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A. Laradji and A. Umar

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: mathdept@kfupm.edu.sa

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A. Laradji and A. Umar
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261
Saudi Arabia
email: alaradji@kfupm.edu.sa and aumar@kfupm.edu.sa

Abstract

The 'Hat Problem' also known as 'Problème des Rencontres' is often presented as follows: At a restaurant n people check their hats in, and when they leave their hats are returned in a random order. In how many ways can it happen that no one receives his own hat back, and further what is the probability (for large n) of such an event E? The surprising answer which is now folklore is $P(E) \to e^{-1}$ (for large n), and the convergence is very rapid. In this article we give a few generalizations (old and new) of this problem by associating it with certain semigroups of transformations.

1 Introduction and Preliminaries

The 'Hat Problem' also known as 'Problème des Rencontres' is often presented as follows: At a restaurant n people check their hats in, and when they leave their hats are returned in a random order. In how many ways can it happen that no one receives his own hat back, and further what is the probability (for large n) of such an event E? There are now several equivalent formulations to the 'Hat Problem', and we give two different forms. The first, known as the 'Matching Problem' may not be relevant to this article but is popular with students. It says: suppose there are n questions in an exam and n possible answers to choose from, each question having a unique answer. In how many ways can a student match these questions and answers so that no question is matched to its correct answer, and further what is the probability of such an event? It is now not difficult to see that in the 'Matching Problem', the number of ways it can happen that no question is matched to its correct answer is equal to the number of ways hats are returned and no one receives his own hat back.

The second, more relevant to the present discourse is the 'Derangement Problem'. First, recall that a permutation σ of X_n is a *derangement* if $\sigma(x) \neq x$ for all x in X_n. Again, it is not difficult to see that in the 'Hat Problem', the number of ways it can happen that no one receives his own hat back is equal to the number of derangements of X_n. As pointed out in [16, p. 85], the 'Derangement Problem' was first solved by Montmort in probabilistic terms (in 1708), and later independently investigated by Euler. In fact Montmort, Euler and Laplace considered a more general situation, see Table 1 below.

As long as each person leaving receives exactly one hat, then we are dealing with partial one-one transformations (equivalently, subpermutations). Thus we can talk of partial derangements, or more generally, partial one-one maps with exactly k fixed points [12].

Investigated by/in	# of persons checking in	# of persons leaving	# of persons receiving own hat	# of hats received by each person leaving	# of claimed hats
	n	n	0	exactly 1	n
Monmort,Euler & Laplace[15]	n	n	$k\ (\leq n)$	exactly 1	n
Hanson,Seyffarth & Weston[3]	n	$r-$fixed persons	$k\ (\leq r \leq n)$	exactly 1	r
Laradji & Umar[11]	n	varies: $(k \leq) m (\leq n)$	$k\ (\leq n)$	exactly 1	varies from k to n
this article	n	varies: $(k \leq) m (\leq n)$	$k\ (\leq m \leq n)$	at least 1	n
this article	n	varies: $(k \leq) m (\leq n)$	$k\ (\leq m \leq n)$	at least 1	varies from k to n

Table 1. *Summary*

Let $X_n = \{1, 2, \cdots, n\}$. For a given (partial) mapping or transformation $\alpha : Y \subseteq X_n \longrightarrow X_n$ we denote its set of fixed points by $F(\alpha) = \{x \in Y : x\alpha = x\}$, its domain Y by $Dom\,\alpha$ and its image set by $Im\,\alpha$. If $Dom\alpha = X_n$ then α is called a *full* transformation, otherwise it is *strictly partial*. The *height* of α is $|\mathrm{Im}\ \alpha|$. The set of permutations of X_n more commonly known as the symmetric group is usually denoted by S_n while the set of partial one-one transformations of X_n more commonly known as the symmetric inverse semigroup is usually denoted by I_n.
Let $f(n, r, k)$ be the number of ways in which $r\ (\leq n)$ persons leave the restaurant such that only $k\ (\leq r)$ of them receive their own hats back. It is not difficult to see that

$$f(n, r, k) = |\{\alpha \in I_n : |\mathrm{Im}\ \alpha| = r\ \wedge |F(\alpha)| = k\}|. \tag{1.1}$$

Moreover, since we can choose the k fixed points in $\binom{n}{k}$ ways and from the remaining $n-k$ elements of X_n no fixed points are required it follows that

$$f(n, r, k) = \binom{n}{k} f(n-k, r-k, 0). \tag{1.2}$$

Thus to compute $f(n, r, k)$ it is sufficient to compute $f(n, r, 0)$. Laradji and Umar [12] used this fact in showing that

Theorem 1.1 *Let $f(n, r, k)$ be as defined in (1.1). Then for $n \geq r \geq k \geq 0$, we have*

$$f(n, r, k) = \frac{n!}{k!(n-r)!} \sum_{m=0}^{r-k} \binom{n-k-m}{r-k-m} \frac{(-1)^m}{m!}.$$

Let $E(n, r, k)$ be the event that when r persons leave the restaurant only k of them receive their own hats back, and let $PE(n, r, k)$ be the probability of such an event occurring. It now follows from Theorem 1.1 and the well-known fact that there are $n!$ permutations of n objects, that

$$PE(n, n, 0) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \to e^{-1}$$

for large n. In fact, it is known (see for example [13]) that

$$PE(n, n, k) = \frac{1}{k!} \sum_{m=0}^{n-k} \frac{(-1)^m}{m!} \to (k!e)^{-1}.$$

Next let $E(n, k)$ be the event that when some persons (at least k of them) leave the restaurant only k of them receive their own hats back, and let $PE(n, k)$ be the probability of such an event occurring. If we denote by $a_{n,k}$ the number of partial one-one maps of X_n having exactly k fixed points and $b_n = |I_n|$ then Laradji and Umar [12] have shown that

$$PE(n, k) = \frac{a_{n,k}}{b_n} \to (k!e)^{-1}.$$

for large n.

2 Further Generalizations of the Hat Problem

Now suppose we allow the persons entering and leaving the restaurant to behave more greedily or less gentlemanly, that's to have possibly more than one hat when they leave and let $F(n, r, k)$ be the number of ways in which $r\,(\leq n)$ persons leave the restaurant such that only $k\,(\leq r)$ of them receive their own hats back and perhaps other persons' else hats, and that all hats are claimed. It is not difficult to see that

$$F(n, r, k) = |\{\alpha \in T_n : |\text{Im } \alpha| = r \,\wedge |F(\alpha)| = k\}|. \tag{2.1}$$

where T_n is the semigroup of full transformations of X_n. Note that there is no corresponding result to Eqn.(1.2) above because of the interference by the greedy lot who take more than one hat. Finding an expression for $F(n, r, k)$ similar to Theorem 1.1 is our immediate objective. In a private communication, Howie and Giraldes showed us how they obtained a recurrence for $F(n, n-1, k)$ via some closely related function.

Proposition 2.1 *Let $F(n, r, k)$ be as defined in (2.1). Then for $n \geq r \geq k \geq 0$, we have*

$$F(n, r, k) = \binom{n}{r} \sum_{i=0}^{r-k} F(r, r-i, k) \sum_{j=i}^{n-r} \binom{n-r}{j} S(j, i) i! (r-i)^{n-r-j}.$$

Proof. Let α (in T_n) be such that $|\text{Im } \alpha| = r$ and $|F(\alpha)| = k$. Then to construct all such α, first we choose the r elements of Im α from X_n, this can be done in $\binom{n}{r}$ ways. Now these r images must map into themselves whilst preserving the k fixed points, this can be done in $\sum_{i=0}^{r-k} F(r, r-i, k)$ ways. However, note that there are $i \in \{0, 1, \cdots, r-k\}$ elements of Im α without pre-images thus far, and $n-r$ elements of $X_n \setminus \text{Im } \alpha$ without images thus far. Next we choose j $(n-r \geq j \geq i)$ pre-images (for the i images without pre-images before now), partition them into i non-empty subsets and tie them to their i images in a one-one fashion, this can be done in $\sum_{j=i}^{n-r} \binom{n-r}{j} S(j,i)i!$. Note that no new fixed points will arise since the two sets: the i images and the $n-r$ elements of $X_n \setminus \text{Im } \alpha$ are disjoint. Now there remains $n-r-j$ elements without images thus far. Finally note that the only way to avoid repetitions is to attach these $n-r-j$ to the $r-i$ firstly used elements of Im α, this can be done in $(r-i)^{n-r-j}$ ways. Hence the result follows. ■

This is not a satisfactory result and therefore needs to be sharpened. Nevertheless, we can easily deduce the following results each of which is either known or can be proved directly.

Corollary 2.2 $F(n, r, r) = \binom{n}{r} r^{n-r}$.

Remark 2.3 From the well-known fact that α (in T_n) is idempotent iff Im $\alpha = F(\alpha)$ iff $|\text{Im } \alpha = F(\alpha)|$, it follows that $F(n, r, r)$ is the number of idempotents of T_n with r fixed points and so by Corollary 2.2, we recover the result of $\sum_{r=0}^{n} \binom{n}{r} r^{n-r}$ is the number of idempotents of T_n [1, Ex 2.2.2(a)] and [17].

Corollary 2.4 $F(n, r, r-1) = \binom{n}{r} r(r-1)[r^{n-r} - (r-1)^{n-r}]$.

Corollary 2.5 $F(n, n-1, k) = n(n-1)F(n-1, n-1, k) + nF(n-1, n-2, k)$.

We propose an alternative expression for $F(n, r, k)$

Proposition 2.6 *Let $F(n, r, k)$ be as defined in (2.1). Then for $n \geq r \geq k \geq 0$, we have*

$$F(n, r, k) = \binom{n}{r} \sum_{j=0}^{r} (-1)^j (r-j)^{n-r} \sum_{i=j}^{r} F(r, r-i, k) \binom{i}{j}.$$

Remark 2.7 The triangular array of numbers $F(n, r, k)$ $(0 \leq k \leq r \leq n)$ except $F(n, n, k)$ are not yet listed in [15]. For some selected values of $F(n, r, k)$ $(k = 0, 1;$ and $r = n-1)$, see Tables 2-4.

n \ r	1	2	3	4	5	6	7		$\sum F(n,r,0)$
1	0								0
2	0	1							1^2
3	0	6	2						2^3
4	0	24	48	9					3^4
5	0	80	480	420	44				4^5
6	0	240	3360	7920	3840	265			5^6
7	0	672	19320	97440	122640	38010	1854		6^7
8	0	1792	98112	934080	2414720	1893360	407904	14833	7^8

Table 2. *F(n, r,0)*

n \ r	1	2	3	4	5	6	7		$\sum F(n,r,1)$
1	1								1
2	2	0							2
3	3	6	3						12
4	4	36	60	8					108
5	5	140	630	460	45				1280
6	6	450	4620	9300	4110	264			18750
7	7	1302	27405	119140	137025	39858	1855		326592
8	8	3528	141960	1172360	2780120	2052792	422744	14832	6588344

Table 3. *F(n, r,1)*

n \ k	0	1	2	3	4	5	6	7	$\sum F(n,n-1,k)$
1	0								0
2	0	2							2
3	6	6	6						18
4	48	60	24	12					144
5	420	460	240	60	20				1200
6	3840	4110	2040	660	120	30			10800
7	38010	39858	19950	6300	1470	210	42		105840
8	407904	422744	211344	68040	15680	2856	336	56	1128960

Table 4. *F(n, n - 1,k)*

Now let $F(n,k)$ be the number of full transformations of X_n having exactly k fixed points. Then it is clear that

$$F(n,k) = \sum_{r=k}^{n} F(n,r,k).$$

Interestingly, a closed formula for $F(n,k)$ exists

Proposition 2.8 *Let $F(n,k)$ be the number of full transformations of X_n having exactly k fixed points. Then $F(n,k) = \binom{n}{k}(n-1)^{n-k}$.*

Proof. Let α (in T_n) be such that $|F(\alpha)| = k$. Then to construct all such α, first we choose the k elements of $F(\alpha)$ from X_n, this can be done in $\binom{n}{k}$ ways. The remaining $n-k$ elements each has $n-1$ degrees of freedom (avoiding itself), this gives rise to $(n-1)^{n-k}$ possibilities. Hence the result follows. ∎

The corresponding 'Generalized Hat Problem' is: Suppose n people check in their hats in a restaurant, and when some (at least k) of them leave, they each grab at least one hat in a random order, and all hats are claimed. In how many ways can it happen that only k $(0 \leq k \leq n)$ of them receive their own hats back (and possibly some other persons' hat(s) as well), and further what is the probability of such an event occurring for large n? The combinatorial question has been answered by Proposition 2.8. The next result provides an answer to the probabilistic question.

Theorem 2.9 *Let $E(n,k)$ be the event that n people check in their hats in a restaurant, and when some (at least k) of them leave, they each grab at least one hat in a random order, such that all hats are claimed, and let $PE(n,k)$ be the probability of such an event occurring. Then for large n, we have $PE(n,k) \to (k!e)^{-1}$.*

Proof. First note that $|T_n| = n^n$ and $F(n,k) = \binom{n}{k}(n-1)^{n-k}$. Thus

$$\begin{aligned} PE(n,k) &= \frac{\binom{n}{k}(n-1)^{n-k}}{n^n} = \frac{n!}{(n-k)!k!}\frac{(n-1)^{n-k}}{n^n} \\ &= \frac{1}{k!}\frac{n!}{(n-1)^k(n-k)!}\left(1-\frac{1}{n}\right)^n \to (k!e)^{-1} \end{aligned}$$

for a fixed k, as $n \to \infty$ ∎

Finally, in the above 'Generalized Hat Problem' if we relax the condition that all hats are claimed, that is to: some hats (may be none) may be unclaimed then we are dealing with partial transformations. Let P_n denote the semigroup of partial transformations of X_n. Then it is well-known that $|P_n| = (n+1)^n$. We also have

Proposition 2.10 *Let $F'(n,k)$ be the number of partial transformations of X_n having exactly k fixed points. Then $F'(n,k) = \begin{pmatrix} n \\ k \end{pmatrix} n^{n-k}$.*

Proof. Let α (in P_n) be such that $|F(\alpha)| = k$. Then to construct all such α, first we choose the k elements of $F(\alpha)$ from X_n, this can be done in $\begin{pmatrix} n \\ k \end{pmatrix}$ ways. The remaining $n-k$ elements each has n degrees of freedom: $n-1$ possible images (avoiding itself) and the extra freedom of not being in the domain of α, this gives rise to n^{n-k} possibilities. Hence the result follows. ■

We again get the surprising result:

Theorem 2.11 *Let $E'(n,k)$ be the event that n people check in their hats in a restaurant, and when some (at least k) of them leave, they each grab at least one hat in a random order, such that some (possibly none) of the hats may be unclaimed, and let $PE'(n,k)$ be the probability of such an event occurring. Then for large n, we have $PE'(n,k) \to (k!e)^{-1}$.*

Proof. First note that $|P_n| = (n+1)^n$ and $F'(n,k) = \begin{pmatrix} n \\ k \end{pmatrix} n^{n-k}$. Thus

$$\begin{aligned} PE(n,k) &= \frac{\begin{pmatrix} n \\ k \end{pmatrix} n^{n-k}}{(n+1)^n} = \frac{n!}{(n-k)!k!} \frac{n^{n-k}}{(n+1)^n} \\ &= \frac{1}{k!} \frac{n!}{n^k(n-k)!} \left(1 - \frac{1}{n+1}\right)^n \to (k!e)^{-1} \end{aligned}$$

for a fixed k, as $n \to \infty$ ■

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