Semigroups of Mappings

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Chapter 1

Some Semigroups of Mappings

1.1 The Full Transformation Semigroup

Any abstract algebraic study is rooted in concrete examples: if convincing examples are in short supply then the study is of little or no interest. The origins of group theory are in the study of permutations, and the symmetric group, the group of all permutations of a set, is rightly an object of importance within the abstract study. The corresponding object in semigroup theory is the full transformation semigroup, the semigroup $T_X$ of all selfmaps of a set $X$. For the moment we shall consider a finite set $X_n = \{1, 2, \ldots, n\}$, and we write $T_{X_n}$ more simply as $T_n$. The order of $T_n$ is $n^n$.

The semigroup $T_X$ has the same universal property as the symmetric group:

**Theorem 1.1** Let $S$ be a semigroup. Then there exists a set $X$ with the property that $S$ is embedded in $T_X$.

**Proof.** Let $X = S \cup \{1\}$, where $1 \notin S$. For each $s$ in $S$, define $\rho_s : X \to X$ by

\[
x \rho_s = x s \text{ if } x \in S, \\
1 \rho_s = s.
\]

It is easy to verify that $\rho_s \rho_t = \rho_{st}$, and so the map $s \mapsto \rho_s$ is a homomorphism from $S$ into $T_X$. It is also one-one, since

\[
\rho_s = \rho_t \Rightarrow (\forall x \in X) \ x \rho_s = x \rho_t \Rightarrow 1 \rho_s = 1 \rho_t \Rightarrow s = t.
\]
1.2 Regular Semigroups

The semigroup \( T_X \) is certainly not a group, but it does have a ‘group-like’ property. A semigroup \( S \) is called regular if

\[
(\forall a \in S) \ (\exists x \in S) \ axa = a.
\]

(1.1)

A regular semigroup must contain idempotent elements – elements \( e \) with the property that \( e^2 = e \). It immediately follows from (1.1) that both \( ax \) and \( xa \) are idempotents. In general they will be different idempotents.

It is not hard to prove

**Theorem 1.2** The semigroup \( T_X \) is regular.

**Proof** Let \( \alpha \in T_X \). Define a mapping \( \xi : X \to X \) as follows. For each \( y \) in \( \text{im} \alpha \), choose an element \( x \) in \( X \) such that \( x\alpha = y \) and let \( y\xi = x \); also, for all \( y \) not in \( \text{im} \alpha \), choose an arbitrary element \( z \) of \( X \), and let \( y\xi = z \). Then it is clear that, for all \( x \) in \( X \),

\[
x\alpha\xi\alpha = x\alpha.
\]

\[\square\]

**Remark 1.3** The statement that \( T_X \) is regular for all sets \( X \) is equivalent to the Axiom of Choice.

The regularity condition (1.1), somewhat surprisingly, is equivalent to the seemingly stronger condition

\[
(\forall a \in S) \ (\exists y \in S) \ (aya = a \text{ and } yay = y);
\]

(1.2)

for we may take \( y = xax \) and observe that

\[
aya = a(xax)a = (axa)(xa) = axa = a,
\]

\[
yay = (xax)a(xax) = x(axa)xax = x(axa)x = xax = y.
\]

An element \( y \) satisfying (1.2) is called an inverse of \( a \).

More specialised classes of regular semigroups have been studied. The most important is the class of inverse semigroups, defined by the condition that every element has a unique inverse.

**Theorem 1.4** A semigroup is an inverse semigroup if and only if it is regular and idempotents commute.

**Proof** Suppose that \( S \) is regular and that idempotents commute. Let \( a', a'' \) be inverses of \( a \). Then

\[
a' = a'a'a = a'(aa'a)a' = (a'a)(a''a)a' = a''aa'a = a''a'a' = a''(a''a) = a''(aa'a)a' = a''aa''a'' = a''aa''a'' = a''aa'' = a''.
\]
Conversely, suppose that inverses are unique, and let $e$, $f$ be idempotents. Let $x$ be the unique inverse of $ef$; that is,

$$efxef = ef, \quad xefx = x.$$  

Then $fxe$ is also an inverse of $ef$, since

$$(ef)(fxe)(ef) = ef^2xe^2f = (ef)x(ef) = ef,$$

$$(fxe)(ef)(fxe) = f(xefxe) = fxe.$$  

Also, $fxe$ is idempotent:

$$(fxe)^2 = f(xefxe) = fxe.$$  

Hence $fxe$ is its own unique inverse, and so $fxe = ef$. In particular, $ef$ is idempotent, and is its own unique inverse. The same holds for $fe$. But

$$(ef)(fe)(ef) = (ef)^2 = ef, \quad (fe)(ef)(fe) = (fe)^2 = fe,$$

and so $ef$ and $fe$ are mutually inverse. Hence $ef = fe$. \hfill \Box

The class of inverse semigroups, like the class of all semigroups, is rooted in a concrete example. Let $X$ be a non-empty set. The symmetric inverse semigroup $I_X$ is the set of all partial one-one mappings of $X$. If $\alpha, \beta \in I_X$, then

$$\text{dom } (\alpha\beta) = (\text{im } \alpha \cap \text{dom } \beta)\alpha^{-1}, \quad \text{im } (\alpha\beta) = (\text{im } \alpha \cap \text{dom } \beta)\beta,$$

and $x(\alpha\beta) = (x\alpha)\beta$ for all $x$ in dom $(\alpha\beta)$. The idempotents of $I_X$ are the partial identity mappings $\text{id}_A$, where $A$ is a subset of $X$. (This includes the empty set.) The Vagner–Preston Theorem [23, 19, 15] establishes a universal property analogous to Theorem 1.1: every inverse semigroup is embeddable in a symmetric inverse semigroup.

Other classes of regular semigroups do not have this direct link with mappings. Orthodox semigroups, regular semigroups in which the set of idempotents is a subsemigroup, have been studied, but there are few examples ‘in nature’. Much commoner are regular idempotent-generated semigroups, called rigs by Clifford. There is no very accessible theory of rigs, but, as we shall see, examples abound.

### 1.3 The Structure of $T_n$

The rank $\text{rank}(S)$ of a semigroup $S$ is defined by

$$\text{rank}(S) = \min \{|A| : \langle A \rangle = S\}.$$  

The rank of $T_n$ is 3: a set of generators is

$$\{(1 \ 2), \ (1 \ 2 \ldots \ n), \ (1_2)\}. \quad (1.3)$$
CHAPTER 1. SOME SEMIGROUPS OF MAPPINGS

The first two generators are cycles, and generate \( S_n \); the third is the singular mapping

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots & n \\
2 & 2 & 3 & \cdots & n
\end{pmatrix}.
\]

The choice of \( \begin{pmatrix} 1 & 2 \\ i & i \end{pmatrix} \) is arbitrary: in fact any element \( \alpha \) such that \(|\text{im} \alpha| = n - 1 \) would do as well.

It is clear that the rank must be at least 3, for two generators are required to generate the symmetric group \( S_n \), and at least one extra 'singular' generator is needed. It is less obvious that these 3 are sufficient: see [15, Chapter 1, Exercises 6 & 7].

For \( r \in X_n \), let

\[
J_{n,r} = \{ \alpha \in T_n : |\text{im} \alpha| = r \}; \tag{1.4}
\]

thus \( J_{n,n} \) is the symmetric group \( S_n \) and, at the other extreme,

\[
J_{n,1} = \{ \kappa_i : i = 1, 2, \ldots, n \},
\]

where, in an obvious and standard notation

\[
\kappa_i = \begin{pmatrix}
1 & 2 & \cdots & n \\
i & i & \cdots & i
\end{pmatrix}.
\]

Let

\[
K_{n,r} = \{ \alpha \in T_n : |\text{im} \alpha| \leq r \} = \bigcup_{s=1}^{r} J_{n,s}. \tag{1.5}
\]

It is easy to see that, for all \( \alpha, \beta \) in \( T_n \),

\[
|\text{im} (\alpha \beta)| \leq \min\{|\text{im} \alpha|, |\text{im} \beta|\}, \tag{1.6}
\]

and it follows immediately that \( K_{n,r} \) is a subsemigroup of \( T_n \). It is indeed an ideal, in the sense that

\[
K_{n,r} T_n \subseteq K_{n,r}, \quad T_n K_{n,r} \subseteq K_{n,r}.
\]

Clearly \( K_{n,n} = T_n \). More interestingly, the set

\[
K_{n,n-1} = T_n \setminus S_n,
\]

the set of singular selfmaps of \( X_n \), is an ideal: we shall denote it by \( \text{Sing}_n \). Its order is \( n^n - n! \).

It is well known that every element of the symmetric group \( S_n \) is a composition of disjoint cycles. Something similar happens to elements of \( T_n \), but the situation is inevitably more complicated. Associated with a mapping \( \alpha \) in \( T_n \) is a digraph with \( n \) vertices, in which there is an edge \( i \rightarrow j \) if and only if \( i \alpha = j \).

Let \( \alpha \in T_n \). For \( i, j \) in \( X_n \), we write \( i \equiv j \) if and only if there exists \( r, s \geq 0 \) such that \( i \alpha^r = j \alpha^s \). This is an equivalence relation, and it partitions \( X_n \) into
disjoint classes, called orbits. The orbits are the connected components of the associated digraph. An example is helpful. Let

$$
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
3 & 3 & 4 & 5 & 6 & 4 & 6 & 9 & 10 & 10 & 12 & 13 & 11 & 14
\end{pmatrix}.
$$

(1.7)

The example shows the different kinds of components that can arise. The component on the left is standard, the next is acyclic, the third is cyclic and the fourth is trivial. For a general \( \alpha \), let \( \text{cycl}(\alpha) \) be the number of cyclic components, and let \( \text{fix}(\alpha) \) be the number of fixed points: this is equal to the number of acyclic components plus the number of trivial components. It is easy to see, working orbit by orbit, that, for the \( \alpha \) given by (1.7),

$$
\alpha = \left[ \begin{array}{cccc}
(6) & (5) & (4) & (3) \\
(3) & (6) & (4) & (3) \\
(2) & (7) & (6) & (9) \\
(1) & (8) & (9) & (10) \\
(11) & (13) & (12) & (1) \\
(12) & (13) & (11) & (12)
\end{array} \right].
$$

Thus \( \alpha \) (and indeed any \( \alpha \) in any Sing\(_n\)) is expressible as a product of idempotents in \( J_{n,n-1} \). The idempotents in \( J_{n,n-1} \) are all of the form \( (i,j) \), where \( i, j \) are distinct elements of \( X_n \), and so the cardinality of the set \( E_{n-1} \) of idempotents in \( J_{n,n-1} \) is \( n(n-1)/2 \).

Returning to our example (1.7) we see that the standard orbit has 7 elements and contributes 7 factors; the acyclic orbit, with 3 elements, contributes 2 factors; the cyclic orbit, with 3 elements, contributes 4 factors; and the the trivial orbit contributes no factors at all. The length of the product is

$$
g(\alpha) = n + \text{cycl}(\alpha) - \text{fix}(\alpha).
$$

(1.8)

We have looked only at an example, but it is true in general that each \( \alpha \) in every Sing\(_n\) is a product of length \( g(\alpha) \) of idempotents in \( J_{n,n-1} \). Moreover, though we shall not go into the proof, this length is best possible, and we have (see [18, 13]) the following theorem:

**Theorem 1.5** Let \( \alpha \in \text{Sing}_n \) and let \( E_{n-1} \) be the set of idempotents in \( J_{n,n-1} \). Then \( g(\alpha) \), as defined by (1.8), has the property that

\[
\alpha \in E_{n-1}^{g(\alpha)}, \quad \alpha \notin E_{n-1}^{g(\alpha)-1}.
\]
This is our first significant example of a rig (regular idempotent-generated semigroup). The number of idempotent generators is \( n(n-1) \), and it is legitimate to ask whether this is the smallest possible set of generators of \( \text{Sing}_n \). It is not, but to clarify this issue we need a little more general theory of semigroups.

### 1.4 Green’s Relations

In a group, each pair \( a, b \) of elements are mutually divisible, both right and left:
\[
a = xb = by, \quad b = ua = av, \quad \text{with} \quad x = ab^{-1}, \quad y = b^{-1}a, \quad u = ba^{-1}, \quad v = a^{-1}b.
\]
Indeed, a semigroup is a group if and only if these divisibility properties hold. We define an equivalence relation \( L \) on a semigroup \( S \) by
\[
a L b \quad \text{if and only if} \quad a = b \quad \text{or} \quad (\exists x, y \in S) a = xb, \quad b = ya.
\]
We can simplify that definition by defining \( S^1 \) to be the semigroup \( S \) with a unity element 1 adjoined if necessary. So we write
\[
a L b \quad \text{if and only if} \quad (\exists x, y \in S^1) a = xb, \quad b = ya.
\]
(1.9)

Similarly
\[
a R b \quad \text{if and only if} \quad (\exists u, v \in S^1) a = bu, \quad b = av;
\]
(1.10)
and again
\[
a J b \quad \text{if and only if} \quad (\exists x, y, u, v \in S^1) a = xby, \quad b = uav.
\]
(1.11)
The equivalence \( L \cap R \) is denoted by \( H \). A further equivalence \( D \) is defined by
\[
a D b \quad \text{if and only if} \quad (\exists c \in S) a L c \quad \text{and} \quad c R b.
\]
(1.12)
Equivalently (not quite obviously),
\[
a D b \quad \text{if and only if} \quad (\exists d \in S) a R d \quad \text{and} \quad d L b.
\]
(1.13)
These five equivalences, known as Green’s relations \([7]\), form a lattice as follows:

In many cases the two relations \( D \) and \( J \) coincide. This happens when the semigroup is finite, but also in other circumstances.
1.4. GREEN’S RELATIONS

What is the ‘Green structure’ of the semigroup \( T_X \)? To answer the question we need to introduce a new notion. Associated with every mapping in \( T_X \) is an equivalence relation \( \ker \alpha \) on \( X \), defined by

\[
\ker \alpha = \{ (x, y) \in X \times X : x \alpha = y \alpha \}. \tag{1.14}
\]

**Theorem 1.6** For elements \( \alpha, \beta \) in \( T_X \),

(i) \( \alpha \mathcal{L} \beta \) if and only if \( \text{im} \alpha = \text{im} \beta \);

(ii) \( \alpha \mathcal{R} \beta \) if and only if \( \ker \alpha = \ker \beta \);

(iii) \( \alpha \mathcal{D} \beta \) if and only if \( |\text{im} \alpha| = |\text{im} \beta| \);

(iv) \( \mathcal{J} = \mathcal{D} \).

**Proof**

(i) Suppose that \( \alpha \mathcal{L} \beta \), which is to say that there exist \( \mu, \nu \) in \( T_X \) such that \( \mu \alpha = \beta \), \( \nu \beta = \alpha \). Then

\[
\text{im} \alpha = X \alpha = X \nu \beta \subseteq X \beta = \text{im} \beta ,
\]

and similarly \( \text{im} \beta \subseteq \text{im} \alpha \).

Conversely, suppose that \( \text{im} \alpha = \text{im} \beta \), and let \( x \in X \). Choose \( x \mu \) to be any element whose image under \( \alpha \) is \( x \beta \). (This is possible since \( \text{im} \alpha = \text{im} \beta \).) It is then clear that \( \mu \alpha = \beta \), and in a similar way we find \( \nu \) such that \( \nu \beta = \alpha \).

(ii) Suppose that \( \alpha \mathcal{R} \beta \), which is to say that there exist \( \xi, \eta \) in \( T_X \) such that \( \alpha \xi = \beta \), \( \beta \eta = \alpha \). Then

\[
(x, y) \in \ker \alpha \Rightarrow x \alpha = y \alpha \Rightarrow x \alpha \xi = y \alpha \xi \Rightarrow x \beta = y \beta \Rightarrow (x, y) \in \ker \beta ,
\]

and so \( \ker \alpha \subseteq \ker \beta \). Similarly, \( \ker \beta \subseteq \ker \alpha \).

Conversely, suppose that \( \ker \alpha = \ker \beta \). Let \( z \) be an element of \( \text{im} \alpha \) and let \( C \) be the \( \ker \alpha \)-class that maps to \( z \) by \( \alpha \). Define \( z \xi \) to be the image of \( C \) under \( \beta \). For any \( x \in X \setminus \text{im} \alpha \) we define \( z \xi \) in an arbitrary way. Then, for each element \( c \) of \( C \),

\[
\alpha \xi = z \xi = c \beta .
\]

This holds for all \( z \) in \( \text{im} \alpha \) and for all the \( \ker \alpha \)-classes, and so \( \alpha \xi = \beta \). In the same way we can find \( \eta \) such that \( \beta \eta = \alpha \), and so \( \alpha \mathcal{R} \beta \).

(iii) For any \( \alpha \) in \( T_X \) we have that \( |X/\ker \alpha| = |\text{im} \alpha| \). If \( \alpha \mathcal{D} \beta \), then there exists \( \gamma \) such that \( \alpha \mathcal{L} \gamma \) and \( \gamma \mathcal{R} \beta \). Hence \( |\text{im} \alpha| = |\text{im} \gamma| = |X/\ker \gamma| = |X/\ker \beta| = |\text{im} \beta| \).

Conversely, suppose that \( |\text{im} \alpha| = |\text{im} \beta| \). Then let \( \gamma \) be an element of \( T_X \) that maps the \( \ker \beta \)-classes onto the image of \( \alpha \). Then \( \text{im} \gamma = \text{im} \alpha \) and \( \ker \gamma = \ker \beta \). Hence \( \alpha \mathcal{L} \gamma \) and \( \gamma \mathcal{R} \beta \), as required.
(iv). If $\xi, \eta$ are arbitrary elements of $T_X$, then, as already observed above in (1.6), we see that

$$|\text{im} (\xi \eta)| \leq |\text{im} \xi| \quad \text{and} \quad |\text{im} (\xi \eta)| \leq |\text{im} \eta|.$$  \hspace{1cm} (1.15)

Suppose that $\alpha \mathcal{J} \beta$. Then there exist $\lambda, \mu, \nu, \rho$ such that

$$\alpha = \lambda \beta \mu, \quad \beta = \nu \alpha \rho.$$  

Hence, by (1.15)

$$|\text{im} \alpha| = |\text{im} (\lambda \beta \mu)| \leq |\text{im} \beta|, \quad |\text{im} \beta| = |\text{im} (\nu \alpha \rho)| \leq |\text{im} \alpha|.$$  

Thus $|\text{im} \alpha| = |\text{im} \beta|$ and so $\alpha \mathcal{D} \beta$. \hfill \Box

As a result of this theorem we can gain useful information about the structure of the $\mathcal{J}$-classes $J_{n,r}$ in the semigroup $T_n$. The number of $\mathcal{L}$-classes is the number of distinct subsets of cardinality $r$: this is the binomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$  

The number of $\mathcal{R}$-classes is the number of equivalences on the set $X_n$ having $r$ classes: this, less well known, is the Stirling number $S(n, r)$ of the second kind. It is clear that

$$S(n, 1) = S(n, n) = 1.$$  \hspace{1cm} (1.16)

Now let $n \geq 2$, and consider an equivalence $\rho$ on $\{1, 2, \ldots, n\}$ having $r$ classes. Either $\{n\}$ is a $\rho$-class or $n$ is part of a larger $\rho$-class. The number of equivalences of the first kind is $S(n-1, r-1)$, and the number of the second kind is $rS(n-1, r)$ (since $n$ may be added to any one of the $r$ classes). Hence

$$S(n, r) = S(n-1, r-1) + rS(n-1, r).$$  \hspace{1cm} (1.17)

A table of small values may be of interest:

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<th>3</th>
<th>4</th>
<th>5</th>
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<td>31</td>
<td>90</td>
<td>65</td>
<td>15</td>
<td>1</td>
</tr>
</tbody>
</table>

From the table one might conjecture that

$$S(n, n-1) = \frac{n(n-1)}{2}, \quad S(n, 2) = 2^{n-1} - 1;$$  \hspace{1cm} (1.18)

both are easily proved by induction.
1.4. GREEN’S RELATIONS

We can now visualise a \( J \)-class \( J_{n,r} \) of \( T_n \) as an ‘eggbox’ in which the \( L \)-classes are the columns, the \( R \)-classes are the rows and the \( H \)-classes are the cells. The number of cells is \( \binom{n}{r} \times S(n,r) \), and each cell contains \( r! \) elements.

Consider \( H \), an \( H \)-class in \( J_{n,r} \) corresponding to the image \( S = \{ a_1, a_2, \ldots, a_r \} \) and the equivalence \( \rho \) with classes \( A_1, A_2, \ldots, A_r \), and suppose that \( S \) is a transversal of \( \rho \). By this we mean that each \( a_i \) in \( S \) belongs to a unique \( \rho \)-class \( A_j \). One possibility is that \( a_i \in A_i \) for all \( i \), and in this case we obtain

\[ \epsilon = \left( \begin{array}{ccc} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{array} \right), \]

the unique idempotent in \( H \). A typical element of \( H \) is

\[ \alpha_\sigma = \left( \begin{array}{ccc} A_1 & A_2 & \cdots & A_r \\ a_{1\sigma} & a_{2\sigma} & \cdots & a_{r\sigma} \end{array} \right), \]

where \( \sigma \) is a permutation of \( \{1, 2, \ldots, r\} \), and, since \( \alpha_\sigma \alpha_\tau = \alpha_{\sigma \tau} \) for all permutations \( \sigma, \tau \) of \( \{1, 2, \ldots, r\} \) it follows that \( H \) is a group, with identity \( \epsilon \), isomorphic to the symmetric group \( S_r \).

Suppose now that \( H = H_{S,\rho} \), determined by the subset \( S = \{ a_1, a_2, \ldots, a_r \} \) and the equivalence \( \rho \) with classes \( A_1, A_2, \ldots, A_r \) is such that \( S \) is not a transversal of \( \rho \). Then, re-labelling if necessary, we may consider two elements

\[ \beta = \left( \begin{array}{ccc} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{array} \right), \quad \gamma = \left( \begin{array}{ccc} A_1 & A_2 & \cdots & A_r \\ c_1 & c_2 & \cdots & c_r \end{array} \right), \]

such that \( b_1, b_2 \in A_1 \). Then \( A_1 \cup A_2 \) maps by \( \beta \gamma \) to a single element \( c_i \). Also, for \( j \geq 3 \), each \( A_j \) maps by \( \beta \gamma \) to some \( c_j' \) (where the elements \( c_j' \) are not necessarily distinct). We conclude:

**Theorem 1.7** Let \( H = H_{S,\rho} \) be an \( H \)-class in \( J_{n,r} \), consisting of all mappings with image \( S \) and kernel \( \rho \). If \( S \) is a transversal of \( \rho \), then \( \text{H}^2 = \text{H} \), and \( \text{H} \) is a group. Otherwise

\[ \text{H}^2 \subseteq I_{n,r} = \bigcup_{1 \leq s \leq n-1} J_{n,s}. \]

Another useful observation is as follows:

**Theorem 1.8** Let \( H_{S,\rho} \) and \( H_{T,\sigma} \) be \( H \)-classes in \( J_{n,r} \). Then \( H_{S,\rho} H_{T,\sigma} = H_{T,\rho} \) if and only if \( H_{S,\sigma} \) is a group.

**Proof**

\[ H_{S,\rho} \quad H_{T,\rho} \]

\[ H_{S,\sigma} \quad H_{T,\sigma} \]
CHAPTER 1. SOME SEMIGROUPS OF MAPPINGS

Let
\[ S = \{s_1, s_2, \ldots, s_r\}, \quad T = \{t_1, t_2, \ldots, t_r\}, \]
and let the equivalences \( \rho \) and \( \sigma \) have classes
\[ A_1, A_2, \ldots, A_r \quad \text{and} \quad B_1, B_2, \ldots, B_r, \]
respectively. Then a typical element of \( H_{S,\rho}H_{T,\sigma} \) is
\[ (A_1 \ A_2 \ \ldots \ A_r) \quad (B_1 \ B_2 \ \ldots \ B_r), \]
and the product lies in \( J_{n,r} \) if and only if \( \{s_1, s_2, \ldots, s_r\} \) is a transversal of \( \sigma \),
that is, by Theorem 1.7, if and only if \( H_{S,\sigma} \) is a group. If this happens, then
\[ H_{S,\rho}H_{T,\sigma} \subseteq H_{T,\rho}. \]
Equality follows, since any member
\[ \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ s_1 & s_2 & \ldots & s_r \end{pmatrix} \quad \begin{pmatrix} B_1 & B_2 & \ldots & B_r \\ t_1 & t_2 & \ldots & t_r \end{pmatrix}, \]
is a product
\[ \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ s_1 & s_2 & \ldots & s_r \end{pmatrix} \quad \begin{pmatrix} B_1 & B_2 & \ldots & B_r \\ t_1 & t_2 & \ldots & t_r \end{pmatrix}. \]

We have already seen that the composition of mappings can only move downwards:
\[ \text{im} (\alpha \beta) \leq \min \{ |\text{im} \alpha|, |\text{im} \beta| \}. \]
There is, however, a limit to how far downwards a product can go. The defect \( \text{def} \alpha \) of an element of \( T_X \) is defined as \( |X \setminus \text{im} \alpha| \). Then:

**Theorem 1.9** Let \( \alpha, \beta \in T_X \). Then
\[ \text{def} (\alpha \beta) \leq \text{def} \alpha + \text{def} \beta. \]

**Proof** If \( y \notin \text{im} (\alpha \beta) \), then either:
1. \( y \in X \setminus \text{im} \beta \); or
2. \( y \in \text{im} \beta \), but \( y \notin (X \setminus \text{im} \alpha)\beta \).

Thus
\[ X \setminus (\alpha \beta) = X \setminus \text{im} \beta \cup (X \setminus \text{im} \alpha) \beta, \]
and the union is disjoint. The result follows, since
\[ |(X \setminus \text{im} \alpha)\beta| \leq |X \setminus \text{im} \alpha| = \text{def} \alpha. \]

If \( X = X_n \), then \( \text{def} \alpha = n - |\text{im} \alpha| \), and so we have the corollary:

**Corollary 1.10** Let \( \alpha, \beta \in T_n \). Then
\[ |\text{im} (\alpha \beta)| \geq |\text{im} \alpha| + |\text{im} \beta| - n. \]
Chapter 2

Rank

2.1 The Rank of $\text{Sing}_n$

If $\alpha, \beta \in J_{n,r}$ then $\text{im}(\beta \alpha) \subseteq \text{im} \alpha$. If this is an equality then $\beta \alpha \in J_{n,r}$; otherwise $|\text{im}(\beta \alpha)| < |\text{im} \alpha|$ and the product lies in a lower $\mathcal{J}$-class. Similarly, $\text{im}(\alpha \beta) \subseteq \text{im} \beta$, and so either $\alpha \beta \in J_{n,r}$, or the product falls into a lower $\mathcal{J}$-class.

This observation is the key to the next result. The rank $\text{rank}(S)$ of a semigroup $S$ is defined by

$$\text{rank}(S) = \min\{|A| : A \subseteq S \text{ and } \langle A \rangle = S\}.$$  \hfill (2.1)

**Theorem 2.1**

$$\text{rank}(K_{n,r}) \geq S(n, r).$$

**Proof** Suppose that $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is a generating set for $K_{n,r}$. By virtue of (1.6), $A$ must contain some elements from $J_{n,r}$; we may suppose that these are $\alpha_1, \alpha_2, \ldots, \alpha_p$, and these elements are the only ones that can generate the elements of $J_{n,r}$. Any product

$$\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}$$

of these elements is either in a lower $\mathcal{J}$-class or is $\mathcal{R}$-equivalent to $\alpha_{i_1}$. Hence the elements $\alpha_1, \alpha_2, \ldots, \alpha_p$ must cover all the $\mathcal{R}$-classes, and so $m \geq p \geq S(n, r)$. \hfill $\square$

**Remark 2.2** A similar argument using $\mathcal{L}$-classes establishes that $\text{rank}(K_{n,r}) \geq \binom{n}{r}$. But we always have $S(n, r) \geq \binom{n}{r}$, and so the argument is superfluous.

In particular we have that

$$\text{rank}(\text{Sing}_n) \geq S(n, n-1) = \frac{n(n-1)}{2}.$$ \hfill (2.2)

In fact this is an equality [14]:

11
Theorem 2.3 With the notation above,  
\[
\text{rank}(\text{Sing}_n) = \frac{n(n-1)}{2}.
\]

Proof The key observation is the following identity concerning idempotents in \(E_{n-1}\): if \(3 \leq m \leq n\) and \(i_1, i_2, \ldots, i_m\) in \(X_n\) are all distinct, then  
\[
\begin{pmatrix}
(i_2, i_3) & \cdots & (i_m, i_1) \\
(i_1, i_2) & \cdots & (i_m, i_1)
\end{pmatrix}^{m-1} = \begin{pmatrix}
(i_1, i_2) \\
(i_1, i_2)
\end{pmatrix}.
\] (2.3)

For  
\[
\begin{pmatrix}
(i_2, i_3) & \cdots & (i_m, i_1) \\
(i_1, i_2) & \cdots & (i_m, i_1)
\end{pmatrix} = \begin{pmatrix}
(i_1, i_2, i_3, i_4, \ldots, i_m) \\
(i_2, i_3, \ldots, i_m)
\end{pmatrix} = \beta\text{ (say)},
\]
and it is then easy to see that  
\[
\beta^{m-1} = \begin{pmatrix}
i_1, i_2, i_3, \ldots, i_m \\
i_2, i_3, \ldots, i_m
\end{pmatrix} = \begin{pmatrix}
i_1, i_2 \\
i_1, i_2
\end{pmatrix}.
\]

We can associate a set \(I\) of elements of \(E_{n-1}\) with a digraph \(\Delta(I)\). The vertices are labelled 1, 2, \ldots, \(n\), and there is an edge \((i, j)\) from \(i\) to \(j\) if and only if \(\binom{i}{j} \in I\). A digraph is strong if, for all vertices \(i, j\) there is a path (observing arrows) from \(i\) to \(j\), and it is complete if, for all \(i \neq j\), at least one of \((i, j)\) and \((j, i)\) is an edge. Then \(I\) is a generating set for \(\text{Sing}_n\) if and only if \(\Delta(I)\) is strong and complete.

Consider the set  
\[
I = \left\{ \binom{i}{j} : i < j, i + j \text{ odd} \right\} \cup \left\{ \binom{i}{j} : i > j, i + j \text{ even} \right\}.
\]

The associated digraph is certainly complete, and is also strong, since, for all \(i < j\) such that \(i + j\) is even,  
\[
(i, i + 1), (i + 1, i + 2), \ldots, (j - 1, j)
\]
is a path from \(i\) to \(j\), and, for all \(i > j\) such that \(i + j\) is odd,  
\[
(j, j + 1), (j + 1, j + 2), \ldots, (i - 1, i)
\]
is a path from \(j\) to \(i\). The cardinality of \(I\) is the number of pairs \((i, j)\) in \(\{1, 2, \ldots, n\}\) such that \(i \neq j\), and this is easily seen to be \(n(n - 1)/2\). We know from (2.2) that the rank of \(\text{Sing}_n\) is at least \(n(n - 1)/2\), and so the rank is \(n(n - 1)/2\), as required.

We have established that \(\text{Sing}_n\) is a rig (a regular idempotent-generated semigroup), and we also know its rank. In fact we know a little more: if \(S\) is an
idempotent-generated semigroup, with set $E$ of idempotents, we can define the idempotent rank $\text{idrank}(S)$ by
\[ \text{idrank}(S) = \min\{|A| : A \subseteq E \text{ and } \langle A \rangle = S\}. \]

It is conceivable that $\text{idrank}(S) > \text{rank}(S)$ (we shall in due course encounter a natural example), but it is clear from the arguments above that
\[ \text{idrank}(\text{Sing}_n) = \text{rank}(\text{Sing}_n) = \frac{n(n-1)}{2}. \]  

2.2 The Rank of $K_{n,r}$

We have seen that $\text{Sing}_n$ is the special case of the semigroup $K_{n,r} = \{ \alpha \in T_n : |\text{im} \alpha| \leq r \}$ for which $r = n-1$. It is natural, therefore, to seek to generalise (2.4) to other values of $r$. We obtain the following result:

**Theorem 2.4**

\[ \text{idrank}(K_{n,r}) = \text{rank}(K_{n,r}) = S(n, r) \]

where $S(n, r)$ is the Stirling number of the second kind, as defined in (1.17).

As in the previous case, we know that the rank must be at least the number of $R$-classes, and so the rank is at least $S(n, r)$. The main burden of the proof, to be found in [17], is to find $S(n, r)$ idempotents that generate $K_{n,r}$. This is quite complicated and it is inappropriate to give details here. To give an indication of how it is done, denote the statement of the theorem by $P(n, r)$, noting that $P(n, n-1)$ is known to be true for every $n$. Then prove that $P(n, 2)$ is true for every $n$. Finally, establish that, for all $1 \leq r \leq n-1$,
\[ P(n-1, r-1) \text{ and } P(n-1, r) \Rightarrow P(n, r). \]

A double induction based on Pascal’s Triangle does the rest. For example, suppose we wish to establish $P(6, 3)$: from $P(4, 2)$ and $P(4, 3)$ we obtain $P(5, 3)$; then from $P(5, 2)$ and $P(5, 3)$ we obtain $P(6, 3)$.

Garba [5] extended this result to cover the larger semigroup $PT_n$ of partial mappings of $X_n$ into itself: if $K'_{n,r}$ is the semigroup $\{ \alpha \in PT_n : |\text{im} \alpha| \leq r \}$, then
\[ \text{idrank}(K'_{n,r}) = \text{rank}(K'_{n,r}) = S(n+1, r+1). \]

2.3 Order-preserving mappings

A mapping in $T_n$ is called order-preserving if, for all $i, j$ in $\{1, 2, \ldots, n\}$,
\[ i \leq j \Rightarrow i\alpha \leq j\alpha. \]
The only non-singular mapping with this property is the identity mapping. The set of all order-preserving mappings is a subsemigroup of $T_n$, denoted by $O_n$. The cardinality of $O_n$ is not immediately obvious, but it can be found by a relatively easy argument.

A typical example, with $n = 8$, is

$$
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 5 & 5 & 6 & 6 & 7 & 7
\end{pmatrix}.
$$

There are occurrences at 2, 3, 5, 6, 7, and we can codify $\alpha$ in terms of occurrences $o$ and gaps $g$:

$$
googogoogoogoog.
$$

In general, $\alpha$ can be coded by a word in the alphabet $\{o, g\}$ containing $n$ $o$’s and $n - 1$ $g$’s. So the number of possible words is

$$
\frac{(2n - 1)!}{n!(n - 1)!} = \binom{2n - 1}{n - 1};
$$

thus

$$
|O_n| = \binom{2n - 1}{n - 1}.
$$

More interesting is the number of idempotents in $O_n$: let us denote this number by $\varphi(n)$. Idempotent maps from $X_n$ to $X_n$ divide into three types:

(a) maps $\alpha$ for which $\text{im } \alpha \subseteq X_{n-1}$;

(b) maps $\alpha$ for which $X_{n-1} \alpha \subseteq X_{n-1}$ and $n\alpha = n$;

(c) maps $\alpha$ for which $X_{n-1} \alpha$ includes $n$

It is not hard to see that there are $\varphi(n-1)$ maps of type (a) and $\varphi(n-1)$ maps of type (b). To count the maps of type (c), notice that, by the order-preserving property, $(n-1)\alpha = n$. Hence, by the idempotent property – which implies that $r\alpha = r$ for every $r$ in $\text{im } \alpha$ – we must have that $n - 1 \notin \text{im } \alpha$. The idempotents $\alpha$ of type (c) are therefore in one-one correspondence with idempotents $\alpha^*$: $X_{n-1} \to X_{n-1}$, where $\alpha^*$ is identical to $\alpha$ except that $i\alpha^* = n - 1$ whenever $i\alpha = n$. The idempotents $\alpha^*$ are in $O_{n-1}$, but not every idempotent in $O_{n-1}$ appears in this way: the ones that are missing are those that map $X_{n-1}$ into $X_{n-2}$ – the type (a) idempotents in $O_{n-1}$ in fact. So there are $\varphi(n-1) - \varphi(n-2)$ idempotents of type (c).

We deduce that

$$
\varphi(n) = 3\varphi(n-1) - \varphi(n-2),
$$

and this, together with the easy observations that $\varphi(1) = 1$ and $\varphi(2) = 3$, enables us to obtain an expression for $\varphi(n)$.

However, things are better than they appear. Consider the Fibonacci numbers $F_n$, defined by

$$
F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \ (n \geq 3).
$$
Then
\[ F_{2n} = F_{2n-1} + F_{2n-2} = 2F_{2n-2} + F_{2n-3} \]
\[ = 3F_{2n-2} + F_{2n-2} + F_{2n-3} \]
\[ = 3F_{2n-2} - (F_{2n-3} + F_{2n-4}) + F_{2n-3} \]
\[ = 3F_{2n-2} - F_{2n-4} . \]

The initial values work out in just the right way, and we deduce

**Theorem 2.5** The number of idempotents in \( O_n \) is the Fibonacci number \( F_{2n} \).

The semigroup \( O_n \) is certainly regular: for example, if
\[ \alpha = \left( \begin{array}{cccc} \{1,2,3\} & \{4\} & \{5,6\} & \{7,8\} \\ 3 & 5 & 7 & 8 \end{array} \right) \] (2.5)
then (not uniquely)
\[ \alpha' = \left( \begin{array}{cccc} \{1,2,3\} & \{4,5\} & \{6,7\} & \{8\} \\ 1 & 4 & 6 & 8 \end{array} \right) \]
is such that \( \alpha \alpha' \alpha = \alpha \) and \( \alpha' \alpha \alpha' = \alpha' \), and it follows from [15, Proposition 2.4.2] that, in \( O_n \),
\[ \alpha \mathcal{L} \beta \text{ if and only if } \text{im} \alpha = \text{im} \beta , \]
\[ \alpha \mathcal{R} \beta \text{ if and only if } \ker \alpha = \ker \beta , \]
\[ \alpha \mathcal{J} \beta \text{ if and only if } |\text{im} \alpha| = |\text{im} \beta | . \]

Thus \( O_n \), like \( T_n \) itself, is the union of \( \mathcal{J} \)-classes \( J_1, J_2, \ldots, J_n \), where \( J_r = \{ \alpha \in O_n : |\text{im} \alpha| = r \} \). In particular, \( J_n \) consists solely of the identity mapping.

Notice that the \( \mathcal{H} \)-classes are trivial: in the example (2.5) above, once the image is fixed to be \( \{3, 5, 7, 8\} \) and the kernel classes are fixed to be \( \{1, 2, 3\}, \{4\}, \{5, 6\} \text{ and } \{7, 8\} \), there is only one order-preserving map possible. Notice too that the kernel classes \( C \) are convex, in the sense that \( x, y \in C \) and \( x \leq z \leq y \) implies that \( z \in C \). If \( \alpha \in J_{n-1} \), then \( \ker \alpha \) has only one non-singleton class, and the only possibilities, \( n - 1 \) in number, are
\[ \{1,2\}, \{2,3\}, \ldots, \{n-1,n\} . \]

Denote by \(|i, i+1|\) the equivalence whose only non-singleton class is \( \{i, i+1\} \).

There are \( n \) possibilities for \( \text{im} \alpha \), namely
\[ X_n \setminus \{1\}, X_n \setminus \{2\}, \ldots, X_n \setminus \{n\} , \]
and so \( |J_{n-1}| = n(n-1) \).

The (unique) element with kernel \(|i, i+1|\) and image \( X_n \setminus \{k\} \) is
\[ \left( \begin{array}{cccc} 1 & \ldots & i & i+1 & \ldots & k & k+1 & \ldots & n \\ 1 & \ldots & i & i & \ldots & k-1 & k+1 & \ldots & n \end{array} \right) \] (2.6)
if $k \geq i + 1$, and is
\[
\begin{pmatrix}
1 & \ldots & k - 1 & k & \ldots & i & i + 1 & \ldots & n \\
1 & \ldots & k - 1 & k + 1 & \ldots & i + 1 & i + 1 & \ldots & n
\end{pmatrix}
\] (2.7)
if $k \leq i$. A useful notation for the decreasing element (2.6) is
\[
[k \to k - 1 \to \cdots \to i];
\]
for the increasing element (2.7) we write
\[
[k \to k + 1 \to \cdots \to i + 1].
\]

There are $n - 1$ decreasing idempotents $[i \to i - 1]$ and $n - 1$ increasing idempotents $[i - 1 \to i]$ ($i = 2, 3, \ldots, n$). Thus $E_1$ the set of all idempotents in $J_{n-1}$, has $2n - 2$ elements in all. From
\[
[k \to k - 1 \to \cdots \to i] = [i + 1 \to i] [i + 2 \to i + 1] \cdots [k \to k - 1]
\]
and
\[
[k \to k + 1 \to \cdots \to i + 1] = [i \to i + 1] [i - 1 \to i] \cdots [k \to k + 1]
\]
we deduce that $\langle E_1 \rangle$ contains the whole of $J_{n-1}$.

We now quote a very old result in [12] to the effect that any element $\alpha$ for which $|\text{im } \alpha| \leq n - 2$ is expressible as a product of elements in $J_{n-1}$. We deduce that $\langle E_1 \rangle = O_n \setminus \{\iota\}$ (where $\iota$ of course denotes the identity mapping of $X_n$). We also note that
\[
\text{rank}(O_n \setminus \{\iota\}) \leq 2n - 2, \quad \text{idrank}(O_n \setminus \{\iota\}) \leq 2n - 2.
\]
In fact, it is shown in in [6] that $O_n \setminus \{\iota\}$ is an example where the rank and the idempotent rank are distinct:

**Theorem 2.6**

\[
\text{rank}(O_n \setminus \{\iota\}) = n, \quad \text{idrank}(O_n \setminus \{\iota\}) = 2n - 2.
\]

We can write $O_n \setminus \{\iota\}$ as $L_{n,n-1}$, where $L_{n,r} = \{\alpha \in O_n : |\text{im } \alpha|\}$. The distinction between rank and idempotent rank disappears when $r \leq n - 2$: Garba [5] showed that, for $1 \leq r \leq n - 2$
\[
\text{rank}(L_{n,r}) = \text{idrank}(L_{n,r}) = \binom{n}{r}.
\]

### 2.4 Order-decreasing transformations

A map $\alpha : X_n \to X_n$ is called order-decreasing if, for all $i$ in $X_n$, $x \alpha \leq x$. It is clear that $1 \alpha = 1$, that $2 \alpha = 1$ or 2, etc., and so $D_n$, the semigroup of all order-decreasing maps, is of cardinality $n!$. Some more interesting combinatorics (as
with the case of $O_n$) arises if we ask for the number of idempotents in $D_n$, which turns out to be the Bell number $B_n$, which can be expressed in terms of Stirling numbers:

$$B_n = \sum_{r=1}^{n} S(n, r).$$

Higgins [8] considered $O_n \cap D_n$, the semigroup of all maps that are both order-preserving and order-decreasing, and showed $|O_n \cap D_n| = C_n$, the Catalan number, defined as

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

This is an interesting area, involving algebra, combinatorics, and even asymptotics, but I have chosen to stop here, and to transfer attention to the full transformation semigroup $T_X$ where $X$ is infinite.
Chapter 3

Infinite sets

Let $X$ be an infinite set, and let $\alpha : X \to X$ be an element of $T_X$, the semigroup of all selfmaps of $X$. We define

\[ \text{fix } \alpha = |\{ x \in X : x\alpha = x \}|, \]
\[ \text{shift } \alpha = |\{ x \in X : x\alpha \neq x \}|, \]
referring to the fixes of $\alpha$ and the shifts of $\alpha$. A map $\alpha : X \to X$ is called singular if $\text{im } \alpha \subset X$.

We define the defect $\text{def } \alpha$ of $\alpha$ by

\[ \text{def } \alpha = |X \setminus \text{im } \alpha|, \]
and the collapse $\text{coll } \alpha$ by

\[ \text{coll } \alpha = \sum_{y \in \text{im } \alpha} (|y\alpha^{-1}| - 1). \]

The defect is easily understood as a measure of how far $\alpha$ is from being onto; the collapse perhaps requires explanation. First, by $y\alpha^{-1}$ we mean the set $\{ x \in X : x\alpha = y \}$. If $\alpha$ is one-one, then $\text{coll } \alpha = 0$; in general we can regard $\text{coll } \alpha$ as a measure of how far $\alpha$ is from being one-one. It is clear that $\alpha \in S_X$, the symmetric group on the set $X$, if and only if $\text{def } \alpha = \text{coll } \alpha = 0$.

We shall eventually consider in more detail the nature of idempotents in $T_X$, but for the moment, note that the only one-one idempotent is $\iota$, the identity map. For if $\epsilon$ is idempotent and one-one, then, for all $x$ in $X$,

\[ (x\epsilon)\epsilon = x\epsilon, \]

and so, by the one-one property, $x\epsilon = x$ for all $x$.

It is reasonable to ask whether the result that all maps in $T_X \setminus G_X$ can be expressed as products of idempotents, but one quickly learns that this is not the case. In the infinite case, let $\alpha$ be one-one but not onto, and suppose that

\[ \alpha = \epsilon_1\epsilon_2 \ldots \epsilon_n, \]
a product of idempotents. Then, for all $x$ in $X$, 

$$x\alpha = x\epsilon_1\epsilon_2 \ldots \epsilon_n = x\epsilon_1^2\epsilon_2 \ldots \epsilon_n = (x\epsilon_1)\alpha.$$ 

Since $\alpha$ is one-one, it follows that $x\epsilon_1 = x$. This holds for all $x$ in $X$, and so $\epsilon_1$ is the identity element $i$. Thus $\epsilon_1$ can be left out. The same argument then applies to $\epsilon_2$, and so all along the line. The conclusion is that $\alpha = i$, and this is a contradiction. A similar argument, affecting the right hand side of the product, applies if $\alpha$ is onto but not one-one.

Let $\epsilon(i \neq i)$ be an idempotent in $T_X$. If $x$ is shifted by $\epsilon$, then $x\epsilon$ is fixed, and we conclude that 

$$\text{def } \epsilon = \text{shift } \epsilon.$$ 

An example may help: let 

$$\epsilon = \begin{pmatrix} 1, 2, 3 & 4, 5, 6 & 7, 8, 9 \\ 3 & 6 & 9 \end{pmatrix}$$ 

Then 

$$X \setminus \text{im } \epsilon = \{ x : x\epsilon \neq x \} = \{ 1, 2, 4, 5, 7, 8 \}$$ 

and so certainly $\text{shift } \epsilon = \text{def } \epsilon = 6$. As for the collapse, we see that 

$$\text{coll } \epsilon = (\{1, 2, 3\} - 1) + \cdots + (\{4, 5, 6\} - 1) + (\{7, 8, 9\} - 1) = 6$$ 

—the same as the shift and defect. This is true in general: for every idempotent in $T_X$, 

$$\text{shift } \epsilon = \text{def } \epsilon = \text{coll } \epsilon.$$ 

This works for an infinite example also: for the idempotent $\eta$ we have 

$$\eta = \begin{pmatrix} 1, 2 & 3, 4 & 5, 6 & \cdots \\ 2 & 4 & 6 & \cdots \end{pmatrix},$$ 

and so $\text{shift } \eta = \text{def } \eta = \text{coll } \eta = \aleph_0$.

It seems reasonable (principally when $X$ is infinite) to refer to an element $\alpha$ of $T_X$ for which 

$$\text{shift } \alpha = \text{def } \alpha = \text{coll } \alpha$$

as balanced. For each cardinal $k$ such that $\aleph_0 \leq k \leq |X|$, let $B_k$ be the set of all elements $\eta$ of $T_X$ such that 

$$\text{shift } \eta = \text{def } \eta = \text{coll } \eta = k.$$ 

Then, not quite obviously, each $B_k$ is a subsemigroup of $T_X$ and so is 

$$B = \bigcup_{\aleph_0 \leq k \leq |X|} B_k.$$ 

The set 

$$F = \{ \alpha \in T_X : \text{shift } \alpha < \infty \text{ and } \text{def } \alpha > 0 \}$$

is also a subsemigroup.

Then we have the following result [11]:
Theorem 3.1 Let $X$ be infinite, and let $E$ be the set of singular idempotents in $T_X$. Then

$$\langle E \rangle = F \cup B.$$  

The result has been refined. Let us denote the set of idempotents of finite, non-zero shift by $E_F$ and the idempotents in $B_k$ by $E_k$.

$$\langle E_F \rangle = F, \quad \langle E_k \rangle = B_k \quad (8_0 \leq k \leq |X|). \quad (3.3)$$

Given a set $A$ of generators for a semigroup $S$, and a positive integer $k$, let $A^{[k]} = A \cup A^2 \cup A^3 \cup \cdots$. In an infinite semigroup it is possible that the ascent

$$A \subset A^{[2]} \subset A^{[3]} \subset \cdots$$

is infinite, and in such a case we say that $S$ has infinite $A$-depth, and write $\Delta_A(S) = \infty$. Otherwise the ascent terminates, and we define the $A$-depth by

$$\Delta_A(S) = \min \{ k \in N : A^{[k]} = S \}.$$  

If $A$ consists of idempotents, it is clear that

$$A \subseteq A^2 \subseteq A^3 \subseteq \cdots,$$

and so $A^{[k]} = A^k$.

Consider an element $\alpha$ in the semigroup $F$ defined by (3.2), and suppose that

$$\alpha = \epsilon_1 \epsilon_2 \cdots \epsilon_k,$$  

(3.4)

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in E_F$. It is easy to show the following properties, where $\alpha, \beta \in F$ and $\epsilon \in E_F$:

$$\text{shift } (\alpha \beta) \leq \text{shift } \alpha + \text{shift } \beta, \quad (3.5)$$

$$\text{shift } \epsilon = \text{def } \epsilon, \quad (3.6)$$

$$\text{def } (\alpha \beta) \geq \text{def } \alpha, \quad (3.7)$$

$$\text{def } (\alpha \beta) \geq \text{def } \beta. \quad (3.8)$$

Returning to (3.4), let us suppose that $\text{def } \alpha = d$. It follows from (3.7) and (3.8) that $\text{def } \epsilon_i \leq d$ for $i = 1, 2, \ldots, k$. Hence, by (3.6), $\text{shift } \epsilon_i \leq d$ for all $i$, and from (3.5) it follows that $\text{shift } \alpha \leq kd$. Thus

$$\frac{\text{shift } \alpha}{\text{def } \alpha} \leq k.$$  

To find an element not belonging to $E^k_F$ we need only find an element for which $\text{shift } \alpha / \text{def } \alpha > k$. Such an element is given by selecting elements $x_1, x_2, \ldots, x_{k+2}$ in $X$, and defining $\alpha$ by

$$x_i = x_{i+1} \quad (i = 1, 2, \ldots, k + 1), \quad x \alpha = x \quad \text{otherwise}.$$
It is clear that shift $\alpha = k + 1$, def $\alpha = \|x_1\| = 1$. We deduce that

$$\Delta_{E_{E'}}(F) = \infty.$$  

Given the highly infinite definition of $B_k$, one might expect that $\Delta_{E_k}(B_k)$ would also be infinite, but the unexpected outcome is that, for all $k$ such that $8_0 \leq k \leq |X|$, 

$$\Delta_{E_k}(B_k) = 4.$$  

The hard bit of this is to prove that the answer is not 3. Several results analogous to Theorem 3.1 have been proved. In particular there is a linear version, concerned with the semigroup $\text{Sing}_L$ of singular self-maps of an infinite-dimensional linear space $L$. For a given $\alpha$, both $\text{im}\, \alpha$ and $\text{fix}\, \alpha = \{v \in L : v\alpha = v\}$ are subspaces, and so is $\ker\, \alpha = \{v \in L : v\alpha = 0\}$. Then

$$\begin{align*}
\text{shift}\, \alpha &= \text{codim}(\text{fix}\, \alpha), \\
\text{def}\, \alpha &= \text{codim}(\text{im}\, \alpha), \\
\text{coll}\, \alpha &= \text{dim}(\ker\, \alpha).
\end{align*}$$

are obvious analogues of the shift, defect and collapse in the set-theoretic case. Then let $F$ be the semigroup of all linear maps $\alpha$ for which $\text{shift}\, \alpha$ is finite and $\text{def}\, \alpha \neq 0$, and let $B = \{\alpha \in \text{Sing}_L : \text{shift}\, \alpha = \text{def}\, \alpha = \text{coll}\, \alpha \geq 8_0\}$. Then we have the direct analogue of Theorem 3.1, established by Reynolds and Sullivan:\[20]

**Theorem 3.2** Let $L$ be an infinite-dimensional linear space and let $\text{Sing}_L$ be the semigroup of all singular linear selfmaps of $L$. Let $E$ be the set of idempotent maps in $\text{Sing}_L$. Then $\langle E \rangle = F \cup B$.

As in the previous case, we can be more precise. For each infinite cardinal $k$ not greater than $\dim L$, let

$$B_k = \{\alpha \in \text{Sing}_L : \text{shift}\, \alpha = \text{def}\, \alpha = \text{coll}\, \alpha = k\} ,$$

and let $E_k = E \cap B_k$. Then

$$\langle E \cap F \rangle = F \quad \text{and, for each } k, \quad \langle E_k \rangle = B_k .$$

The analogy breaks down at this point, for

$$\Delta_{E_k}(B_k) = 3 .$$

Theorems 3.1 and 3.2 have been generalised by Fountain and Lewin [4] to the case of an *independence algebra*, a universal algebra concept that includes both the set case and the linear case, and Fountain [3] was later able to ‘explain’ in universal algebra terms the cause of the different values of $\Delta_{E_k}(B_k)$ in the two cases described. I will not go into details, but the existence of a zero in the linear case has something to do with the result.
Chapter 4

Relative Rank

4.1 Rank and Relative Rank

The material of this section comes from [9] and [16].

It is clear that the notion of the rank of $T_X$ is of no interest in the case where $X$ is infinite. For any semigroup $S$ with order $k > \aleph_0$, all one can say is that the rank of $S$ is $k$. Even if $|X| = \aleph_0$, the smallest infinite cardinal, the semigroup $T_X$ is of order $2^{\aleph_0}$. So its rank is also $2^{\aleph_0}$.

There is, however, another notion that yields some interesting—and in some cases slightly surprising—results. For a semigroup $S$ and a subset $A$ of $S$, the relative rank of $S$ modulo $A$ is given by

$$\text{rank}(S : A) = \min\{|B| : \langle A \cup B \rangle = S\}.$$ (4.1)

So, for example, if $X_n = \{1, 2, \ldots, n\}$ is finite, then

$$\text{rank}(T_n : S_n) = 1,$$ (4.2)

since, together with the permutations in $S_n$, we need just one element of $J_{n,n-1}$ to generate $T_n$. (See (1.3)). Again, if $E_n$ is the set of idempotents of $T_n$, then

$$\text{rank}(T_n : E_n) = 2.$$

Certain properties are clear:

$$\text{rank}(S : \emptyset) = \text{rank}S, \quad \text{rank}(S : S) = 0, \quad \text{rank}(S : \langle A \rangle) = \text{rank}(S : A),$$

and $\text{rank}(S : A) = 0$ if and only if $\langle A \rangle = S$. Also, if $G$ is a group and $N$ is a normal subgroup of $G$, then $\text{rank}(G : N) = \text{rank}(G/N)$.

In exploring the notion for infinite transformation semigroups, we shall require the previously encountered measures shift, defect and collapse. We also need a new parameter. Let $\alpha \in T_X$, where $X$ is infinite. Let

$$K(\alpha) = \{x \in X : |x\alpha^{-1}| = |X| \};$$
then the infinite contraction index \( \text{con} \alpha \) is defined as \( |K(\alpha)| \). It is the number of \((\ker \alpha)\)-classes of size \(|X|\).

We will require a definition concerning cardinal numbers. A cardinal \( k \) is singular if there exist sets \( Y \) and \( Z_y (y \in Y) \) such that

\[
|Y| < k, \quad |Z_y| < k \quad (y \in Y),
\]

but

\[
\left| \bigcup_{y \in Y} Z_y \right| = k.
\]

(An example of a singular cardinal is \( \aleph_\omega \), defined as the limit of the sequence \( \aleph_0, \aleph_1, \aleph_2, \ldots \)). A cardinal that is not singular is called regular. The following lemma is useful:

**Lemma 4.1** Let \( \alpha, \beta \in T_X \), where \( X \) is infinite. Then

(i) \( \text{def} (\alpha \beta) \leq \text{def} (\alpha) + \text{def} (\beta) \);

(ii) if \( |X| \) is a regular cardinal, then \( \text{con} (\alpha \beta) \leq \text{con} \alpha + \text{con} \beta \).

**Proof** Part (i) is already established in Theorem 1.9. To prove Part (ii), suppose that \( x \in K(\alpha \beta) \). Then

\[
|X| = |x(\alpha \beta)^{-1}| = |x\beta^{-1}\alpha^{-1}| = \left| \bigcup_{y \in x\beta^{-1}} y\alpha^{-1} \right|.
\]

Since \(|X|\) is regular, means that either \(|x\beta^{-1}| = |X|\), or \(|y\alpha^{-1}| = |X|\) for some \( y \) in \( x\beta^{-1} \). The the former case we have \( x \in K(\beta) \), while in the latter case \( x \in K(\alpha)\beta \). We have shown that

\[
K(\alpha \beta) \subseteq K(\alpha)\beta \cup K(\beta);
\]

hence

\[
\text{con} (\alpha \beta) = |K(\alpha \beta)| \leq |K(\alpha)\beta| + |K(\beta)| \leq |K(\alpha)| + |K(\beta)| = \text{con} \alpha + \text{con} \beta.
\]

□

From (4.2) we know that \( \text{rank}(T_X : S_X) = 1 \) when \( X \) is finite. This is not the case for \( X \) infinite:

**Lemma 4.2** Let \( X \) be infinite and let \( \mu \in T_X \). Then \( \langle S_X, \mu \rangle \neq T_X \). Consequently \( \text{rank}(T_X, S_X) > 1 \).

**Proof** Suppose, for a contradiction, that \( \langle S_X, \mu \rangle = T_X \), and let \( \alpha \) be a one-one mapping from \( X \) into a proper subset of \( X \). Decompose \( \alpha \) into a product of elements from \( S_X \cup \{\mu\} \):

\[
\alpha = \beta_1\beta_2 \ldots \beta_k.
\]
Since $\alpha$ is not a bijection, at least one of the $\beta_i$ must be $\mu$; let $\beta_j$ be the first such. Then
\[
\beta_j^{-1} \ldots \beta_1^{-1} \alpha = \mu \beta_{j+1} \ldots \beta_k.
\]
The left hand side of the above equality is certainly one-one, and hence so is $\mu$. Hence $\langle S_X, \mu \rangle$ must consist entirely of one-one mappings, and so cannot coincide with $T_X$.

When $X$ is finite the top $J$-class of $T_X$, namely
\[
J = \{ \alpha \in T_X : |\text{im } \alpha| = |X| \},
\tag{4.3}
\]
is simply the symmetric group $S_X$. This is no longer the case when $X$ is infinite, for there are many non-bijections whose image has cardinality $|X|$. In fact we have

**Lemma 4.3** Let $J$ be defined by (4.3). Then $\langle J \rangle = T_X$, and so $\text{rank}(T_X : J) = 0$.

**Proof** Let $\alpha \in T_X$. Write $X$ as a disjoint union $Y \cup Z$, where $|X| = |Y| = |Z|$. Let $\beta : X \to Y$ be an arbitrary bijection. Then $\beta \in J$, since $|\text{im } \beta| = |Y| = |X|$. Let $\gamma$ in $T_X$ be defined by
\[
x\gamma = \begin{cases} 
x \beta^{-1} \alpha & \text{if } x \in Y \\
x & \text{if } x \in Z.
\end{cases}
\]
Since $Z \subseteq \text{im } \gamma$ and $|Z| = |X|$, we have $\gamma \in J$.

Now, since $\beta$ maps $X$ into $Y$,
\[
x\beta \gamma = x\beta \beta^{-1} \alpha = x\alpha
\]
for all $x$ in $X$. Thus $\alpha = \beta \gamma \in \langle J \rangle$.

We can now prove the following result:

**Theorem 4.4** $\text{rank}(T_X : S_X) = 2$.

**Proof** Let $\mu$ be an injection (one-one) of defect $|X|$ and let $\nu$ be a surjection (onto) with the property that all the $\langle \ker \nu \rangle$-classes have cardinality $|X|$. In view of Lemmas 4.2 and 4.3, it is enough to show that $\langle S_X, \mu, \nu \rangle$ contains $J$, the top $J$-class of $T_X$.

Let $\alpha \in T_X$ be an arbitrary mapping in $J$; that is, $|\text{im } \alpha| = |X|$. We shall show that $\alpha$ can be expressed as a product of $\mu, \nu$ and two bijections $\pi$ and $\sigma$ which we define below.

Let $\Lambda$ be an index set of cardinality $|X|$, partitioned into two disjoint subsets $\Lambda_1, \Lambda_2$, both of cardinality $|X|$:
\[
\Lambda = \Lambda_1 \cup \Lambda_2, \quad \Lambda_1 \cap \Lambda_2 = \emptyset, \quad |\Lambda_1| = |\Lambda_2| = |X|.
\]
Since $\con = |X|$, we can index the ker-$\nu$-classes having cardinality $|X|$ using $\Lambda$; so suppose that the classes of cardinality $|X|$ are $B_{\lambda}$ ($\lambda \in \Lambda$). By the same token, since $|\text{im } \alpha| = |X|$, the ker-$\alpha$-classes can be indexed by $\Lambda_1$; denote them by $C_{\lambda}$ ($\lambda \in \Lambda_1$).

We now define $\pi$. First, since $|B_{\lambda}| = |X|$ for each $\lambda \in \Lambda_1$ there exists, for each $\lambda \in \Lambda_1$, an injection $\pi_{\lambda} : C_{\lambda} \mu \to B_{\lambda}$ ($\lambda \in \Lambda_1$).

Let

$$\pi = \bigcup_{\lambda \in \Lambda_1} \pi_{\lambda}.$$

Then $\pi$ is a partial injection. Since the domain of $\pi$ is $\text{im } \mu$, we can say that $|X \setminus \text{dom } \pi| = |X \setminus \text{im } \mu| = |X|$.

Note also that $\text{im } \pi \subseteq \bigcup_{\lambda \in \Lambda_1} B_{\lambda}$, and so $|X \setminus \text{im } \pi| \geq \left| \bigcup_{\lambda \in \Lambda_2} B_{\lambda} \right| = |X|$. Hence $\pi$ can be extended to a bijection $\pi \in S_X$.

Next, we define the bijection $\sigma$. Recall that, for each $\lambda \in \Lambda$, the set $B_{\lambda}$ is a ker-$\nu$-class. Hence $B_{\lambda} \nu$ is a single element. Similarly, for each $\lambda \in \Lambda_1$, $C_{\lambda} \alpha$ is a single element. By the Axiom of Choice, we select, for each $\lambda \in \Lambda_1$, an element $z_{\lambda}$ in $(C_{\lambda} \alpha)\nu^{-1}$. (The set $(C_{\lambda} \alpha)\nu^{-1}$ is certainly non-empty, since $\nu$ is a surjection; indeed, the definition of $\nu$ ensures that $|(C_{\lambda} \alpha)\nu^{-1}| = |X|$.) We define a mapping $\sigma : \{B_{\lambda} \nu : \lambda \in \Lambda_1\} \to X$ by the rule that

$$(B_{\lambda} \nu)\sigma = z_{\lambda} \quad (\lambda \in \Lambda_1).$$

For $\lambda$ in $\Lambda_2$ we have $B_{\lambda} \nu \notin \text{dom } \sigma$, and so $|X \setminus \text{dom } \sigma| = |X|$. Note also that $\text{im } \sigma$ contains at most one element from each $(C_{\lambda} \alpha)\nu^{-1}$, and so $|X \setminus \text{im } \sigma| = |X|$. Hence $\sigma$ can be extended to a bijection $\sigma \in S_X$.

We claim now that $\alpha = \mu \nu \sigma \nu$.

Let $x \in X$ Then $x \in C_{\lambda}$ for some $\lambda$ in $\Lambda_1$. Then $x \mu \in C_{\lambda} \mu$ and so

$$x \mu \nu \in (C_{\lambda} \mu)\nu = B_{\lambda} \nu,$$

a singleton. It follows that $x \mu \nu \sigma \in (C_{\lambda} \alpha)\nu^{-1}$, and so

$$x \mu \nu \sigma \nu = C_{\lambda} \alpha = x \alpha.$$

It follows that $(S_X, \mu, \nu) = T_X$, and so rank($T_X : S_X$) = 2. \qed

A similar argument establishes that, if $X$ is infinite, rank($T_X : E_X$) = 2, where $E_X$ is the set of idempotents in $T_X$. 

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CHAPTER 4. RELATIVE RANK

Since $\con = |X|$, we can index the ker-$\nu$-classes having cardinality $|X|$ using $\Lambda$; so suppose that the classes of cardinality $|X|$ are $B_{\lambda}$ ($\lambda \in \Lambda$). By the same token, since $|\text{im } \alpha| = |X|$, the ker-$\alpha$-classes can be indexed by $\Lambda_1$; denote them by $C_{\lambda}$ ($\lambda \in \Lambda_1$).

We now define $\pi$. First, since $|B_{\lambda}| = |X|$ for each $\lambda \in \Lambda_1$ there exists, for each $\lambda \in \Lambda_1$, an injection $\pi_{\lambda} : C_{\lambda} \mu \to B_{\lambda}$ ($\lambda \in \Lambda_1$).

Let

$$\pi = \bigcup_{\lambda \in \Lambda_1} \pi_{\lambda}.$$

Then $\pi$ is a partial injection. Since the domain of $\pi$ is $\text{im } \mu$, we can say that $|X \setminus \text{dom } \pi| = |X \setminus \text{im } \mu| = |X|$.

Note also that $\text{im } \pi \subseteq \bigcup_{\lambda \in \Lambda_1} B_{\lambda}$, and so $|X \setminus \text{im } \pi| \geq \left| \bigcup_{\lambda \in \Lambda_2} B_{\lambda} \right| = |X|$. Hence $\pi$ can be extended to a bijection $\pi \in S_X$.

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$$(B_{\lambda} \nu)\sigma = z_{\lambda} \quad (\lambda \in \Lambda_1).$$

For $\lambda$ in $\Lambda_2$ we have $B_{\lambda} \nu \notin \text{dom } \sigma$, and so $|X \setminus \text{dom } \sigma| = |X|$. Note also that $\text{im } \sigma$ contains at most one element from each $(C_{\lambda} \alpha)\nu^{-1}$, and so $|X \setminus \text{im } \sigma| = |X|$. Hence $\sigma$ can be extended to a bijection $\sigma \in S_X$.

We claim now that $\alpha = \mu \nu \sigma \nu$.

Let $x \in X$ Then $x \in C_{\lambda}$ for some $\lambda$ in $\Lambda_1$. Then $x \mu \in C_{\lambda} \mu$ and so

$$x \mu \nu \in (C_{\lambda} \mu)\nu = B_{\lambda} \nu,$$

a singleton. It follows that $x \mu \nu \sigma \in (C_{\lambda} \alpha)\nu^{-1}$, and so

$$x \mu \nu \sigma \nu = C_{\lambda} \alpha = x \alpha.$$

It follows that $(S_X, \mu, \nu) = T_X$, and so rank($T_X : S_X$) = 2. \qed

A similar argument establishes that, if $X$ is infinite, rank($T_X : E_X$) = 2, where $E_X$ is the set of idempotents in $T_X$. 

4.2 Countable versus uncountable

Within $T_X$ we have encountered relative ranks of 0, 1 and 2; and of course we also have an uncountable relative rank, namely $\text{rank}(T_X : \emptyset)$. Can we find anything in between?

Nikola Ruškuc proved the following theorem [10]:

**Theorem 4.5** Let $X$ be an infinite set. Then any countable subset $S$ of $T_X$ is contained in a 2-generated subsemigroup of $T_X$.

**Proof** Write $S$ as $\{\theta_1, \theta_2, \ldots\}$. Partition $X$ into a countable disjoint union of infinitely many sets $X_0, X_1, X_2, \ldots$, all of cardinality $|X|$. Then partition $X_0$ into $X_{01}, X_{02}, X_{03}, \ldots$, again all of cardinality $|X|$.

Let $\beta$ be a mapping in $T_X$ that maps $X_n$ bijectively onto $X_{n+1}$ for all $n \geq 0$. Let $\gamma$ be a mapping in $T_X$, mapping $X_n$ bijectively onto $X_{0n}$ for all $n \geq 1$. Although we have yet to define $\gamma$ in $X_0$, we can see that the mapping

$$\delta_n = \beta \gamma \beta_n \gamma$$

is a well-defined bijection of $X$ onto $X_{0n}$. (This is because $\text{im} \beta \cap X_0 = \emptyset$.) We may then complete the definition of $\gamma$ by defining

$$x \delta_n \gamma = x \theta_n \ (x \in X, n \in \mathbb{N}),$$

Thus

$$\theta_n = \delta_n \gamma = \beta \gamma \beta_n \gamma^2,$$

and so $S = \langle \beta, \gamma \rangle$. $\square$

As a consequence, we have the following result concerning relative ranks:

**Corollary 4.6** Let $X$ be an infinite set. The relative rank of a subset $T_X$ over a subset $S$ is either uncountable or at most 2.

**Proof** If $T_X \setminus S$ is uncountable, then $\text{rank}(T_X : S)$ is uncountable. If $T_X \setminus S$ is countable then it can be generated by at most two of its elements, and so $\text{rank}(T_X : S) \leq 2$. $\square$

We were quite proud of Theorem 4.5, but then we discovered that it was first proved by Sierpiński [21] in 1935. His proof was more complicated, but Banach [1] in the same year produced the more elegant proof shown above. Well, at least we were in good company, although nearly 70 years late! The result does not seem to have entered the consciousness of the mathematical community, for the much-quoted result of Evans [2] that every countable semigroup is embeddable in a semigroup of rank 2 follows immediately from Theorem 4.5.

There were, however, some reasonable questions. For example, we defined a contraction in $T_N$ to be a map $\alpha$ such that $|\alpha i - j\alpha| \leq |i - j|$ for all $i, j \in \mathbb{N}$. Then we showed that $\text{rank}(T_N : \mathcal{C})$ is uncountable.
CHAPTER 4. RELATIVE RANK

The huge semigroup $B_X$, consisting of all binary relations on the set $X$, contains $T_X$, also the symmetric inverse semigroup $I_X$; and also the larger set $P_X$ of partial mappings from $X$ to $X$. (If $|X| = n$ then

$$|T_X| = n^n, \quad |P_X| = (n + 1)^n, \quad |B_X| = 2^{n^2}.\)$$

Slightly surprisingly, though one learns not be be too surprised when dealing with infinite sets, if $X$ is infinite,

$$\text{rank}(B_X : T_X) = \text{rank}(B_X : P_X) = \text{rank}(B_X : I_X) = 1, \quad \text{rank}(B_X : S_X) = 2.$$
Bibliography


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