King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 357

Sep 2006

Semigroups of Mappings

John M. Howie

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

Semigroups of Mappings

John M. Howie

School of Mathematics and Statistics
University of St Andrews
North Haugh
St Andrews, Fife KY16 9SS
United Kingdom

Chapter 1

Some Semigroups of Mappings

1.1 The Full Transformation Semigroup

Any abstract algebraic study is rooted in concrete examples: if convincing examples are in short supply then the study is of little or no interest. The origins of group theory are in the study of permutations, and the *symmetric group*, the group of all permutations of a set, is rightly an object of importance within the abstract study. The corresponding object in semigroup theory is the *full transformation semigroup*, the semigroup $\mathcal{T}_X$ of all selfmaps of a set X. For the moment we shall consider a finite set $X_n = \{1, 2, \ldots, n\}$, and we write $\mathcal{T}_{X_n}$ more simply as T_n. The order of T_n is n^n.

The semigroup $\mathcal{T}_X$ has the same universal property as the symmetric group:

Theorem 1.1 *Let S be a semigroup. Then there exists a set X with the property that S is embedded in $\mathcal{T}_X$.*

Proof. Let $X = S \cup \{1\}$. where $1 \notin S$. For each s in S, define $\rho_s : X \to X$ by

$$\begin{aligned} x\rho_s &= xs \text{ if } x \in S\,, \\ 1\rho_s &= s\,. \end{aligned}$$

It is easy to verify that $\rho_s\rho_t = \rho_{st}$, and so the map $s \mapsto \rho_s$ is a homomorphism from S into $\mathcal{T}_X$. It is also one-one, since

$$\rho_s = \rho_t \Rightarrow (\forall x \in X)\, x\rho_s = x\rho_t \Rightarrow 1\rho_s = 1\rho_t \Rightarrow s = t\,.$$

□

1.2 Regular Semigroups

The semigroup $\mathcal{T}_X$ is certainly not a group, but it does have a 'group-like' property. A semigroup S is called *regular* if

$$(\forall a \in S)\,(\exists x \in S)\; axa = a\,. \tag{1.1}$$

A regular semigroup must contain *idempotent* elements – elements e with the property that $e^2 = e$. It immediately follows from (1.1) that both ax and xa are idempotents. In general they will be different idempotents.

It is not hard to prove

Theorem 1.2 *The semigroup $\mathcal{T}_X$ is regular.*

Proof Let $\alpha \in \mathcal{T}_X$. Define a mapping $\xi : X \to X$ as follows. For each y in $\operatorname{im}\alpha$, choose an element x in X such that $x\alpha = y$ and let $y\xi = x$; also, for all y *not* in $\operatorname{im}\alpha$, choose an arbitrary element z of X, and let $y\xi = z$. Then it is clear that, for all x in X.

$$x\alpha\xi\alpha = x\alpha\,.$$

□

Remark 1.3 *The statement that $\mathcal{T}_X$ is regular for all sets X is equivalent to the Axiom of Choice.*

The regularity condition (1.1), somewhat surprisingly, is equivalent to the seemingly stronger condition

$$(\forall a \in S)\,(\exists y \in S)\,(aya = a \text{ and } yay = y)\,; \tag{1.2}$$

for we may take $y = xax$ and observe that

$$\begin{aligned} aya &= a(xax)a = (axa)(xa) = axa = a\,,\\ yay &= (xax)a(xax) = x(axa)xax = x(axa)x = xax = y\,. \end{aligned}$$

An element y satisfying (1.2) is called an *inverse* of a.

More specialised classes of regular semigroups have been studied. The most important is the class of inverse semigroups, defined by the condition that every element has a *unique* inverse.

Theorem 1.4 *A semigroup is an inverse semigroup if and only if it is regular and idempotents commute.*

Proof Suppose that S is regular and that idempotents commute. Let a', a'' be inverses of a. Then

$$\begin{aligned} a' &= a'aa' = a'(aa''a)a' = (a'a)(a''a)a' = a''aa'aa' = a''aa'\\ &= a''(aa''a)a' = a''(aa'')(aa') = a''aa'aa'' = a''(aa'a)a'' = a''aa'' = a''\,. \end{aligned}$$

Conversely, suppose that inverses are unique, and let e, f be idempotents. Let x be the unique inverse of ef.; that is,

$$efxef = ef\,, \quad xefx = x\,.$$

Then fxe is also an inverse of ef, since

$$\begin{aligned}(ef)(fxe)(ef) &= ef^2xe^2f = (ef)x(ef) = ef\,,\\ (fxe)(ef)(fxe) &= f(xefx)e = fxe\,.\end{aligned}$$

Also, fxe is idempotent:

$$(fxe)^2 = f(xefx)e = fxe\,.$$

Hence fxe is its own unique inverse, and so $fxe = ef$. In particular, ef is idempotent, and is its own unique inverse. The same holds for fe. But

$$(ef)(fe)(ef) = (ef)^2 = ef\,, \quad (fe)(ef)(fe) = (fe)^2 = fe\,,$$

and so ef and fe are mutually inverse. Hence $ef = fe$. □

The class of inverse semigroups, like the class of all semigroups, is rooted in a concrete example. Let X be a non-empty set. The *symmetric inverse semigroup* $\mathcal{I}_X$ is the set of all partial one-one mappings of X. If $\alpha, \beta \in \mathcal{I}_X$, then

$$\mathrm{dom}\,(\alpha\beta) = (\mathrm{im}\,\alpha \cap \mathrm{dom}\,\beta)\alpha^{-1}\,, \quad \mathrm{im}\,(\alpha\beta) = (\mathrm{im}\,\alpha \cap \mathrm{dom}\,\beta)\beta\,,$$

and $x(\alpha\beta) = (x\alpha)\beta$ for all x in $\mathrm{dom}\,(\alpha\beta)$. The idempotents of $\mathcal{I}_X$ are the partial identity mappings id_A, where A is a subset of X. (This includes the empty set.) The Vagner–Preston Theorem [23, 19, 15] establishes a universal property analogous to Theorem 1.1: every inverse semigroup is embeddable in a symmetric inverse semigroup.

Other classes of regular semigroups do not have this direct link with mappings. *Orthodox* semigroups, regular semigroups in which the set of idempotents is a subsemigroup, have been studied, but there are few examples 'in nature'. Much commoner are *regular idempotent-generated* semigroups, called *rigs* by Clifford. There is no very accessible theory of rigs, but, as we shall see, examples abound.

1.3 The Structure of T_n

The *rank* $\mathrm{rank}(S)$ of a semigroup S is defined by

$$\mathrm{rank}(S) = \min\{|A| : \langle A\rangle = S\}\,.$$

The rank of T_n is 3: a set of generators is

$$\{(1\,2)\,,\ (1\,2\,\ldots\,n)\,,\ \tbinom{1}{2}\}\,. \qquad (1.3)$$

The first two generators are cycles, and generate S_n; the third is the singular mapping

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 2 & 3 & \dots & n \end{pmatrix} .$$

The choice of $\binom{1}{2}$ is arbitrary: in fact any element α such that $|\mathrm{im}\,\alpha| = n-1$ would do as well.

It is clear that the rank must be at least 3, for two generators are required to generate the symmetric group S_n, and at least one extra 'singular' generator is needed. It is less obvious that these 3 are sufficient: see [15, Chapter 1, Exercises 6 & 7].

For $r \in X_n$, let

$$J_{n,r} = \{\alpha \in T_n : |\mathrm{im}\,\alpha| = r\}\,; \tag{1.4}$$

thus $J_{n,n}$ is the symmetric group S_n and, at the other extreme,

$$J_{n,1} = \{\kappa_i : i = 1, 2, \dots, n\}\,,$$

where, in an obvious and standard notation

$$\kappa_i = \begin{pmatrix} 1 & 2 & \dots & n \\ i & i & \dots & i \end{pmatrix} .$$

Let

$$K_{n,r} = \{\alpha \in T_n : |\mathrm{im}\,\alpha| \le r\} = \bigcup_{s=1}^{r} J_{n,s}\,. \tag{1.5}$$

It is easy to see that, for all α, β in T_n,

$$|\mathrm{im}\,(\alpha\beta)| \le \min\{|\mathrm{im}\,\alpha|, |\mathrm{im}\,\beta|\}\,, \tag{1.6}$$

and it follows immediately that $K_{n,r}$ is a subsemigroup of T_n. It is indeed an *ideal*, in the sense that

$$K_{n,r}T_n \subseteq K_{n,r}\,, \quad T_nK_{n,r} \subseteq K_{n,r}\,.$$

Clearly $K_{n,n} = T_n$. More interestingly, the set

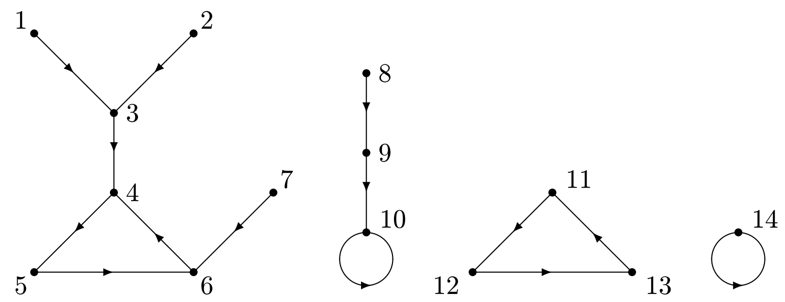
$$K_{n,n-1} = T_n \setminus S_n\,,$$

the set of *singular* selfmaps of X_n, is an ideal: we shall denote it by Sing_n. Its order is $n^n - n!$.

It is well known that every element of the symmetric group S_n is a composition of disjoint cycles. Something similar happens to elements of T_n, but the situation is inevitably more complicated. Associated with a mapping α in T_n is a digraph with n vertices, in which there is an edge $i \longrightarrow j$ if and only if $i\alpha = j$. Let $\alpha \in T_n$. For i, j in X_n, we write $i \equiv j$ if and only if there exists $r, s \ge 0$ such that $i\alpha^r = j\alpha^s$. This is an equivalence relation, and it partitions X_n into disjoint classes, called *orbits*. The orbits are the connected components of the associated digraph. An example is helpful. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 3 & 3 & 4 & 5 & 6 & 4 & 6 & 9 & 10 & 10 & 12 & 13 & 11 & 14 \end{pmatrix}. \tag{1.7}$$

The associated digraph is



The example shows the different kinds of components that can arise. The component on the left is *standard*, the next is *acyclic*, the third is *cyclic* and the fourth is *trivial*. For a general α, let $\mathrm{cycl}(\alpha)$ be the number of cyclic components, and let $\mathrm{fix}(\alpha)$ be the number of fixed points: this is equal to the number of acyclic components plus the number of trivial components. It is easy to see, working orbit by orbit, that, for the α given by (1.7),

$$\alpha = \left[\binom{6}{3}\binom{5}{6}\binom{4}{5}\binom{3}{4}\binom{1}{3}\binom{2}{3}\binom{7}{6}\right]\left[\binom{9}{10}\binom{8}{9}\right]\left[\binom{11}{1}\binom{13}{11}\binom{12}{13}\binom{1}{12}\right].$$

Thus α (and indeed *any* α in any Sing_n) is expressible as a product of idempotents in $J_{n,n-1}$. The idempotents in $J_{n,n-1}$ are all of the form $\binom{i}{j}$, where i, j are distinct elements of X_n, and so the cardinality of the set E_{n-1} of idempotents in $J_{n,n-1}$ is $n(n-1)/2$.

Returning to our example (1.7) we see that the standard orbit has 7 elements and contributes 7 factors; the acyclic orbit, with 3 elements, contributes 2 factors; the cyclic orbit, with 3 elements, contributes 4 factors; and the the trivial orbit contributes no factors at all. The length of the product is

$$g(\alpha) = n + \mathrm{cycl}(\alpha) - \mathrm{fix}(\alpha)\,. \tag{1.8}$$

We have looked only at an example, but it is true in general that each α in every Sing_n is a product of length $g(\alpha)$ of idempotents in $J_{n,n-1}$. Moreover, though we shall not go into the proof, this length is best possible, and we have (see [18, 13]) the following theorem:

Theorem 1.5 *Let $\alpha \in \mathrm{Sing}_n$ and let E_{n-1} be the set of idempotents in $J_{n,n-1}$. Then $g(\alpha)$, as defined by (1.8), has the property that*

$$\alpha \in E_{n-1}^{g(\alpha)}\,, \quad \alpha \notin E_{n-1}^{g(\alpha)-1}\,.$$

□

This is our first significant example of a rig (regular idempotent-generated semigroup). The number of idempotent generators is $n(n-1)$, and it is legitimate to ask whether this is the smallest possible set of generators of Sing_n. It is not, but to clarify this issue we need a little more general theory of semigroups.

1.4 Green's Relations

In a group, each pair a, b of elements are mutually divisible, both right and left: $a = xb = by$, $b = ua = av$, with $x = ab^{-1}$, $y = b^{-1}a$, $u = ba^{-1}$, $v = a^{-1}b$. Indeed, a semigroup is a group if and only if these divisibility properties hold. We define an equivalence relation $\mathcal{L}$ on a semigroup S by

$$a \,\mathcal{L}\, b \text{ if and only if } a = b \text{ or } (\exists x, y \in S)\; a = xb,\ b = ya\,.$$

We can simplify that definition by defining S^1 to be the semigroup S with a unity element 1 adjoined if necessary. So we write

$$a \,\mathcal{L}\, b \text{ if and only if } (\exists x, y \in S^1)\; a = xb,\ b = ya\,. \tag{1.9}$$

Similarly

$$a \,\mathcal{R}\, b \text{ if and only if } (\exists u, v \in S^1)\; a = bu,\ b = av\,; \tag{1.10}$$

and again

$$a \,\mathcal{J}\, b \text{ if and only if } (\exists x, y, u, v \in S^1)\; a = xby,\ b = uav\,. \tag{1.11}$$

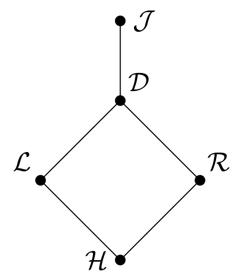
The equivalence $\mathcal{L} \cap \mathcal{R}$ is denoted by $\mathcal{H}$. A further equivalence $\mathcal{D}$ is defined by

$$a \,\mathcal{D}\, b \text{ if and only if } (\exists c \in S)\; a \,\mathcal{L}\, c \text{ and } c \,\mathcal{R}\, b\,. \tag{1.12}$$

Equivalently (not quite obviously),

$$a \,\mathcal{D}\, b \text{ if and only if } (\exists d \in S)\; a \,\mathcal{R}\, d \text{ and } d \,\mathcal{L}\, b\,. \tag{1.13}$$

These five equivalences, known as *Green's relations* [7], form a lattice as follows:



In many cases the two relations $\mathcal{D}$ and $\mathcal{J}$ coincide. This happens when the semigroup is finite, but also in other circumstances.

What is the 'Green structure' of the semigroup $\mathcal{T}_X$? To answer the question we need to introduce a new notion. Associated with every mapping in $\mathcal{T}_X$ is an equivalence relation $\ker\alpha$ on X, defined by

$$\ker\alpha = \{(x, y) \in X \times X : x\alpha = y\alpha\}\,. \tag{1.14}$$

Theorem 1.6 *For elements α, β in $\mathcal{T}_X$,*

(i) $\alpha \,\mathcal{L}\, \beta$ *if and only if* $\operatorname{im}\alpha = \operatorname{im}\beta$*;*

(ii) $\alpha \,\mathcal{R}\, \beta$ *if and only if* $\ker\alpha = \ker\beta$*;*

(iii) $\alpha \,\mathcal{D}\, \beta$ *if and only if* $|\operatorname{im}\alpha| = |\operatorname{im}\beta|$*;*

(iv) $\mathcal{J} = \mathcal{D}$*.*

Proof
(i) Suppose that $\alpha \,\mathcal{L}\, \beta$, which is to say that there exist μ, ν in $\mathcal{T}_X$ such that $\mu\alpha = \beta$, $\nu\beta = \alpha$. Then

$$\operatorname{im}\alpha = X\alpha = X\nu\beta \subseteq X\beta = \operatorname{im}\beta\,,$$

and similarly $\operatorname{im}\beta \subseteq \operatorname{im}\alpha$.

Conversely, suppose that $\operatorname{im}\alpha = \operatorname{im}\beta$, and let $x \in X$. Choose $x\mu$ to be any element whose image under α is $x\beta$. (This is possible since $\operatorname{im}\alpha = \operatorname{im}\beta$.) It is then clear that $\mu\alpha = \beta$, and in a similar way we find ν such that $\nu\beta = \alpha$.

(ii) Suppose that $\alpha \,\mathcal{R}\, \beta$, which is to say that there exist ξ, η in $\mathcal{T}_X$ such that $\alpha\xi = \beta$, $\beta\eta = \alpha$. Then

$$(x, y) \in \ker\alpha \;\Rightarrow\; x\alpha = y\alpha \;\Rightarrow\; x\alpha\xi = y\alpha\xi \;\Rightarrow\; x\beta = y\beta \;\Rightarrow\; (x, y) \in \ker\beta\,,$$

and so $\ker\alpha \subseteq \ker\beta$. Similarly, $\ker\beta \subseteq \ker\alpha$.

Conversely, suppose that $\ker\alpha = \ker\beta$. Let z be an element of $\operatorname{im}\alpha$ and let C be the $\ker\alpha$-class that maps to z by α. Define $z\xi$ to be the image of C under β. For any x in $X \setminus \operatorname{im}\alpha$ we define $z\xi$ in an arbitrary way. Then, for each element c of C,

$$c\alpha\xi = z\xi = c\beta\,.$$

This holds for all z in $\operatorname{im}\alpha$ and for all the $\ker\alpha$-classes, and so $\alpha\xi = \beta$. In the same way we can find η such that $\beta\eta = \alpha$, and so $\alpha \,\mathcal{R}\, \beta$.

(iii) For any α in $\mathcal{T}_X$ we have that $|X/\ker\alpha| = |\operatorname{im}\alpha|$. If $\alpha \,\mathcal{D}\, \beta$, then there exists γ such that $\alpha \,\mathcal{L}\, \gamma$ and $\gamma \,\mathcal{R}\, \beta$. Hence

$$|\operatorname{im}\alpha| = |\operatorname{im}\gamma| = |X/\ker\gamma| = |X/\ker\beta| = |\operatorname{im}\beta|\,.$$

Conversely, suppose that $|\operatorname{im}\alpha| = |\operatorname{im}\beta|$. Then let γ be an element of $\mathcal{T}_X$ that maps the $\ker\beta$-classes onto the image of α. Then $\operatorname{im}\gamma = \operatorname{im}\alpha$ and $\ker\gamma = \ker\beta$. Hence $\alpha \,\mathcal{L}\, \gamma$ and $\gamma \,\mathcal{R}\, \beta$, as required.

(iv). If ξ, η are arbitrary elements of $\mathcal{T}_X$, then, as already observed above in (1.6), we see that

$$|\mathrm{im}\,(\xi\eta)| \leq |\mathrm{im}\,\xi| \text{ and } |\mathrm{im}\,(\xi\eta)| \leq |\mathrm{im}\,\eta|\,. \tag{1.15}$$

Suppose that $\alpha\,\mathcal{J}\,\beta$. Then there exist λ, μ, ν, ρ such that

$$\alpha = \lambda\beta\mu\,,\ \beta = \nu\alpha\rho\,.$$

Hence, by (1.15)

$$|\mathrm{im}\,\alpha| = |\mathrm{im}\,(\lambda\beta\mu)| \leq |\mathrm{im}\,\beta|\,,\ |\mathrm{im}\,\beta| = |\mathrm{im}\,(\nu\alpha\rho)| \leq |\mathrm{im}\,\alpha|\,.$$

Thus $|\mathrm{im}\,\alpha| = |\mathrm{im}\,\beta|$ and so $\alpha\,\mathcal{D}\,\beta$. □

As a result of this theorem we can gain useful information about the structure of the $\mathcal{J}$-classes $J_{n,r}$ in the semigroup T_n. The number of $\mathcal{L}$-classes is the number of distinct subsets of cardinality r: this is the binomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}\,.$$

The number of $\mathcal{R}$-classes is the number of equivalences on the set X_n having r classes: this, less well known, is the *Stirling number* $S(n,r)$ of the second kind. It is clear that

$$S(n,1) = S(n,n) = 1\,. \tag{1.16}$$

Now let $n \geq 2$, and consider a equivalence ρ on $\{1, 2, \ldots, n\}$ having r classes. Either $\{n\}$ is a ρ-class or n is part of a larger ρ-class. The number of equivalences of the first kind is $S(n-1, r-1)$, and the number of the second kind is $rS(n-1, r)$ (since n may be added to any one of the r classes). Hence

$$S(n,r) = S(n-1, r-1) + rS(n-1, r)\,. \tag{1.17}$$

A table of small values may be of interest:

	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

From the table one might conjecture that

$$S(n, n-1) = \frac{n(n-1)}{2}\,, \quad S(n,2) = 2^{n-1} - 1\,; \tag{1.18}$$

both are easily proved by induction.

We can now visualise a $\mathcal{J}$-class $J_{n,r}$ of T_n as an 'eggbox' in which the $\mathcal{L}$-classes are the columns, the $\mathcal{R}$-classes are the rows and the $\mathcal{H}$-classes are the cells. The number of cells is $\binom{n}{r} \times S(n,r)$, and each cell contains $r!$ elements.

Consider H, an $\mathcal{H}$-class in $J_{n,r}$ corresponding to the image

$$S = \{a_1, a_2, \ldots, a_r\}$$

and the equivalence ρ with classes $A_1, A_2, \ldots, A_r$, and suppose that S is a *transversal* of ρ. By this we mean that each a_i in S belongs to a unique ρ-class A_j. One possibility is that $a_i \in A_i$ for all i, and in this case we obtain

$$\epsilon = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ a_1 & a_2 & \ldots & a_r \end{pmatrix},$$

the unique idempotent in H. A typical element of H is

$$\alpha_\sigma = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ a_{1\sigma} & a_{2\sigma} & \ldots & a_{r\sigma} \end{pmatrix},$$

where σ is a permutation of $\{1, 2, \ldots, r\}$, and, since $\alpha_\sigma \alpha_\tau = \alpha_{\sigma\tau}$ for all permutations σ, τ of $\{1, 2, \ldots, r\}$ it follows that H is a group, with identity ϵ, isomorphic to the symmetric group S_r.

Suppose now that $H = H_{S,\rho}$, determined by the subset $S = \{a_1, a_2, \ldots, a_r\}$ and the equivalence ρ with classes $A_1, A_2, \ldots, A_r$ is such that S is *not* a transversal of ρ. Then, re-labelling if necessary, we may consider two elements

$$\beta = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}, \quad \gamma = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix},$$

such that $b_1, b_2 \in A_1$. Then $A_1 \cup A_2$ maps by $\beta\gamma$ to a single element c_i. Also, for $j \geq 3$, each A_j maps by $\beta\gamma$ to some $c_{j'}$ (where the elements $c_{j'}$ are not necessarily distinct). We conclude:

Theorem 1.7 *Let $H = H_{S,\rho}$ be an $\mathcal{H}$-class in $J_{n,r}$, consisting of all mappings with image S and kernel ρ. If S is a transversal of ρ, then $H^2 = H$, and H is a group. Otherwise*

$$H^2 \subseteq I_{n,r} = \bigcup_{1 \leq s \leq n-1} J_{n,s}\,.$$

□

Another useful observation is as follows:

Theorem 1.8 *Let $H_{S,\rho}$ and $H_{T,\sigma}$ be $\mathcal{H}$-classes in $J_{n,r}$. Then $H_{S,\rho} H_{T,\sigma} = H_{T,\rho}$ if and only if $H_{S,\sigma}$ is a group.*

Proof

$H_{S,\rho}$	$H_{T,\rho}$
$H_{S,\sigma}$	$H_{T,\sigma}$

Let

$$S = \{s_1, s_2, \ldots, s_r\}\,,\ T = \{t_1, t_2, \ldots, t_r\}\,,$$

and let the equivalences ρ and σ have classes

$$A_1, A_2, \ldots, A_r \ \text{ and } \ B_1, B_2, \ldots, B_r\,,$$

respectively. Then a typical element of $H_{S,\rho}H_{T,\sigma}$ is

$$\begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ s_1 & s_2 & \ldots & s_r \end{pmatrix} \begin{pmatrix} B_1 & B_2 & \ldots & B_r \\ t_1 & t_2 & \ldots & t_r \end{pmatrix},$$

and the product lies in $J_{n,r}$ if and only if $\{s_1, s_2, \ldots, s_r\}$ is a transversal of σ, that is, by Theorem 1.7, if and only if $H_{S,\sigma}$ is a group. If this happens, then $H_{S,\rho}H_{T,\sigma} \subseteq H_{T,\rho}$. Equality follows, since any member

$$\begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ t_1 & t_2 & \ldots & t_r \end{pmatrix}$$

is a product

$$\begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ s_1 & s_2 & \ldots & s_r \end{pmatrix} \begin{pmatrix} B_1 & B_2 & \ldots & B_r \\ t_1 & t_2 & \ldots & t_r \end{pmatrix}.$$

□

We have already seen that the composition of mappings can only move downwards:

$$\text{im}\,(\alpha\beta)| \leq \min\{|\text{im}\,\alpha|, |\text{im}\,\beta|\}\,.$$

There is, however, a limit to how far downwards a product can go. The *defect* def α of an element of $\mathcal{T}_X$ is defined as $|X \setminus \text{im}\,\alpha|$. Then:

Theorem 1.9 *Let $\alpha, \beta \in \mathcal{T}_X$. Then*

$$\text{def}\,(\alpha\beta) \leq \text{def}\,\alpha + \text{def}\,\beta\,.$$

Proof If $y \notin \text{im}\,(\alpha\beta)$, then either:

1. $y \in X \setminus \text{im}\,\beta$; or
2. $y \in \text{im}\,\beta$, but $y \in (X \setminus \text{im}\,\alpha)\beta$.

Thus

$$X \setminus \text{im}\,(\alpha\beta) = X \setminus \text{im}\,\beta \cup (X \setminus \text{im}\,\alpha)\beta\,,$$

and the union is disjoint. The result follows, since

$$|(X \setminus \text{im}\,\alpha)\beta| \leq |X \setminus \text{im}\,\alpha| = \text{def}\,\alpha\,.$$

□

If $X = X_n$, then def $\alpha = n - |\text{im}\,\alpha|$, and so we have the corollary:

Corollary 1.10 *Let $\alpha, \beta \in T_n$. Then*

$$|\text{im}\,(\alpha\beta)| \geq |\text{im}\,\alpha| + |\text{im}\,\beta| - n\,.$$

□

Chapter 2

Rank

2.1 The Rank of Sing_n

If $\alpha, \beta \in J_{n,r}$ then $\mathrm{im}\,(\beta\alpha) \subseteq \mathrm{im}\,\alpha$. If this is an equality then $\beta\alpha \,\mathcal{L}\, \alpha$; otherwise $|\mathrm{im}\,(\beta\alpha)| < |\mathrm{im}\,\alpha|$ and the product lies in a lower $\mathcal{J}$-class. Similarly, $\mathrm{im}\,(\alpha\beta) \subseteq \mathrm{im}\,\beta$, and so either $\alpha\beta \,\mathcal{R}\, \alpha$, or the product falls into a lower $\mathcal{J}$-class.

This observation is the key to the next result. The *rank* $\mathrm{rank}(S)$ of a semigroup S is defined by

$$\mathrm{rank}(S) = \min\{|A| : A \subseteq S \text{ and } \langle A \rangle = S\}\,. \tag{2.1}$$

Theorem 2.1

$$\mathrm{rank}(K_{n,r}) \geq S(n,r)\,.$$

Proof Suppose that $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is a generating set for $K_{n,r}$. By virtue of (1.6), A must contain some elements from $J_{n,r}$; we may suppose that these are $\alpha_1, \alpha_2, \ldots, \alpha_p$, and these elements are the only ones that can generate the elements of $J_{n,r}$. Any product

$$\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}$$

of these elements is either in a lower $\mathcal{J}$-class or is $\mathcal{R}$-equivalent to α_{i_1}. Hence the elements $\alpha_1, \alpha_2, \ldots, \alpha_p$ must cover all the $\mathcal{R}$-classes, and so $m \geq p \geq S(n,r)$. □

Remark 2.2 *A similar argument using $\mathcal{L}$-classes establishes that* $\mathrm{rank}(K_{n,r}) \geq \binom{n}{r}$. *But we always have $S(n,r) \geq \binom{n}{r}$, and so the argument is superfluous.*

In particular we have that

$$\mathrm{rank}(\mathrm{Sing}_n) \geq S(n,n-1) = \frac{n(n-1)}{2}\,. \tag{2.2}$$

In fact this is an equality [14]:

Theorem 2.3 *With the notation above,*

$$\text{rank}(\text{Sing}_n) = \frac{n(n-1)}{2}.$$

Proof The key observation is the following identity concerning idempotents in E_{n-1}: if $3 \le m \le n$ and $i_1, i_2, \dots i_m$ in X_n are all distinct , then

$$\left[\binom{i_2}{i_1}\binom{i_3}{i_2}\cdots\binom{i_m}{i_{m-1}}\binom{i_1}{i_m}\right]^{m-1} = \binom{i_1}{i_2}. \tag{2.3}$$

For

$$\binom{i_2}{i_1}\binom{i_3}{i_2}\cdots\binom{i_m}{i_{m-1}}\binom{i_1}{i_m} = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & \dots & i_m \\ i_m & i_m & i_2 & i_3 & \dots & i_{m-1} \end{pmatrix} = \beta \text{ (say)},$$

and it is then easy to see that

$$\beta^{m-1} = \begin{pmatrix} i_1 & i_2 & i_3 & \dots & i_m \\ i_2 & i_2 & i_3 & \dots & i_m \end{pmatrix} = \binom{i_1}{i_2}.$$

We can associate a set I of elements of E_{n-1} with a digraph $\Delta(I)$. The vertices are labelled $1, 2, \dots, n$, and there is an edge (i, j) from i to j if and only if $\binom{i}{j} \in I$. A digraph is *strong* if, for all vertices i, j there is a path (observing arrows) from i to j, and it is *complete* if, for all $i \ne j$, at least one of (i, j) and (j, i) is an edge. Then I is a generating set for Sing_n if and only if $\Delta(I)$ is strong and complete.

Consider the set

$$I = \left\{\binom{i}{j} : i < j, i + j \text{ odd}\right\} \cup \left\{\binom{i}{j} : i > j, i + j \text{ even}\right\}.$$

The associated digraph is certainly complete, and is also strong, since, for all $i < j$ such that $i + j$ is even,

$$(i, i + 1), (i + 1, i + 2), \dots, (j - 1, j)$$

is a path from i to j, and, for all $i > j$ such that $i + j$ is odd,

$$(j, j + 1), (j + 1, j + 2), \dots (i - 1, i)$$

is a path from j to i. The cardinality of I is the number of pairs (i, j) in $\{1, 2, \dots, n\}$ such that $i \ne j$, and this is easily seen to be $n(n - 1)/2$. We know from (2.2) that the rank of Sing_n is at least $n(n - 1)/2$, and so the rank is $n(n - 1)/2$, as required. □

We have established that Sing_n is a rig (a regular idempotent-generated semigroup), and we also know its rank. In fact we know a little more: if S is an idempotent-generated semigroup, with set E of idempotents, we can define the *idempotent rank* idrank(S) by

$$\text{idrank}(S) = \min\{|A| : A \subseteq E \text{ and } \langle A \rangle = S\}\,.$$

It is conceivable that idrank$(S) >$ rank(S) (we shall in due course encounter a natural example), but it is clear from the arguments above that

$$\text{idrank}(\text{Sing}_n) = \text{rank}(\text{Sing}_n) = \frac{n(n-1)}{2}\,. \tag{2.4}$$

2.2 The Rank of $K_{n,r}$

We have seen that Sing_n is the special case of the semigroup

$$K_{n,r} = \{\alpha \in T_n : |\text{im}\,\alpha| \le r\}$$

for which $r = n-1$. It is natural, therefore, to seek to generalise (2.4) to other values of r. We obtain the following result:

Theorem 2.4

$$\text{idrank}(K_{n,r}) = \text{rank}(K_{n,r}) = S(n,r)$$

where $S(n,r)$ is the Stirling number of the second kind, as defined in (1.17).

As in the previous case, we know that the rank must be at least the number of $\mathcal{R}$-classes, and so the rank is at least $S(n,r)$. The main burden of the proof, to be found in [17], is to find $S(n,r)$ idempotents that generate $K_{n,r}$. This is quite complicated and it is inappropriate to give details here. To give an indication of how it is done, denote the statement of the theorem by $\mathbf{P(n,r)}$, noting that $\mathbf{P(n,n-1)}$ is known to be true for every n. Then prove that $\mathbf{P(n,2)}$ is true for every n. Finally, establish that, for all $1 \le r \le n-1$,

$$\mathbf{P(n-1,r-1)} \text{ and } \mathbf{P(n-1,r)} \;\Rightarrow\; \mathbf{P(n,r)}\,.$$

A double induction based on Pascal's Triangle does the rest. For example, suppose we wish to establish $\mathbf{P(6,3)}$.: from $\mathbf{P(4,2)}$ and $\mathbf{P(4,3)}$ we obtain $\mathbf{P(5,3)}$; then from $\mathbf{P(5,2)}$ and $\mathbf{P(5,3)}$ we obtain $\mathbf{P(6,3)}$.

Garba [5] extended this result to cover the larger semigroup PT_n of *partial* mappings of X_n into itself: if $K'_{n,r}$ is the semigroup $\{\alpha \in PT_n : |\text{im}\,\alpha| \le r\}$, then

$$\text{idrank}(K'_{n,r}) = \text{rank}(K'_{n,r}) = S(n+1,r+1)\,.$$

2.3 Order-preserving mappings

A mapping in T_n is called *order-preserving* if, for all i,j in $\{1,2,\ldots,n\}$,

$$i \le j \;\Rightarrow i\alpha \le j\alpha.$$

The only non-singular mapping with this property is the identity mapping. The set of all order-preserving mappings is a subsemigroup of T_n, denoted by O_n. The cardinality of O_n is not immediately obvious, but it can be found by a relatively easy argument

A typical example, with $n = 8$, is

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 5 & 5 & 6 & 6 & 7 & 7 \end{pmatrix}.$$

There are *occurrences* at 2, 3, 5, 6, 7, and we can codify α in terms of occurrences o and gaps g:

$$gogoggoogoogoog\,.$$

In general, α can be coded by a word in the alphabet $\{o, g\}$ containing n o's and $n-1$ g's. So the number of possible words is

$$\frac{(2n-1)!}{n!(n-1)!} = \binom{2n-1}{n-1};$$

thus

$$|O_n| = \binom{2n-1}{n-1}.$$

More interesting is the number of idempotents in O_n: let us denote this number by $\varphi(n)$. Idempotent maps from X_n to X_n divide into three types:

(a) maps α for which $\operatorname{im}\alpha \subseteq X_{n-1}$;

(b) maps α for which $X_{n-1}\alpha \subseteq X_{n-1}$ and $n\alpha = n$;

(c) maps α for which $X_{n-1}\alpha$ includes n

It is not hard to see that there are $\varphi(n-1)$ maps of type (a) and $\varphi(n-1)$ maps of type (b). To count the maps of type (c), notice that, by the order-preserving property, $(n-1)\alpha = n$. Hence, by the idempotent property – which implies that $r\alpha = r$ for every r in $\operatorname{im}\alpha$ – we must have that $n-1 \notin \operatorname{im}\alpha$. The idempotents α of type (c) are therefore in one-one correspondence with idempotents $\alpha^* : X_{n-1} \to X_{n-1}$, where α^* is identical to α except that $i\alpha^* = n-1$ whenever $i\alpha = n$. The idempotents α^* are in O_{n-1}, but not every idempotent in O_{n-1} appears in this way: the ones that are missing are those that map X_{n-1} into X_{n-2} – the type (a) idempotents in O_{n-1} in fact. So there are $\varphi(n-1)-\varphi(n-2)$ idempotents of type (c).

We deduce that

$$\varphi(n) = 3\varphi(n-1) - \varphi(n-2)\,,$$

and this, together with the easy observations that $\varphi(1) = 1$ and $\varphi(2) = 3$, enables us to obtain an expression for $\varphi(n)$.

However, things are better than they appear. Consider the Fibonacci numbers F_n, defined by

$$F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}\ (n \geq 3)\,.$$

Then

$$\begin{aligned} F_{2n} &= F_{2n-1} + F_{2n-2} = 2F_{2n-2} + F_{2n-3} \\ &= 3F_{2n-2} - F_{2n-2} + F_{2n-3} \\ &= 3F_{2n-2} - (F_{2n-3} + F_{2n-4}) + F_{2n-3} \\ &= 3F_{2n-2} - F_{2n-4}\,. \end{aligned}$$

The initial values work out in just the right way, and we deduce

Theorem 2.5 *The number of idempotents in O_n is the Fibonacci number F_{2n}.*

The semigroup O_n is certainly regular: for example, if

$$\alpha = \begin{pmatrix} \{1,2,3\} & \{4\} & \{5,6\} & \{7,8\} \\ 3 & 5 & 7 & 8 \end{pmatrix} \tag{2.5}$$

then (not uniquely)

$$\alpha' = \begin{pmatrix} \{1,2,3\} & \{4,5\} & \{6,7\} & \{8\} \\ 1 & 4 & 6 & 8 \end{pmatrix}$$

is such that $\alpha\alpha'\alpha = \alpha$ and $\alpha'\alpha\alpha' = \alpha'$, and it follows from [15, Proposition 2.4.2] that, in O_n,

$$\alpha\,\mathcal{L}\,\beta \text{ if and only if } \operatorname{im}\alpha = \operatorname{im}\beta\,,$$

$$\alpha\,\mathcal{R}\,\beta \text{ if and only if } \ker\alpha = \ker\beta\,,$$

$$\alpha\,\mathcal{J}\,\beta \text{ if and only if } |\operatorname{im}\alpha| = |\operatorname{im}\beta|\,.$$

Thus O_n, like T_n itself, is the union of $\mathcal{J}$-classes $J_1, J_2, \ldots, J_n$, where $J_r = \{\alpha \in O_n : |\operatorname{im}\alpha| = r\}$. In particular, J_n consists solely of the identity mapping.

Notice that the $\mathcal{H}$-classes are trivial: in the example (2.5) above, once the image is fixed to be $\{3, 5, 7, 8\}$ and the kernel classes are fixed to be $\{1, 2, 3\}$, $\{4\}$, $\{5, 6\}$ and $\{7.8\}$, there is only one order-preserving map possible.. Notice too that the kernel classes C are *convex*, in the sense that $x, y \in C$ and $x \le z \le y$ implies that $z \in C$. If $\alpha \in J_{n-1}$, then $\ker\alpha$ has only one non-singleton class, and the only possibilities, $n-1$ in number, are

$$\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\,.$$

Denote by $|i, i+1|$ the equivalence whose only non-singleton class is $\{i, i+1\}$.

There are n possibilities for $\operatorname{im}\alpha$, namely

$$X_n \setminus \{1\}, X_n \setminus \{2\}, \ldots, X_n \setminus \{n\}\,,$$

and so $|J_{n-1}| = n(n-1)$.

The (unique) element with kernel $|i, i+1|$ and image $X_n \setminus \{k\}$ is

$$\begin{pmatrix} 1 & \ldots & i & i+1 & \ldots & k & k+1 & \ldots & n \\ 1 & \ldots & i & i & \ldots & k-1 & k+1 & \ldots & n \end{pmatrix} \tag{2.6}$$

if $k \geq i+1$, and is

$$\begin{pmatrix} 1 & \dots & k-1 & k & \dots & i & i+1 & \dots & n \\ 1 & \dots & k-1 & k+1 & \dots & i+1 & i+1 & \dots & n \end{pmatrix} \tag{2.7}$$

if $k \leq i$. A useful notation for the *decreasing element* (2.6) is

$$[k \to k-1 \to \cdots \to i]\,;$$

for the *increasing element* (2.7) we write

$$[k \to k+1 \to \cdots \to i+1]\,.$$

There are $n-1$ decreasing idempotents $[i \to i-1]$ and $n-1$ increasing idempotents $[i-1 \to i]$ $(i = 2, 3, \dots, n)$. Thus E_1 the set of all idempotents in J_{n-1}, has $2n-2$ elements in all. From

$$[k \to k-1 \to \cdots \to i] = [i+1 \to i]\,[i+2 \to i+1] \dots [k \to k-1]$$

and

$$[k \to k+1 \to \cdots \to i+1] = [i \to i+1]\,[i-1 \to i] \dots [k \to k+1]$$

we deduce that $\langle E_1 \rangle$ contains the whole of J_{n-1}.

We now quote a very old result in [12] to the effect that any element α for which $|\operatorname{im}\alpha| \leq n-2$ is expressible as a product of elements in J_{n-1}. We deduce that $\langle E_1 \rangle = O_n \setminus \{\iota\}$ (where ι of course denotes the identity mapping of X_n). We also note that

$$\operatorname{rank}(O_n \setminus \{\iota\}) \leq 2n-2\,, \quad \operatorname{idrank}(O_n \setminus \{\iota\}) \leq 2n-2\,.$$

In fact, it is shown in in [6] that $O_n \setminus \{\iota\}$ is an example where the rank and the idempotent rank are distinct:

Theorem 2.6

$$\operatorname{rank}(O_n \setminus \{\iota\}) = n\,, \quad \operatorname{idrank}(O_n \setminus \{\iota\}) = 2n-2\,.$$

We can write $O_n \setminus \{\iota\}$ as $L_{n,n-1}$, where $L_{n,r} = \{\alpha \in O_n : |\operatorname{im}\alpha|\}$. The distinction between rank and idempotent rank disappears when $r \leq n-2$: Garba [5] showed that, for $1 \leq r \leq n-2$

$$\operatorname{rank}(L_{n,r}) = \operatorname{idrank}(L_{n,r}) = \binom{n}{r}\,.$$

2.4 Order-decreasing transformations

A map $\alpha : X_n \to X_n$ is called *order-decreasing* if, for all i in X_n, $x\alpha \leq x$. It is clear that $1\alpha = 1$, that $2\alpha = 1$ or 2, etc., and so D_n, the semigroup of all order-decreasing maps, is of cardinality $n!$. Some more interesting combinatorics (as with the case of O_n) arises if we ask for the number of idempotents in D_n, which turns out to be the Bell number B_n, which can be expressed in terms of Stirling numbers:

$$B_n = \sum_{r=1}^{n} S(n,r)\,.$$

Higgins [8] considered $O_n \cap D_n$, the semigroup of all maps that are both order-preserving and order-decreasing, and showed $|O_n \cap D_n| = C_n$, the *Catalan number*, defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n}\,.$$

This is an interesting area, involving algebra, combinatorics, and even asymptotics, but I have chosen to stop here, and to transfer attention to the full transformation semigroup T_X where X is infinite.

Chapter 3

Infinite sets

Let X be an infinite set, and let $\alpha : X \to X$ be an element of T_X, the semigroup of all selfmaps of X. We define

$$\text{fix}\,\alpha = |\{x \in X : x\alpha = x\}|\,,\ \text{shift}\,\alpha = |\{x \in X : x\alpha \neq x\}|\,,$$

referring to the *fix* of α and the *shift* of α. A map $\alpha : X \to X$ is called singular if $\text{im}\,\alpha \subset X$.

We define the *defect* $\text{def}\,\alpha$ of α by

$$\text{def}\,\alpha = |X \setminus \text{im}\,\alpha|\,,$$

and the *collapse* $\text{coll}\,\alpha$ by

$$\text{coll}\,\alpha = \sum_{y \in \text{im}\,\alpha} (|y\alpha^{-1}| - 1)\,.$$

The defect is easily understood as a measure of how far α is from being onto; the collapse perhaps requires explanation. First, by $y\alpha^{-1}$ we mean the set $\{x \in X : x\alpha = y\}$. If α is one-one, then $\text{coll}\,\alpha = 0$; in general we can regard $\text{coll}\,\alpha$ as a measure of how far α is from being one-one. It is clear that $\alpha \in S_X$, the symmetric group on the set X, if and only if $\text{def}\,\alpha = \text{coll}\,\alpha = 0$.

We shall eventually consider in more detail the nature of idempotents in T_X, but for the moment, note that the only one-one idempotent is ι, the identity map. For if ϵ is idempotent and one-one, then, for all x in X,

$$(x\epsilon)\epsilon = x\epsilon\,,$$

and so, by the one-one property, $x\epsilon = x$ for all x.

It is reasonable to ask whether the result that all maps in $T_X \setminus G_X$ can be expressed as products of idempotents, but one quickly learns that this is not the case. In the infinite case, let α be one-one but not onto, and suppose that

$$\alpha = \epsilon_1\epsilon_2 \ldots \epsilon_n\,,$$

a product of idempotents. Then, for all x in X,

$$x\alpha = x\epsilon_1\epsilon_2 \dots \epsilon_n = x\epsilon_1^2\epsilon_2 \dots \epsilon_n = (x\epsilon_1)\alpha\,.$$

Since α is one-one, it follows that $x\epsilon_1 = x$. This holds for all x in X, and so ϵ_1 is the identity element ι. Thus ϵ_1 can be left out. The same argument then applies to ϵ_2, and so all along the line. The conclusion is that $\alpha = \iota$, and this is a contradiction. A similar argument, affecting the right hand side of the product, applies if α is onto but not one-one.

Let $\epsilon(\neq \iota)$ be an idempotent in T_X. If x is shifted by ϵ, then $x\epsilon$ is fixed, and we conclude that

$$\operatorname{def} \epsilon = \operatorname{shift} \epsilon\,.$$

An example may help: let

$$\epsilon = \begin{pmatrix} \{1,2,3\} & \{4,5,6\} & \{7,8,9\} \\ 3 & 6 & 9 \end{pmatrix}$$

Then

$$X \setminus \operatorname{im} \epsilon = \{x : x\epsilon \neq x\} = \{1,2,4,5,7,8\}$$

and so certainly $\operatorname{shift} \epsilon = \operatorname{def} \epsilon = 6$. As for the collapse, we see that

$$\operatorname{coll} \epsilon = (|\{1,2,3\}| - 1) + \cdots + (|\{4,5,6\}| - 1) + (|\{7,8,9\}| - 1) = 6$$

—the same as the shift and defect. This is true in general: for every idempotent in T_X,

$$\operatorname{shift} \epsilon = \operatorname{def} \epsilon = \operatorname{coll} \epsilon\,.$$

This works for an infinite example also: for the idempotent η we have

$$\eta = \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \dots \\ 2 & 4 & 6 & \dots \end{pmatrix},$$

and so $\operatorname{shift} \eta = \operatorname{def} \eta = \operatorname{coll} \eta = \aleph_0$.

It seems reasonable (principally when X is infinite) to refer to an element α of T_X for which

$$\operatorname{shift} \alpha = \operatorname{def} \alpha = \operatorname{coll} \alpha$$

as *balanced*. For each cardinal $\mathbf{k}$ such that $\aleph_0 \leq \mathbf{k} \leq |X|$, let $B_{\mathbf{k}}$ be the set of all elements η of T_X such that

$$\operatorname{shift} \eta = \operatorname{def} \eta = \operatorname{coll} \eta = \mathbf{k}\,.$$

Then, not quite obviously, each $B_{\mathbf{k}}$ is a subsemigroup of T_X and so is

$$B = \bigcup_{\aleph_0 \leq \mathbf{k} \leq |X|} B_{\mathbf{k}}\,. \tag{3.1}$$

The set

$$F = \{\alpha \in T_X : \operatorname{shift} \alpha < \infty \text{ and } \operatorname{def} \alpha > 0\} \tag{3.2}$$

is also a subsemigroup.

Then we have the following result [11]:

Theorem 3.1 *Let X be infinite, and let E be the set of singular idempotents in T_X. Then*

$$\langle E\rangle = F \cup B\,.$$

The result has been refined. Let us denote the set of idempotents of finite, non-zero shift by E_F and the idempotents in $B_{\mathbf{k}}$ by $E_{\mathbf{k}}$.

$$\langle E_F\rangle = F, \quad \langle E_{\mathbf{k}}\rangle = B_{\mathbf{k}} \quad (\aleph_0 \le \mathbf{k} \le |X|)\,. \tag{3.3}$$

Given a set A of generators for a semigroup S, and a positive integer k, let

$$A^{[k]} = A \cup A^2 \ldots \cup A^k\,.$$

In an infinite semigroup it is possible that the ascent

$$A \subset A^{[2]} \subset A^{[3]} \subset \cdots$$

is infinite, and in such a case we say that S has infinite A-depth, and write $\Delta_A(S) = \infty$. Otherwise the ascent terminates, and we define the A-depth by

$$\Delta_A(S) = \min\{k \in \mathbf{N} : A^{[k]} = S\}\,.$$

If A consists of idempotents, it is clear that

$$A \subseteq A^2 \subseteq A^3 \subseteq \cdots\,,$$

and so $A^{[k]} = A^k$.

Consider an element α in the semigroup F defined by (3.2), and suppose that

$$\alpha = \epsilon_1\epsilon_2\ldots\epsilon_k\,, \tag{3.4}$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in E_F$. It is easy to show the following properties, where $\alpha, \beta \in F$ and $\epsilon \in E_F$:

$$\begin{aligned}
\text{shift}\,(\alpha\beta) &\le \text{shift}\,\alpha + \text{shift}\,\beta\,, && (3.5)\\
\text{shift}\,\epsilon &= \text{def}\,\epsilon\,, && (3.6)\\
\text{def}\,(\alpha\beta) &\ge \text{def}\,\alpha\,, && (3.7)\\
\text{def}\,(\alpha\beta) &\ge \text{def}\,\beta\,. && (3.8)
\end{aligned}$$

Returning to (3.4), let us suppose that $\text{def}\,\alpha = d$. It follows from (3.7) and (3.8) that $\text{def}\,\epsilon_i \le d$ for $i = 1, 2, \ldots, k$. Hence, by (3.6), $\text{shift}\,\epsilon_i \le d$ for all i, and from (3.5) it follows that $\text{shift}\,\alpha \le kd$. Thus

$$\frac{\text{shift}\,\alpha}{\text{def}\,\alpha} \le k\,.$$

To find an element not belonging to E_F^k we need only find an element for which $\text{shift}\,\alpha/\text{def}\,\alpha > k$. Such an element is given by selecting elements $x_1, x_2, \ldots, x_{k+2}$ in X, and defining α by

$$x_i = x_{i+1}\ (i = 1, 2, \ldots, k+1)\,, \quad x\alpha = x \ \text{ otherwise}\,.$$

It is clear that shift $\alpha = k+1$, def $\alpha = |\{x_1\}| = 1$. We deduce that

$$\Delta_{E_F}(F) = \infty\,.$$

Given the highly infinite definition of $B_{\mathbf{k}}$, one might expect that $\Delta_{E_{\mathbf{k}}}(B_{\mathbf{k}})$ would also be infinite, but the unexpected outcome is that, for all $\mathbf{k}$ such that $\aleph_0 \le \mathbf{k} \le |X|$,

$$\Delta_{E_{\mathbf{k}}}(B_{\mathbf{k}}) = 4\,.$$

The hard bit of this is to prove that the answer is not 3.

Several results analogous to Theorem 3.1 have been proved. In particular there is a linear version, concerned with the semigroup Sing L of singular self-maps of an infinite-dimensional linear space L. For a given α, both im α and fix $\alpha = \{v \in L : v\alpha = v\}$ are subspaces, and so is $\ker\alpha = \{v \in L : v\alpha = 0\}$. Then

$$\begin{aligned} \text{shift}\,\alpha &= \text{codim}(\text{fix}\,\alpha)\,, \\ \text{def}\,\alpha &= \text{codim}(\text{im}\,\alpha)\,, \\ \text{coll}\,\alpha &= \dim(\ker\alpha)\,. \end{aligned}$$

are obvious analogues of the shift, defect and collapse in the set-theoretic case. Then let F be the semigroup of all linear maps α for which shift α is finite and def $\alpha \neq 0$, and let $B = \{\alpha \in \text{Sing}\,L : \text{shift}\,\alpha = \text{def}\,\alpha = \text{coll}\,\alpha \ge \aleph_0\}$. Then we have the direct analogue of Theorem 3.1, established by Reynolds and Sullivan[20]:

Theorem 3.2 *Let L be an infinite-dimensional linear space and let* Sing L *be the semigroup of all singular linear selfmaps of L. Let E be the set of idempotent maps in* Sing L. *Then $\langle E\rangle = F \cup B$.*

As in the previous case, we can be more precise. For each infinite cardinal $\mathbf{k}$ not greater than dim L, let

$$B_{\mathbf{k}} = \{\alpha \in \text{Sing}\,L : \text{shift}\,\alpha = \text{def}\,\alpha = \text{coll}\,\alpha = \mathbf{k}\}\,,$$

and let $E_{\mathbf{k}} = E \cap B_{\mathbf{k}}$. Then

$$\langle E \cap F\rangle = F \text{ and, for each } \mathbf{k},\ \langle E_{\mathbf{k}}\rangle = B_{\mathbf{k}}\,.$$

The analogy breaks down at this point, for

$$\Delta_{E_{\mathbf{k}}}(B_{\mathbf{k}}) = 3\,.$$

Theorems 3.1 and 3.2 have been generalised by Fountain and Lewin [4] to the case of an *independence algebra*, a universal algebra concept that includes both the set case and the linear case, and Fountain [3] was later able to 'explain' in universal algebra terms the cause of the different values of $\Delta_{E_{\mathbf{k}}}(B_{\mathbf{k}})$ in the two cases described. I will not go into details, but the existence of a zero in the linear case has something to do with the result.

Chapter 4

Relative Rank

4.1 Rank and Relative Rank

The material of this section comes from [9] and [16].

It is clear that the notion of the rank of T_X is of no interest in the case where X is infinite. For any semigroup S with order $\mathbf{k} > \aleph_0$, all one can say is that the rank of S is $\mathbf{k}$. Even if $|X| = \aleph_0$, the smallest infinite cardinal, the semigroup T_X is of order $2^{\aleph_0}$. So its rank is also $2^{\aleph_0}$.

There is, however, another notion that yields some interesting—and in some cases slightly surprising—results. For a semigroup S and a subset A of S, the *relative rank of S modulo A* is given by

$$\operatorname{rank}(S : A) = \min\{|B| : \langle A \cup B \rangle\} = S\,. \tag{4.1}$$

So, for example, if $X_n = \{1, 2, \dots, n\}$ is finite, then

$$\operatorname{rank}(T_n : S_n) = 1\,, \tag{4.2}$$

since, together with the permutations in S_n, we need just one element of $J_{n,n-1}$ to generate T_n. (See (1.3)). Again, if E_n is the set of idempotents of T_n, then

$$\operatorname{rank}(T_n : E_n) = 2\,.$$

Certain properties are clear:

$$\operatorname{rank}(S : \emptyset) = \operatorname{rank} S\,, \quad \operatorname{rank}(S : S) = 0\,, \quad \operatorname{rank}(S : \langle A \rangle) = \operatorname{rank}(S : A)\,,$$

and $\operatorname{rank}(S : A) = 0$ if and only if $\langle A \rangle = S$. Also, if G is a group and N is a normal subgroup of G, then $\operatorname{rank}(G : N) = \operatorname{rank}(G/N)$.

In exploring the notion for infinite transformation semigroups, we shall require the previously encountered measures *shift, defect* and *collapse.* We also need a new parameter. Let $\alpha \in T_X$, where X is infinite. Let

$$K(\alpha) = \{x \in X : |x\alpha^{-1}| = |X|\}\,;$$

then the *infinite contraction index* $\operatorname{con}\alpha$ is defined as $|K(\alpha)|$. It is the number of $(\ker\alpha)$-classes of size $|X|$.

We will require a definition concerning cardinal numbers. A cardinal $\mathbf{k}$ is *singular* if there exist sets Y and $Z_y (y \in Y)$ such that

$$|Y| < \mathbf{k}\,, \qquad |Z_y| < \mathbf{k} \quad (y \in Y)\,,$$

but

$$\Big|\bigcup_{y \in Y} Z_y\Big| = \mathbf{k}\,.$$

(An example of a singular cardinal is $\aleph_\omega$, defined as the limit of the sequence $\aleph_0, \aleph_1, \aleph_2, \ldots$.) A cardinal that is not singular is called *regular*. The following lemma is useful:

Lemma 4.1 *Let $\alpha, \beta \in T_X$, where X is infinite. Then*

(i) $\operatorname{def}(\alpha\beta) \le \operatorname{def}(\alpha) + \operatorname{def}(\beta)$*;*

(ii) *if $|X|$ is a regular cardinal, then* $\operatorname{con}(\alpha\beta) \le \operatorname{con}\alpha + \operatorname{con}\beta$*.*

Proof Part (i) is already established in Theorem 1.9. To prove Part (ii), suppose that $x \in K(\alpha\beta)$. Then

$$|X| = |x(\alpha\beta)^{-1}| = |x\beta^{-1}\alpha^{-1}| = \Big|\bigcup_{y \in x\beta^{-1}} y\alpha^{-1}\Big|\,.$$

Since $|X|$ is regular, means that either $|x\beta^{-1}| = |X|$, or $|y\alpha^{-1}| = |X|$ for some y in $x\beta^{-1}$. The the former case we have $x \in K(\beta)$, while in the latter case $x \in K(\alpha)\beta$. We have shown that

$$K(\alpha\beta) \subseteq K(\alpha)\beta \cup K(\beta)\,;$$

hence

$$\operatorname{con}(\alpha\beta) = |K(\alpha\beta)| \le |K(\alpha)\beta| + |K(\beta)| \le |K(\alpha)| + |K(\beta)| = \operatorname{con}\alpha + \operatorname{con}\beta\,.$$

□

¿From (4.2) we know that $\operatorname{rank}(T_X : S_X) = 1$ when X is finite. This is not the case for X infinite:

Lemma 4.2 *Let X be infinite and let $\mu \in T_X$. Then $\langle S_X, \mu\rangle \ne T_X$. Consequently* $\operatorname{rank}(T_X, S_X) > 1$*.*

Proof Suppose, for a contradiction, that $\langle S_X, \mu\rangle = T_X$, and let α be a one-one mapping from X into a proper subset of X. Decompose α into a product of elements from $S_X \cup \{\mu\}$:

$$\alpha = \beta_1\beta_2\ldots\beta_k\,.$$

Since α is not a bijection, at least one of the β_i must be μ; let β_j be the first such. Then

$$\beta_{j-1}^{-1} \dots \beta_1^{-1} \alpha = \mu \beta_{j+1} \dots \beta_k \,.$$

The left hand side of the above equality is certainly one-one, and hence so is μ. Hence $\langle S_X, \mu \rangle$ must consist entirely of one-one mappings, and so cannot coincide with T_X. □

When X is finite the top $\mathcal{J}$-class of T_X, namely

$$J = \{\alpha \in T_X : |\operatorname{im} \alpha| = |X|\} \,, \tag{4.3}$$

is simply the symmetric group S_X. This is no longer the case when X is infinite, for there are many non-bijections whose image has cardinality $|X|$. In fact we have

Lemma 4.3 *Let J be defined by (4.3). Then $\langle J \rangle = T_X$, and so* $\operatorname{rank}(T_X : J) = 0$.

Proof Let $\alpha \in T_X$. Write X as a disjoint union $Y \cup Z$, where $|X| = |Y| = |Z|$. Let $\beta : X \to Y$ be an arbitrary bijection. Then $\beta \in J$, since $|\operatorname{im} \beta| = |Y| = |X|$. Let γ in T_X be defined by

$$x\gamma = \begin{cases} x\beta^{-1}\alpha & \text{if } x \in Y \\ x & \text{if } x \in Z. \end{cases}$$

Since $Z \subseteq \operatorname{im} \gamma$ and $|Z| = |X|$, we have $\gamma \in J$.

Now, since β maps X into Y,

$$x\beta\gamma = x\beta\beta^{-1}\alpha = x\alpha$$

for all x in X. Thus $\alpha = \beta\gamma \in \langle J \rangle$. □

We can now prove the following result:

Theorem 4.4 $\operatorname{rank}(T_X : S_X) = 2$.

Proof Let μ be an injection (one-one) of defect $|X|$ and let ν be a surjection (onto) with the property that all the $(\ker \nu)$-classes have cardinality $|X|$. In view of Lemmas 4.2 and 4.3, it is enough to show that $\langle S_X, \mu, \nu \rangle$ contains J, the top $\mathcal{J}$-class of T_X.

Let $\alpha \in T_X$ be an arbitrary mapping in J; that is, $|\operatorname{im} \alpha| = |X|$. We shall show that α can be expressed as a product of μ, ν and two bijections π and σ which we define below.

Let Λ be an index set of cardinality $|X|$, partitioned into two disjoint subsets Λ_1, Λ_2, both of cardinality $|X|$:

$$\Lambda = \Lambda_1 \cup \Lambda_2 \,, \quad \Lambda_1 \cap \Lambda_2 = \emptyset \,, \quad |\Lambda_1| = |\Lambda_2| = |X| \,.$$

Since $\operatorname{con}\nu = |X|$, we can index the $\ker\nu$-classes having cardinality $|X|$ using Λ; so suppose that the classes of cardinality $|X|$ are B_λ ($\lambda \in \Lambda$). By the same token, since $|\operatorname{im}\alpha| = |X|$, the $\ker\alpha$-classes can be indexed by Λ_1; denote them by C_λ ($\lambda \in \Lambda_1$).

We now define π. First, since $|B_\lambda| = |X|$ for each λ in Λ_1 there exists, for each λ in Λ_1, an injection

$$\pi_\lambda : C_\lambda\,\mu \to B_\lambda \quad (\lambda \in \Lambda_1)\,.$$

Let

$$\overline{\pi} = \bigcup_{\lambda\in\Lambda_1} \pi_\lambda\,.$$

Then $\overline{\pi}$ is a partial injection. Since the domain of $\overline{\pi}$ is $\operatorname{im}\mu$, we can say that

$$|X \setminus \operatorname{dom}\overline{\pi}| = |X \setminus \operatorname{im}\mu| = |X|\,.$$

Note also that

$$\operatorname{im}\overline{\pi} \subseteq \bigcup_{\lambda\in\Lambda_1} B_\lambda\,,$$

and so

$$|X \setminus \operatorname{im}\overline{\pi}| \geq \Big|\bigcup_{\lambda\in\Lambda_2} B_\lambda\Big| = |X|\,.$$

Hence $\overline{\pi}$ can be extended to a bijection $\pi \in S_X$.

Next, we define the bijection σ. Recall that, for each λ in Λ, the set B_λ is a $\ker\nu$-class. Hence $B_\lambda\nu$ is a single element. Similarly, for each λ in Λ_1, $C_\lambda\alpha$ is a single element. By the Axiom of Choice, we select, for each λ in Λ_1, an element z_λ in $(C_\lambda\alpha)\nu^{-1}$. (The set $(C_\lambda\alpha)\nu^{-1}$ is certainly non-empty, since ν is a surjection; indeed, the definition of ν ensures that $|(C_\lambda\alpha)\nu^{-1}| = |X|$.) We define a mapping $\overline{\sigma} : \{B_\lambda\nu : \lambda \in \Lambda_1\} \to X$ by the rule that

$$(B_\lambda\nu)\overline{\sigma} = z_\lambda \quad (\lambda \in \Lambda_1)\,.$$

For λ in Λ_2 we have $B_\lambda\nu \notin \operatorname{dom}\overline{\sigma}$, and so $|X \setminus \operatorname{dom}\overline{\sigma}| = |X|$. Note also that $\operatorname{im}\overline{\sigma}$ contains at most one element from each $(C_\lambda\alpha)\nu^{-1}$, and so $|X\setminus\operatorname{im}\overline{\sigma}| = |X|$. Hence $\overline{\sigma}$ can be extended to a bijection σ in S_X.

We claim now that

$$\alpha = \mu\pi\nu\sigma\nu\,.$$

Let $x \in X$ Then $x \in C_\lambda$ for some λ in Λ_1.. Then $x\mu \in C_\lambda\,\mu$ and so

$$x\mu\pi\nu \in (C_\lambda\,\mu\pi_\lambda)\nu = B_\lambda\nu\,,$$

a singleton. It follows that $x\mu\pi\nu\sigma \in (C_\lambda\alpha)\nu^{-1}$, and so

$$x\mu\pi\nu\sigma\nu = C_\lambda\alpha = x\alpha\,.$$

It follows that $\langle S_X, \mu, \nu\rangle = T_X$, and so $\operatorname{rank}(T_X : S_X) = 2$. □

A similar argument establishes that. if X is infinite, $\operatorname{rank}(T_X : E_X) = 2$, where E_X is the set of idempotents in T_X.

4.2 Countable versus uncountable

Within T_X we have encountered relative ranks of 0, 1 and 2; and of course we also have an uncountable relative rank, namely $\mathrm{rank}(T_X : \emptyset)$. Can we find anything in between?

Nikola Ruškuc proved the following theorem [10]:

Theorem 4.5 *Let X be an infinite set. Then any countable subset S of T_X is contained in a 2-generated subsemigroup of T_X.*

Proof Write S as $\{\theta_1, \theta_2, \ldots\}$. Partition X into a countable disjoint union of infinitely many sets $X_0, X_1, X_2, \ldots$, all of cardinality $|X|$. Then partition X_0 into $X_{01}, X_{02}, X_{03}, \ldots$, again all of cardinality $|X|$.

Let β be a mapping in T_X that maps X_n bijectively onto X_{n+1} for all $n \geq 0$. Let γ be a mapping in T_X, mapping X_n bijectively onto X_{0n} for all $n \geq 1$. Although we have yet to define γ in X_0, we can see that the mapping

$$\delta_n = \beta\gamma\beta^n\gamma$$

is a well-defined bijection of X onto X_{0n}. (This is because $\mathrm{im}\,\beta \cap X_0 = \emptyset$.) We may then complete the definition of γ by defining

$$x\delta_n\gamma = x\theta_n \ (x \in X, n \in \mathbf{N}),.$$

Thus

$$\theta_n = \delta_n\gamma = \beta\gamma\beta^n\gamma^2 ,$$

and so $S = \langle \beta, \gamma \rangle$ □

As a consequence, we have the following result concerning relative ranks:

Corollary 4.6 *Let X be an infinite set. The relative rank of a subset T_X over a subset S is either uncountable or at most 2.*

Proof If $T_X \setminus S$ is uncountable, then $\mathrm{rank}(T_X : S)$ is uncountable. If $T_X \setminus S$ is countable then it can be generated by at most two of its elements, and so $\mathrm{rank}(T_X : S) \leq 2$. □

We were quite proud of Theorem 4.5, but then we discovered that it was first proved by Sierpiński [21] in 1935. His proof was more complicated, but Banach [1] in the same year produced the more elegant proof shown above. Well, at least we were in good company, although nearly 70 years late! The result does not seem to have entered the consciousness of the mathematical community, for the much-quoted result of Evans [2] that every countable semigroup is embeddable in a semigroup of rank 2 follows immediately from Theorem 4.5.

There were, however, some reasonable questions. For example, we defined a *contraction* in $\mathcal{T}_{\mathbf{N}}$ to be a map α such that $|i\alpha - j\alpha| \leq |i - j|$ for all $i, j \in \mathbf{N}$. Then we showed that $\mathrm{rank}(\mathcal{T}_{\mathbf{N}} : \mathcal{C})$ is uncountable.

The huge semigroup B_X, consisting of all binary relations on the set X, contains T_X, also the symmetric inverse semigroup I_X; and also the larger set P_X of *partial* mappings from X to X. (If $|X| = n$ then

$$|T_X| = n^n\,, \quad |P_X| = (n+1)^n\,, \quad |B_X| = 2^{n^2}\,.)$$

Slightly surprisingly, though one learns not be be too surprised when dealing with infinite sets, if X is infinite,

$$\mathrm{rank}(B_X : T_X) = \mathrm{rank}(B_X : P_X) = \mathrm{rank}(B_X : I_X) = 1\,, \quad \mathrm{rank}(B_X : S_X) = 2\,.$$

Bibliography

[1] S. Banach, Sur un théorème de M. Sierpiński, *Fund. Math.* **25** (1935), 5–6.

[2] T. Evans, Embedding theorems for multiplicative systems and projective geometries, *Proc. American Math. Soc.* **3** (1952), 614–620.

[3] John Fountain, The depth of the semigroup of balanced endomorphisms of an independence algebra, *Mathematika* **41** (1994), 199–208.

[4] John Fountain and Andrew Lewin, Products of endomorphisms of an independence algebra of infinite rank, *Proc. Cambridge Phil. Soc.* **114** (1993), 303–319

[5] Goje Garba, On the idempotent ranks of certain semigroups of order-preserving transformations. *Portugal. Math.* **51** (1994), 185–204.

[6] Gracinda M. S. Gomes and John M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum* **45** (1992), 272–282.

[7] J. A. Green, On the structure of semigroups, *Ann. Math.* **54** (1951), 163–172.

[8] Peter M. Higgins, Combinatorial results for semigroups of order-preserving transformations, *Math. Proc. Cambridge Phil. Soc.* **113** (1993), 281–296.

[9] Peter M. Higgins, John M. Howie and Nikola Ruškuc, Generators and factorisations of transformation semigroups, *Proc. Royal Soc. Edinburgh A* **128** (1998), 1355–1369.

[10] Peter M. Higgins, John M. Howie, J. D. Mitchell and Nikola Ruškuc, Countable versus uncountable ranks in infinite semigroup of transformations and relations, *Proc. Edinburgh Math. Soc.* **46** (2003), 531–544.

[11] John M. Howie, The subsemigroup generated by the idempotents of a full transformation semigroup, *J. London Math. Soc.* **41** (1966), 707–716.

[12] John M. Howie, Products of idempotents in certain semigroups of transformations, *Proc. Edinburgh Math. Soc.* (2) **17** (1971), 223–236.

[13] John M. Howie, Products of idempotents in finite full transformation semigroups, *Proc. Royal Soc. Edinburgh A* **86** (1980), 243–254.

[14] John M. Howie, Idempotent generators in finite full transformation semigroups, *Proc. Royal Soc. Edinburgh A* **81** (1978), 317–323.

[15] John M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, 1995.

[16] John M. Howie, N. Ruškuc and Peter M. Higgins, On relative ranks of full transformation semigroups, *Comm. Algebra* **26** (1998), 733–748.

[17] John M. Howie and Robert B. McFadden, Idempotent rank in finite full transformation semigroups, *Proc. Royal Soc Edinburgh A* **114** (1990), 161–167.

[18] Nobuko Iwahori, A length formula in a semigroup of mappings, *J. Fac. Sci. Univ. Tokyo, Section 1A (Mathematics)* **24** (1977), 255–260.

[19] G. B. Preston, Representations of inverse semigroups, *J. London Math. Soc.* **29** (1954), 411–419.

[20] M. A. Reynolds and R. P. Sullivan, Products of idempotent linear transformations, *Proc. Royal Soc. Edinburgh A* **100** (1985), 123–138.

[21] W. Sierpiński, Sur les suites infinies de fonctions définies dans les ensembles quelconques, *Fund. Math.* **24** (1935), 209–212.

[22] Abdullahi Umar, Semigroups of order-decreasing finite full transformations, *Proc. Royal Soc. Edinburgh A* **120** (1992), 129–142.

[23] V. V. Vagner, Generalized groups, *Dokl. Akad. Nauk SSSR* **94** (1952), 1119–1122.

April 2006