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Some Distributional Results in Two Correlated Chisquare Variables

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Abstract Ratios of two independent chisquare variables are widely used in statistical tests of hypotheses. This paper ushers a horizon of statistical investigation where the assumption of independence is not met. Moments of the product and ratio of correlated chisquare variables are outlined. Distributions of the sum and product of two correlated chisquares are also derived.

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1. Introduction

Let $X_1, X_2, \cdots X_N\ (N > 2)$ be a two-dimensional independent normal random vectors with mean vector $\bar{X} = (\bar{X}_1, \bar{X}_2)'$ so that the sums of squares and cross product matrix is given by $\sum_{j=1}^{N}(X_j - \bar{X})(X_j - \bar{X})' = A$. Let the matrix A for the bivariate case be denoted by

$A = (a_{ik}), i = 1,2; k = 1,2$ where $a_{ii} = ms_i^2 = \sum_{j=1}^{N}(X_{ij} - \bar{X}_i)^2, m = N - 1, (i = 1,2)$ and

$a_{12} = \sum_{j=1}^{N}(X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) = mrs_1s_2$. Also let the elements of the matrix $\Sigma = (\sigma_{ik})$,

$i = 1,2; k = 1,2$ where $\sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2, \sigma_{12} = \rho\sigma_1\sigma_2$ with $\sigma_1 > 0, \sigma_2 > 0$. The quantity ρ $(-1 < \rho < 1)$ is the product moment correlation coefficient between X_1 and X_2.

Fisher (1915) derived the distribution of the bivariate Wishart matrix in order to study the distribution of correlation coefficient for a bivariate normal sample. Wishart (1928) obtained the distribution of Wishart matrix as the joint distribution of sample variances and covariances from multivariate normal population. The bivariate matrix A is said to have a Wishart distribution with parameters $m = N - 1$ and $\Sigma(2\times 2) > 0$, written as $A \sim W_2(m, \Sigma)$ if its probability density function is given by

$$f_*(A) = \frac{|A|^{(m-3)/2} \exp\left(-\frac{1}{2} tr\Sigma^{-1}A\right)}{2^m \sqrt{\pi}\, |\Sigma|^{m/2} \prod_{i=1}^{2}\Gamma\left((m+1-i)/2\right)}, \quad A > 0, m > 2$$

(See e.g. Anderson, 2003, 252). The pdf of the elements of A can be written as

$$f(a_{11},a_{22},a_{12})=\frac{(1-\rho^2)^{-m/2}(\sigma_1\sigma_2)^{-m}}{2^m\sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)}\left(a_{11}a_{22}-a_{12}^2\right)^{(m-3)/2}$$
$$\times\exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2}-\frac{a_{22}}{2(1-\rho^2)\sigma_2^2}+\frac{\rho a_{12}}{2(1-\rho^2)\sigma_1\sigma_2}\right) \quad (1.1)$$

$a_{11}>0, a_{22}>0, -\infty<a_{12}<\infty, m>2, -1<\rho<1$ (Anderson, 2003, 123).

Because of its important role in multivariate statistical analysis, various authors have given different derivations. See the references in Gupta and Nagar (2000, 87-88) for a good update on the moments of Wishart distribution.

In this paper we deduce some properties of a bivariate chisquare distribution of $U=mS_1^2/\sigma_1^2$ and $V=mS_2^2/\sigma_2^2$ introduced by Joarder (2005a). In particular, moments of the product and ratio of two correlated chisquare variables are also derived. Distributions of the sum and product of correlated chisqaures are also derived.

Ratios of two independent chisquares are widely used in statistical tests of hypotheses. This paper ushers a horizon of statistical investigation where the assumption of independence is not met. In corollary 2.6, we have provided moments of the ratio of correlated chisquare variables. Further investigation is needed to utilize these moments to derive the distribution of the ratio of chisquares or many other interesting properties by the inverse Mellin transformation along Provost (1986).

In case the correlation coefficient $\rho=0$, the probability density function (pdf) of U and V becomes that of the product of two independent chisquare variables each with m degrees of freedom, and the product moments obtained for the correlated case are found to be in agreement with the special situation of independence. We refer to Kotz, Balakrishnan and Johnson (2000) for other type of bivariate chisquare distribution.

For any nonnegative integer k, the following notations will be used in sequel:

$a_{\{k\}}=a(a+1)(a+2)\cdots(a+k-1),$
$a^{\{k\}}=a(a-1)\cdots(a-k+1).$

2. The Density Function of the Bivariate Chisquare Distribution

The first three of the theorems are due to Joarder (2005a).

Theorem 2.1 The random variables U and V are said to have a correlated bivariate chisquare distribution each with m degrees of freedom, if its probability density function is given by

$$f(u,v)=\frac{(uv)^{m/2-1}\,e^{\frac{-(u+v)}{2(1-\rho^2)}}}{2^m\sqrt{\pi}\,\Gamma(m/2)(1-\rho^2)^{m/2}}\sum_{k=0}^{\infty}\left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^k\frac{\Gamma\left((k+1)/2\right)}{k!\Gamma\left((k+m)/2\right)}, \quad (2.1)$$

$m = N - 1 > 2$, $-1 < \rho < 1$. Note that the above pdf can also be written as

$$f(u,v) = \frac{(1-\rho^2)^{-m/2}(uv)^{m/2-1}}{2^m\ \Gamma^2(m/2)} \exp\left(\frac{-(u+v)}{2(1-\rho^2)}\right)\ E\left[\exp\left(\frac{\rho\sqrt{uv}\ \sqrt{Y}}{1-\rho^2}\right)\right]$$

where Y has a beta distribution $B(a,b)$ with parameters $a = 1/2$ and $b = (m-1)/2$.

In case $\rho = 0$, the pdf of the joint probability distribution in Theorem 2.1, would be that of the product of two independent chisquare random variables given by

$$f(u,v) = \frac{(uv)^{m/2-1}\ e^{-(u+v)/2}}{2^m\ \Gamma^2(m/2)}\ ,\ u > 0, v > 0.$$

Theorem 2.2 For $m > \max(a,b)$ and $-1 < \rho < 1$, the (a,b) th product moment $E(U^a V^b)$ of the distribution of U and V, is given by

$$\mu'(a,b;\rho) = \frac{2^{a+b}(1-\rho^2)^{a+b+m/2}}{\sqrt{\pi}\ \Gamma(m/2)} \sum_{k=0}^{\infty} \frac{(2\rho)^k}{k!}\ \Gamma\left(\frac{k+m}{2}+a\right)\Gamma\left(\frac{k+m}{2}+b\right)\frac{\Gamma\left((k+1)/2\right)}{\Gamma\left((k+m)/2\right)}.$$

The following theorem is due to Joarder (2006).

Theorem 2.3 Let $b_{k,m} = \frac{2^k}{k!}\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k+m}{2}\right)$ and $L(m,\rho) = \sqrt{\pi}\ \Gamma(m/2)\left(1-\rho^2\right)^{-m/2}$.

Then for $m > 2, -1 < \rho < 1$ and, we have

$$(i)\sum_{k=0}^{\infty} \rho^k b_{k,m} = L(m,\rho)$$

$$(ii)\sum_{k=0}^{\infty} k b_{k,m} = m\rho^2(1-\rho^2)^{-1}L(m,\rho) = w_1(m,\rho)L(m,\rho)$$

$$(iii)\sum_{k=0}^{\infty} k^{\{2\}}\rho^k b_{k,m} = \left(m(m+1)\rho^4 + m\rho^2\right)(1-\rho^2)^{-2}L(m,\rho) = w_{(2)}(m,\rho)L(m,\rho)$$

$$(iv)\sum_{k=0}^{\infty} k^{\{3\}}\rho^k b_{k,m} = w_{(3)}(m,\rho)L(m,\rho) \text{ where}$$

$$w_3(m,\rho) = \left((m^3+3m^2+2m)\rho^6 + (3m^2+6m)\rho^4\right)(1-\rho^2)^{-3}$$
$$= \left(m(m+1)(m+2)\rho^6 + 3m(m+2)\rho^4\right)(1-\rho^2)^{-3}L(m,\rho) = w_{(3)}(m)L(m,\rho)$$

$$(v)\sum_{k=0}^{\infty} k^{\{4\}}\rho^k\ b_{k,m} = w_{(4)}(m,\rho)L(m,\rho) \text{ where}$$

$$w_{(4)}(m,\rho) = \left[(m^4+6m^3+11m^2+6m)\rho^8 + (6m^3+30m^2+36m)\rho^6 + (3m^2+6m)\rho^4\right](1-\rho^2)^{-4}$$
$$= \left[m(m+1)(m+2)(m+3)\rho^8 + 6m(m+2)(m+3)\rho^6 + 3m(m+2)\rho^4\right](1-\rho^2)^{-4}$$

$$(vi)\sum_{k=0}^{\infty} k^2\rho^k b_{k,m} = (m^2\rho^4 + 2m\rho^2)(1-\rho^2)^{-2}L(m,\rho) = w_2(m,\rho)L(m,\rho)$$

$(vii)\sum_{k=0}^{\infty} k^3\rho^k b_{k,m} = (m^3\rho^6 + (6m^2+4m)\rho^4 + 4m\rho^2)(1-\rho^2)^{-3}L(m,\rho) = w_3(m,\rho)L(m,\rho)$

$(viii)\sum_{k=0}^{\infty} k^4\rho^k \; b_{k,m} = w_4(m,\rho)L(m,\rho)$ where

$w_4(m,\rho) = \left[m^4\rho^8 + (12m^3+16m^2+8m)\rho^6 + (28m^2+32m)\rho^4 + 8m\rho^2\right](1-\rho^2)^{-4}.$

For any nonnegative integer a, $\mu'(a,0;\rho) = \dfrac{2^a\Gamma(m/2+a)}{\Gamma(m/2)} = 2^a(m/2)_{\{a\}}$ which is the a th moment of usual chisquare distribution with m degrees of freedom. Similarly, $\mu'(0,b;\rho) = 2^b(m/2)_{\{b\}}$ **.** When both orders a and b are negative, it is difficult to get closed form expressions for product moments $\mu'(a,b;\rho) = E(U^aV^b)$. If $\rho = 0$, then $\mu'(a,b;0) = \mu'(a,0;0)\mu'(0,b;0)$. It is observed that if b is a nonnegative integer, then $\mu'(1,b;\rho) = 2^b(m/2)_{\{b\}}(m+2b\rho^2)$.

The following theorem follows from Theorem 2.2.

Theorem 2.4 Let U and V have the bivariate chisquare distribution with pdf given by Theorem 2.1. For (i) nonnegative integers a and b, with $m > 2$, or (ii) $a > 0, b < 0$ with $m + b > 0$, the (a,b) th product moment of the distribution of U and V, is given by

$$\mu'(a,b;\rho) = 2^{a+b}\left(\frac{m}{2}\right)_{\{b\}}\sum_{j=0}^{a}\binom{a}{j}(1+b-a+j)_{\{a-j\}}\left(\frac{m}{2}+a-j\right)_{\{j\}}\rho^{2(a-j)}$$

where $-1 < \rho < 1$.

Corollary 2.1 Let $\mu'(a,b,l;\rho) = E\left(S_1^{2a}S_2^{2b}R^l\right)$ be the product moments of the bivariate Wishart distribution with pdf given by (1.1). Then the (a,b)-th product moment of sample variances S_1^2 and S_2^2 can be calculated by

$$E\left(S_1^{2a}S_2^{2b}\right) = \mu'(a,b,0;\rho) = \frac{\sigma_1^{2a}\sigma_2^{2b}}{m^{a+b}}\mu'(a,b;\rho).$$

The following corollary is obvious from Theorem 2.2.

Corollary 2.2 Let U and V have a bivariate chisquare distribution with pdf given by (2.1). Also let $\mu(a,b) = E\left((U-EU)^a(V-EV)^b\right)$ be the centered product moment between U and V. If a and b are of the same sign, then

$(i)E(U^aV^b) = E(U^bV^a)$
$(ii)\mu(a,b) = \mu(b,a).$

Corollary 2.3 For $m > 2, -1 < \rho < 1$, the a th moment of $W = UV$ is given by

$$E(W^a) = \frac{4^a(1-\rho^2)^{2a}}{L(m,\rho)} \sum_{k=0}^{\infty} \left(\frac{k+m}{2}\right)_{\{a\}} \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+m}{2}+a\right) \frac{\Gamma\left((k+1)/2\right)}{\Gamma\left((k+m)/2\right)}$$

where $L(m,\rho)$ is defined in Theorem 2.3. In case a is an integer,

$$E(W^a) = 4^a \left(\frac{m}{2}\right)_{\{a\}} \sum_{j=0}^{a} \binom{a}{j} (1+j)_{\{a-j\}} \left(\frac{m}{2}+a-j\right)_{\{j\}} \rho^{2(a-j)}$$

(cf. Joarder, 2005a).

In case $\rho = 0$, then W will be the product of two independent chisquare random variables each with m degrees of freedom and evidently the resulting moments are in agreement with that situation. In particular, if $\rho = 0$, the mean and variance of W will be $E(W) = m$ and $Var(W) = 4m^2(m+1)$ respectively.

Let $H = U/V$, the ratio of two correlated chisqaure variables U and V that have probability density function in Theorem 2.1. Then the following corollary follows from Theorem 2.2.

Corollary 2.6 For $m > 2a, -1 < \rho < 1$, the a-th moment of H is given by

$$E\left(H^a\right) = \frac{1}{L(m,\rho)} \sum_{k=0}^{\infty} \frac{(2\rho)^k}{k!} \Gamma\left(\frac{k+m}{2}+a\right) \Gamma\left(\frac{k+m}{2}-a\right) \Gamma\left(\frac{k+1}{2}\right) \left(\Gamma\left(\frac{k+m}{2}\right)\right)^{-1}$$

where $L(m,\rho)$ is defined in Theorem 2.3. In case a is a nonnegative integer, we have

$$E(H^a) = \frac{\Gamma(m/2-a)}{\Gamma(m/2)} \sum_{j=0}^{a} \binom{a}{j} (-2a+j+1)_{\{a-j\}} \left((m/2)+a-j\right)_{\{j\}} \rho^{2(a-j)}.$$

In case $\rho = 0$, then H will be the ratio of two independent chisquare variables each m degrees of freedom, and the a-th moment will be simply

$$E\left(H^a\right) = (m/2)_{\{a\}} / (m/2-a)_{\{a\}}, \quad m > 2a$$

which are evidently in agreement with the situation. In particular, if $\rho = 0$, the mean and variance of H will be

$$E(H) = \frac{m}{m-2}, \; m > 2, \text{ and}$$

$$Var(H) = \frac{4m(m-1)}{(m-2)^2(m-4)}, \; m > 4$$

respectively.

If X and Y are two random variables, then it is well known that the independence between X and Y implies uncorrelation, but the converse is sometimes true , e.g. in case X and Y have a bivarite normal distribution. The following corollary is another example along the line.

Corollary 3.1 Let U and V have a bivariate chisquare distribution with pdf given by (2.1). If the product moment correlation between U and V vanishes, then they are independent.

Proof. Since

$$E(UV) = \mu'(1,1) = m(m+2\rho^2),\ E(U) = \mu'(1,0) = m,\ E(V) = \mu'(0,1) = m,$$
$$E(U^2) = \mu'(2,0) = m(m+2),\ E(V^2) = \mu'(0,2) = m(m+2),$$

it can be checked that

$$Var(U) = m(m+2) - m^2 = 2m,$$
$$Var(V) = m(m+2) - m^2 = 2m,$$
$$Cov(U,V) = E(UV) - E(U)E(V) = m(m+2\rho^2) - m(m) = 2m\rho^2$$

and hence the product moment correlation between U and V is given by

$$Corr(U,V) = \frac{Cov(UV)}{[Var(U)Var(V)]^{1/2}} = \rho^2$$

Note that if $\rho = 0$, the pdf in (2.1) becomes the product of that of the two independent chisquare distributions each with m degrees of freedom. In view of the above corollary we have the following:

Corollary 3.2 Let X and Y have a bivariate normal distribution with correlation coefficient ρ. If $h_1(X_1, X_2, \cdots, X_N)$ and $h_2(Y_1, Y_2, \cdots, Y_N)$ are two functions of $(X_1, X_2, \cdots, X_N)$ and $(Y_1, Y_2, \cdots, Y_N)$ respectively, with correlation coefficient $h(\rho)$ such that $h(\rho) = 0$ implies $\rho = 0$, then X and Y are independent.

4. On a Chisquare Distribution with a Nuisance Parameter

Theorem 4.1 A random variable U is said to have a chisquare distribution with a nuisance parameter ρ if it has the pdf given by

$$g(u) = \frac{u^{m/2-1}\, e^{\frac{-u}{2(1-\rho^2)}}}{2^{m/2}\,\Gamma(m/2)\sqrt{\pi}} \sum_{k=0}^{\infty} \left(\frac{\rho\sqrt{2u}}{\sqrt{1-\rho^2}} \right)^k \frac{\Gamma((k+1)/2)}{k!}\ ,\ u > 0,\ m > 2, -1 < \rho < 1.$$

When $\rho = 0$, the above reduces to the pdf of chisquare distribution with m degrees of freedom. If $\rho = 1/\sqrt{2}$, a special case of the chisquare distribution has the following pdf :

$$g(u)=\frac{u^{m/2-1}\,e^{-u}}{2^{m/2}\,\Gamma(m/2)\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(4u)^{k/2}}{k\,!}\Gamma\left(\frac{k+1}{2}\right),\quad u>0,\ m>2.$$

Theorem 4.2 The a th moment of the distribution of U is given by

$$\mu'(a;\rho)=2^{a}\,\frac{\Gamma((m/2)+a)}{\Gamma(m/2)}$$

which is the a th moment of the usual chisquare random variable with m degrees of freedom.

Proof. By the use of Theorem 2.3 in Theorem 2.2 , we have the following:

$$\begin{aligned}\mu'(a;\rho)&=\frac{2^{a}\left(1-\rho^{2}\right)^{a}}{L(m,\rho)}\sum_{k=0}^{\infty}\frac{(2\rho)^{k}}{k\,!}\,\Gamma\big((k+m+2a)/2\big)\Gamma\big((k+1)/2\big)\\&=\frac{2^{a}\left(1-\rho^{2}\right)^{a}}{L(m,\rho)}\sum_{k=0}^{\infty}\frac{\rho^{k}}{k\,!}\,b_{k,m+2a}\\&=\frac{2^{a}\left(1-\rho^{2}\right)^{a+m/2}}{L(m,\rho)}L(m+2a,\rho)\\&=2^{a}\left(1-\rho^{2}\right)^{a+m/2}\left[\frac{\Gamma((m/2)+a)}{\Gamma(m/2)}\left(1-\rho^{2}\right)^{-a-m/2}\right]\end{aligned}$$

since $L(m,\rho)=\sqrt{\pi}\;\Gamma(m/2)\left(1-\rho^{2}\right)^{-m/2}$, $m>2$, and $-1<\rho<1$.

5. Some Functions of Correlated Chisquare Varibales

Theorem 5.1 Let U and V be two correlated chisquare variables with pdf given by Theorem 2.1. Then the joint pdf of $Y=U+V$ and $W=UV$ is given by

$$f_5(y,w)=\frac{w^{m/2-1}\,e^{\frac{-y}{2(1-\rho^2)}}(y^{2}-4w)^{-1/2}}{2^{m-1}\sqrt{\pi}\;\Gamma(m/2)(1-\rho^{2})^{m/2}}\sum_{l=0}^{\infty}\left(\frac{\rho\sqrt{w}}{1-\rho^{2}}\right)^{l}\frac{\Gamma\big((l+1)/2\big)}{l\,!\Gamma\big((l+m)/2\big)},$$

$y>2\sqrt{w}$, $m>2,-1<\rho<1$.

Proof. Let $y=h_1(u,v)=u+v$, $w=h_2(u,v)=uv$, $u=h_1^{-1}(y,w),v=h_2^{-1}(y,w)$ in (2.1) so that the joint pdf of Y and W is given by

$$f_6(y,w)=g_1(h_1^{-1}(y,w),h_2^{-1}(y,w))\,|J_1|+g_2(h_1^{-1}(y,w),h_2^{-1}(y,w))\,|J_2|$$

where g_1 and g_2 are the pdf of Y and W in the domain $D_1=\{(u,v),u>v\}$ and $D_2=\{(u,v):u<v\}$ respectively and $|J_i|,(i=1,2)$ is the Jacobians of transformations. In $D_1=\{(u,v):u>v\}$ we have $2u=y+\sqrt{y^{2}-4w}$, $2v=y-\sqrt{y^{2}-4w}$ and

$$\frac{\partial u}{\partial y}=\frac{1}{2}+\frac{1}{2}y(y^2-4w)^{-1/2},\quad \frac{\partial u}{\partial w}=-(y^2-4w)^{-1/2},$$

$$\frac{\partial v}{\partial y}=\frac{1}{2}-\frac{1}{2}y(y^2-4w)^{-1/2},\quad \frac{\partial v}{\partial w}=(y^2-4w)^{-1/2}$$

so that $\frac{\partial u}{\partial y}\frac{\partial v}{\partial w}-\frac{\partial u}{\partial w}\frac{\partial v}{\partial y}=(y^2-4w)^{-1/2}$

yielding $J(u,v\to y,w)=\left(y^2-4v\right)^{-1/2}, y>2\sqrt{v}$. In $D_2=\{(u,v):u<v\}$, we have $2u=y-\sqrt{y^2-4w}, 2v=y+\sqrt{y^2-4w}$ and as above, it can be proved that

$$\frac{\partial u}{\partial y}\frac{\partial v}{\partial w}-\frac{\partial u}{\partial w}\frac{\partial v}{\partial y}=-(y^2-4w)^{-1/2}$$

so that the Joacobian of the transformation is $|J(u,v\to y,w)|=\left(y^2-4w\right)^{-1/2}, y>2\sqrt{w}$. Then the probability density function of Y and V follows from Theorem 2.1.

Corollary 5.1 If $\rho=0$, then the joint distribution of sum and product of two chisquare variables would be

$$f_7(y,w)=\frac{(yw)^{m/2-1}e^{-y/2}\left(y^2-4w\right)^{-1/2}}{2^{m-1}\,\Gamma^2(m/2)}\quad,\quad 0<w<y^2/4, 0<y,\ m=N-1>2.$$

Theorem 5.2 Let U and V be two correlated chisquare variables with pdf given by Theorem 2.1. Then the pdf of $Y=U+V$ is given by

$$f_8(y)=\frac{y^{m-1}e^{-y/(2-2\rho^2)}}{2^{2m-1}\,\Gamma(m/2)\left(1-\rho^2\right)^{m/2}}\sum_{l=0}^{\infty}\frac{\left(\rho y/\left(2-2\rho^2\right)\right)^l\,\Gamma((l+1)/2)}{l!\Gamma((l+m+1)/2)},\ y>0$$

Proof. It follows from Theorem 2.1 that

$$f_9(y)=\frac{\left(1-\rho^2\right)^{-m/2}}{2^{m-1}\,\Gamma(m/2)\sqrt{\pi}}\ e^{-y/(2-2\rho^2)}$$
$$\times\sum_{l=0}^{\infty}\frac{\left(\rho/\left(1-\rho^2\right)\right)^l\,\Gamma\left((l+1)/2\right)}{l!\Gamma\left((l+m)/2\right)}\int_0^{y^2/4}w^{(l+m-2)/2}\left(y^2-4w\right)^{-1/2}dw$$

which can be simplified as

$$f_{10}(y)=\frac{\left(1-\rho^2\right)^{-m/2}e^{-y/(2-2\rho^2)}}{2^{m-1}\,\Gamma(m/2)\sqrt{\pi}}$$
$$\times\sum_{l=0}^{\infty}\frac{\left(\rho/\left(1-\rho^2\right)\right)^l\,\Gamma\left((l+1)/2\right)}{l!\Gamma\left((l+m)/2\right)}\left[\frac{y^{l+m-1}}{2^{m+l}}\frac{\Gamma((l+m)/2)\Gamma(1/2)}{\Gamma(l+m+1/2)}\right].$$

If $\rho = 0$, the pdf in Theorem 5.2 matches, as expected, with χ^2_{2m}.

Theorem 5.3 Let U and V be two correlated chisquare variables with pdf given by Theorem 2.1. Then the pdf of $W = UV$ is given by

$$f_{11}(w) = \frac{w^{(m-2)/2}}{2^{m-1}\ \Gamma(m/2)\left(1-\rho^2\right)^{m/2}\sqrt{\pi}} \int_{2\sqrt{w}}^{\infty} \left(y^2 - 4w\right)^{-1/2} e^{-y/(2-2\rho^2)} dy$$
$$\times \sum_{l=0}^{\infty} \frac{\left(\rho\sqrt{w}/\left(2-2\rho^2\right)\right)^l \Gamma\left((l+1)/2\right)}{l!\,\Gamma\left((l+m)/2\right)},\ w > 0.$$

Proof. It follows from Theorem 2.1 that

$$f_{12}(w) = \frac{w^{(m-2)/2}}{2^{m-1}\ \Gamma(m/2)\left(1-\rho^2\right)^{m/2}\sqrt{\pi}} \int_{2\sqrt{w}}^{\infty} \left(y^2 - 4w\right)^{-1/2} e^{-y/(2-2\rho^2)} dy$$
$$\times \sum_{l=0}^{\infty} \frac{\left(\rho\sqrt{w}/\left(2-2\rho^2\right)\right)^l \Gamma\left((l+1)/2\right)}{l!\,\Gamma\left((l+m)/2\right)}.$$

If W is the product of two independent chisquare variables with d.f. m_1 and m_2, then

$$f_{15}(w) = \frac{w^{(m_1+m_2)/4-1}}{2^{m_1+m_2/2-1}\Gamma(m_1/2)\Gamma(m_2/2)} K_{(m_1-m_2)/4}(\sqrt{w}),\ w > 0$$

(Springer, 1979, 365) where $K_\alpha(x)$ is the modified Bessel function of the third kind of order α which is neither zero nor a positive integer (Erdelyi, 1959, (13), p.5). Notice that the above density function does not allow the two chisquares having the same degrees of freedom. The following theorem provides a different form of the density of the product of two independent chisquare variables with the same degrees of freedom.

Theorem 5.4 In case U and V are independent, the pdf of $W = UV$ is given by

$$f_{14}(w) = \frac{\sqrt{3}\ w^{(m-3)/2}}{\sqrt{\pi}\ 2^{m-3/2}\Gamma^2(m/2)} K_1(\sqrt{w})$$

where $K_\alpha(x) = \dfrac{\sqrt{\pi}x^\alpha}{2^\alpha(\alpha+1/2)} \displaystyle\int_1^\infty e^{-xt}(t^2-1)^{\alpha-1/2}dt = \left(\frac{\pi}{2x}\right)^{1/2} {}_1F_1(0,\alpha,2x)$.

Proof. From Theorem 5.3 we have

$$f_{13}(w) = \frac{w^{m/2-1}}{2^{m-1}\Gamma^2(m/2)} \int_{2\sqrt{w}}^{\infty} \left(y^2 - 4w\right)^{-1/2} e^{-y/2} dy$$

$$= \frac{w^{m/2-1}}{2^{m-1}\Gamma^2(m/2)} \int_{2\sqrt{w}}^{\infty} \left[4w\left(\frac{y^2}{4w} - 1\right)\right]^{-1/2} e^{-y/2} dy .$$

Letting $\frac{y}{2\sqrt{w}} = x, dy = 2\sqrt{w}\, dx$, we have

$$f_{14}(w) = \frac{w^{m/2-1}}{2^{m-1}\Gamma^2(m/2)} \int_{1}^{\infty} \left(t^2 - 1\right)^{-1/2} e^{-\sqrt{w}\, t} dt ,$$

$$= \frac{\sqrt{3}\, w^{(m-3)/2}}{2^{m-3/2}\Gamma^2(m/2)\sqrt{\pi}} K_1(\sqrt{w}).$$

The moments of $H = U/V$, the ratio of two correlated chisquare variables, is outlined in Corollary 2.6. But further investigation is required to derive the distribution of $H = U/V$, along Provost (1986).

6. Estimation of ρ^2

Theorem 5.1 Let X_1 and X_2 have a bivariate normal distribution with $\sigma_1^2 = \sigma_2^2$. Then an unbiased estimators of ρ^2 is given by

$$H_* = \frac{m-2}{2}\left(1 - \frac{S_1^2}{S_2^2}\right) + 1$$

with variance

$$Var(H_*) = \frac{1}{m-4}\left[(1-\rho^2)m^2 - (1+4\rho^2 - 5\rho^4)m + 8\rho^2(1-\rho^2)\right]$$

Proof. From Corollary 2.6 we have

$$E(H) = \frac{m - 2\rho^2}{m-2}.$$

Then the unbiasedness of H_* follows from

$$E\left(\frac{U}{V}\right) = \frac{m - 2\rho^2}{m-2}$$

where $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$. The variance of H_* follows from Corollary 2.6 by virtue of

$$Var(H_*) = \frac{(m-2)^2}{4} Var(H).$$

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