^{-1}(V) = U \cap f^{-1}(V) \subseteq U \cap Int\left[C\left(Int\left(f^{-1}(V)\right)\right]\right] \subseteq Int\left[C\left(U \cap Int\left(f^{-1}(V)\right)\right)\right] \subseteq Int\left[C\left(Int\left(U \cap (f^{-1}(V))\right)\right)\right]$. Hence $f|U$ is $\alpha-I$-continuous.
**Definition 20.** [Abd El-Monsef et al., 1992] A function \( f : (X, \tau, I) \to (Y, \sigma) \) is called \( I – \text{continuous} \) if the inverse image of each open set in \( (Y, \sigma) \) is an \( I – \text{open} \) set in \( (X, \tau, I) \).

**Theorem 21.** A function \( f : (X, \tau, I) \to (Y, \sigma) \) is \( \alpha – I – \text{continuous} \) if its restrictions by an open cover \( \{U_i : i \in \nabla\} \) of \( X \) are \( I – \text{continuous} \).

**Proof.** Let \( V \in \sigma \). Then \( f^{-1}(V) = \bigcup_{i \in \nabla} \left[ f^{-1}(V) \cap U_i \right] = \bigcup_{i \in \nabla} \left[ (f|U_i)^{-1}(V) \right] \). By hypothesis, \( (f|U_i)^{-1}(V) \) is \( I – \text{open} \) and so is \( \alpha – I – \text{open} \). Then by Proposition 3, \( f^{-1}(V) \) is \( \alpha – I – \text{open} \). We conclude that \( f \) is \( \alpha – I – \text{continuous} \).

**Theorem 22.** Prove that any function \( f : (X, \tau, I) \to (Y, \sigma) \) is \( \alpha – I – \text{continuous} \) if and only if its graph function is \( \alpha – I – \text{continuous} \).

**Proof.** Let \( f : (X, \tau, I) \to (Y, \sigma) \) be \( \alpha – I – \text{continuous} \), \( x \in X \) and \( H \) an open set in \( X \times Y \) containing \( g(x) \), where \( g : X \to X \times Y \) is defined by \( g(x) = (x, f(x)) \), for every \( x \in X \). Then there exist \( U \in \tau \) and \( V \in \sigma \) such that \( g(x) = (x, f(x)) \in U \times V \subseteq H \). By hypothesis, there exists \( W \in \alpha IO(X, x) \) such that \( f(W) \subseteq V \). Proposition 9 shows that \( x \in U \cap W \in \alpha IO(X, \tau) \) which is contained in \( U \). So, \( (U \cap W) \times V \subseteq U \times V \subseteq H \) and hence \( g(U \cap W) \subseteq H \). This means that \( g : (X, \tau, I) \to (X \times Y, \tau_{X \times Y}) \) is \( \alpha – I – \text{continuous} \).

Conversely, let \( x \in X \) and \( V \in \sigma \) containing \( f(x) \), then \( X \times V \in \tau_{X \times Y} \). \( \alpha – I – \text{continuity} \) of \( g \) shows that there exists \( W \in \alpha IO(X, x) \) such that \( g(W) \subseteq X \times V \). This projection of both sides gives \( f(W) \subseteq V \). Hence \( f \) is \( \alpha – I – \text{continuous} \) and this completes the proof.

**Definition 23.** A space \((X, \tau)\) is \( \alpha – I – \text{compact} \) if for every \( \alpha – I – \text{open} \) cover \( \{W_i : i \in \nabla\} \), there exists a finite subset \( \nabla_0 \) of \( \nabla \) such that \( X – \bigcup \{U_i : i \in \nabla_0\} \in I \).
Lemma 24. [Newcomb, 1967]. For any function \( f : (X, \tau, I) \to (Y, \sigma) \), \( f(I) \) is an ideal on \( Y \).

Theorem 25. Let \( f : (X, \tau, I) \to (Y, \sigma) \) be an \( \alpha - I \)-continuous surjective function. Suppose that \( (X, \tau) \) is \( \alpha - I \)-compact. Then \( (Y, \sigma) \) is \( \alpha - f(I) \)-compact.

Proof. Let \( f : (X, \tau, I) \to (Y, \sigma) \) be an \( \alpha - I \)-continuous surjection and \( \{V_i : i \in \mathbb{V}\} \) be an open cover of \( Y \). Then \( \{f^{-1}(V_i) : i \in \mathbb{V}\} \) is an \( \alpha - I \)-open cover of \( X \). From the assumption, there exists a finite subset \( \mathbb{V}_0 \) of \( \mathbb{V} \) such that \( X - \bigcup \{f^{-1}(V_i) : i \in \mathbb{V}_0\} \in I \). Therefore, \( Y - \bigcup \{V_i : i \in \mathbb{V}_0\} \in f(I) \) which shows that \( (Y, \sigma) \) is \( \alpha - f(I) \)-compact.

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