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Abstract

Prime objects were defined as generalization of simple objects in the categories of rings (modules). In this paper we introduce and investigate what turns out to be a suitable generalization of simple corings (simple comodules), namely *fully coprime corings* (*fully coprime comodules*). Moreover, we consider several *primeness* notions in the category of comodules of a given coring and investigate their relations with the fully coprimeness and the simplicity of these comodules. These notions are applied then to study primeness and coprimeness properties of a given coring, considered as an object in its category of right (left) comodules.

1 Introduction

Prime ideals play a central role in the Theory of Rings. In particular, *localization of commutative rings* at prime ideals is an essential tool in Commutative Algebra. One goal of this paper is to introduce a suitable dual notion of *coprimeness* for corings over arbitrary (not necessarily commutative) ground rings as a first step towards developing a theory of *colocalization of corings*, which seems till now to be far from reach.

The classical notion of a prime ring was generalized, in different ways, to introduce prime objects in the category of modules of a given ring (see [Wis96, Section 13]). A main goal of this paper is to introduce *coprime* objects, which generalize simple objects, in the category of corings (comodules). As there are several *primeness* properties of modules of a given ring, we are led as well to several *primeness* and *coprimeness* properties of comodules of a coring. We investigate these different properties and clarify the relations between them.

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Coprime coalgebras over base fields were introduced by R. Nekooei and L. Torkezadeh in [NT01] as a generalization of simple coalgebras: simple coalgebras are coprime; and finite dimensional coprime coalgebras are simple. These coalgebras, which we call here *fully coprime*, were defined using the so called *wedge product* of subcoalgebras and can be seen as dual to prime algebras: a coalgebra C over a base field is coprime if and only if its dual algebra C^* is prime. Coprime coalgebras were considered also by P. Jara et. al. in their study of representation theory of coalgebras and path coalgebras [JMR].

For a coring \mathcal{C} over a QF ring A such that ${}_A\mathcal{C}$ (\mathcal{C}_A) is projective, we observe in Proposition 5.15 that if $K, L \subseteq \mathcal{C}$ are any A -subbimodules that are right (left) \mathcal{C} -coideals as well and satisfy suitable purity conditions, then the *wedge product* $K \wedge L$, in the sense of [Swe69], is nothing but their *internal coproduct* $(K :_{\mathcal{C}^r} L)$ ($(K :_{\mathcal{C}^l} L)$) in the category of right (left) \mathcal{C} -comodules, in the sense of [RRW05]. This observation suggests extending the notion of fully coprime coalgebras over base fields to *fully coprime corings* over arbitrary ground rings by replacing the wedge product of subcoalgebras with the internal coproduct of subbicomodules. We also extend that notion to *fully coprime comodules* using the internal coproduct of fully invariant subcomodules. Using the internal coproduct of a bicoideal of a coring (a fully invariant subcomodule of a comodule) with itself enables us to introduce *fully cosemiprime corings* (*fully cosemiprime comodules*). Dual to prime radicals of rings (modules), we introduce and investigate the *fully coprime coradicals* of corings (comodules).

This article is divided as follows: after this first introductory section, we give in the second section some definitions and recall some needed results from the Theory of Rings and Modules as well as from the Theory of Corings and Comodules. As a coalgebra C over a base field is fully coprime if and only if its dual algebra $C^* \simeq \text{End}^C(C)^{op}$ is prime, see [NT01, Proposition 1.2], we devote the third section to the study of primeness properties of the ring of \mathcal{C} -colinear endomorphisms $E_M^{\mathcal{C}} := \text{End}^{\mathcal{C}}(M)^{op}$ of a given right \mathcal{C} -comodule M of a coring \mathcal{C} . Given a coring \mathcal{C} , we say a non-zero right \mathcal{C} -comodule M is *E-prime* (respectively *E-semiprime*, *completely E-prime*, *completely E-semiprime*), provided the ring $E_M^{\mathcal{C}} := \text{End}^{\mathcal{C}}(M)^{op}$ is prime (respectively semiprime, domain, reduced). In case M is *self-cogenerator*, Theorem 3.17 provides sufficient and necessary conditions for M to be E-prime (respectively E-semiprime, completely E-prime, completely E-semiprime). Under suitable conditions, we clarify in Theorem 3.30 the relation between E-prime and irreducible comodules. In the fourth section we present and study *fully coprime* (*fully cosemiprime*) *comodules* using the *internal coproduct* of fully invariant subcomodules. Let \mathcal{C} be a coring and M be a non-zero right \mathcal{C} -comodule. A fully invariant non-zero \mathcal{C} -subcomodule $K \subseteq M$ will be called *fully M -coprime* (*fully M -cosemiprime*), iff for any (equal) fully invariant \mathcal{C} -subcomodules $X, Y \subseteq M$ with $K \subseteq (X :_M^{\mathcal{C}} Y)$, we have $K \subseteq X$ or $K \subseteq Y$, where $(X :_M^{\mathcal{C}} Y)$ is the internal coproduct of X, Y in the category of right \mathcal{C} -comodules. We call the non-zero right \mathcal{C} -comodule M *fully coprime* (*fully cosemiprime*), iff M is fully M -coprime (fully M -cosemiprime). The notion of fully coprimeness (fully cosemiprimeness) in the category of left \mathcal{C} -comodules is defined analogously. Theorem 4.11 clarifies the relation between fully coprime (fully cosemiprime) and E-prime (E-semiprime) comodules under suitable conditions. We define the *fully coprime spectrum* of M as the class of all fully M -coprime \mathcal{C} -subcomodules of M and the *fully coprime coradical* of M as the sum of all fully M -coprime \mathcal{C} -subcomodules. In Proposition 4.12 we clarify the

relation between the fully coprime coradical of M and the prime radical of $E_M^{\mathcal{C}}$, in case M is intrinsically injective self-cogenerator and $E_M^{\mathcal{C}}$ is right Noetherian. Fully coprime comodules turn to be a generalization of simple comodules: simple comodules are trivially fully coprime; and Theorem 4.16 (2) shows that if the ground ring A is right Artinian and ${}_A\mathcal{C}$ is locally projective, then a non-zero finitely generated self-injective self-cogenerator right \mathcal{C} -comodule M is fully coprime if and only if M is simple as a $({}^*\mathcal{C}, E_M^{\mathcal{C}})$ -bimodule. Under suitable conditions, we clarify in Theorem 4.21 the relation between fully coprime and irreducible comodules. In the fifth section we introduce and study several primeness and coprimeness properties of a non-zero coring \mathcal{C} , considered as an object in the category $\mathbb{M}^{\mathcal{C}}$ of right \mathcal{C} -comodules and as an object in the category ${}^{\mathcal{C}}\mathbb{M}$ of left \mathcal{C} -comodules. We define the internal coproducts of \mathcal{C} -bicoideals, i.e. $(\mathcal{C}, \mathcal{C})$ -subbicomodules of \mathcal{C} , in $\mathbb{M}^{\mathcal{C}}$ and in ${}^{\mathcal{C}}\mathbb{M}$ and use them to introduce the notions of *fully coprime (fully cosemiprime) \mathcal{C} -bicoideals* and *fully coprime (fully cosemiprime) corings*. Moreover, we introduce and study the *fully coprime spectrum* and the *fully coprime coradical* of \mathcal{C} in $\mathbb{M}^{\mathcal{C}}$ (in ${}^{\mathcal{C}}\mathbb{M}$) and clarify their relations with the prime spectrum and the prime radical of \mathcal{C}^* (${}^*\mathcal{C}$). We investigate several coprimeness (cosemiprimeness) and primeness (semiprimeness) notions for \mathcal{C} and clarify their relations with the simplicity (semisimplicity) of the coring under consideration. In Theorems 5.4 we give sufficient and necessary conditions for the dual ring \mathcal{C}^* (${}^*\mathcal{C}$) of \mathcal{C} to be prime (respectively semiprime, domain, reduced). In case the ground ring A is a QF ring, ${}_A\mathcal{C}, \mathcal{C}_A$ are locally projective and \mathcal{C}^* is right Artinian, ${}^*\mathcal{C}$ is left Artinian, we show in Theorem 5.25 that \mathcal{C}^r is fully coprime if and only if \mathcal{C} is simple if and only if \mathcal{C}^l is fully coprime.

Throughout, R is a commutative ring with $1_R \neq 0_R$, A is an arbitrary but fixed unital R -algebra and \mathcal{C} is a non-zero A -coring. All rings have unities preserved by morphisms of rings and all modules are unital. Let T be a ring and denote with ${}_T\mathbb{M}$ (\mathbb{M}_T) the category of left (right) T -modules. For a left (right) T -module M , we denote with $\sigma[{}_T M] \subseteq {}_T\mathbb{M}$ ($\sigma[M_T] \subseteq \mathbb{M}_T$) the *full* subcategory of *M -subgenerated* left (right) T -modules; see [Wis91] and [Wis96].

2 Preliminaries

In this section we introduce some definitions, remarks and lemmas to which we refer later.

2.1. ([Z-H76]) An A -module W is called *locally projective* (in the sense of B. Zimmermann-Huisgen [Z-H76]), if for every diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F & \xrightarrow{\iota} & W & & \\
 & & \searrow & & \searrow & & \\
 & & & & L & \xrightarrow{\pi} & N \longrightarrow 0 \\
 & & g' \circ \iota & \searrow & g' & & \\
 & & & & & &
 \end{array}$$

with exact rows and F f.g.: for every A -linear map $g : W \rightarrow N$, there exists an A -linear map $g' : W \rightarrow L$, such that the entstanding parallelogram is commutative. Note that every projective A -module is locally projective. Moreover, every locally projective A -module is flat and A -cogenerated.

Prime and coprime modules

Definition 2.2. Let T be a ring. A proper ideal $P \triangleleft T$ is called

prime, iff for any two ideals $I, J \triangleleft T$ with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$;

semiprime, iff for any ideal $I \triangleleft T$ with $I^2 \subseteq P$, we have $I \subseteq P$;

completely prime, iff for any $f, g \in P$ with $fg \in P$, either $f \in P$ or $g \in P$;

completely semiprime, iff for any $f \in T$ with $f^2 \in P$, we have $f \in P$.

The ring T is called *prime* (respectively *semiprime*, *domain*, *reduced*), iff $(0_T) \triangleleft T$ is prime (respectively semiprime, completely prime, completely semiprime).

2.3. Let T be a ring. With $\text{Max}(T)$ (resp. $\text{Max}_r(T)$, $\text{Max}_l(T)$) we denote the class of maximal two-sided T -ideals (resp. maximal right T -ideals, maximal left T -ideals) and with $\text{Sepc}(T)$ the prime spectrum of T consisting of all prime ideals of T . The *Jacobson radical* of T is denoted by $\text{Jac}(T)$ and the *prime radical* of T by $\text{Prad}(T)$. Notice that the ring T is semiprime if and only if $\text{Prad}(T) = 0$.

There are various notions of prime and coprime modules in the literature; see [Wis96, Section 13] for more details. In this paper we adopt the notion of *prime modules* due to R. Johnson [Joh53] and its dual notion of *coprime modules* considered recently by S. Annin [Ann].

Definition 2.4. Let T be a ring. A non-zero T -module M will be called

prime, iff $\text{ann}_T(K) = \text{ann}_T(M)$ for every non-zero T -submodule $0 \neq K \subseteq M$;

coprime, iff $\text{ann}_T(M/K) = \text{ann}_T(M)$ for every proper T -submodule $K \subsetneq M$;

diprime, iff $\text{ann}_T(K) = \text{ann}_T(M)$ or $\text{ann}_T(M/K) = \text{ann}_T(M)$ for every non-trivial T -submodule $0 \neq K \subsetneq M$;

strongly prime, iff $M \in \sigma[K]$ for every non-zero T -submodule $0 \neq K \subseteq M$.

It's well known that for every prime (coprime) T -module M , the associated quotient ring $\bar{T} := T/\text{ann}_T(M)$ is prime. In fact we have more:

Proposition 2.5. ([Lom05, Proposition 1.1]) *Let T be a ring and M be a non-zero T -module. Then the following are equivalent:*

1. $\bar{T} := T/\text{ann}_T(M)$ is a prime ring;
2. M is diprime;
3. For every fully invariant T -submodule $K \subseteq M$ that is M -generated as an $\text{End}_T(M)$ -module, $\text{ann}_T(K) = \text{ann}_T(M)$ or $\text{ann}_T(M/K) = \text{ann}_T(M)$.

Remark 2.6. Let T be a ring and consider the following conditions for a non-zero T -module M :

$$\text{ann}_T(M/K) \neq \text{ann}_T(M) \text{ for every non-zero } T\text{-submodule } 0 \neq K \subseteq M \quad (*)$$

$$\text{ann}_T(K) \neq \text{ann}_T(M) \text{ for every proper } T\text{-submodule } K \subsetneq M \quad (**).$$

We introduce condition $(**)$ as dual to condition $(*)$, which is due to Wisbauer [Wis96, Section 13]. Modules satisfying either of these conditions allow further conclusions from the primeness (coprimeness) properties: by Proposition 2.5, a T -module M satisfying condition $(*)$ (condition $(**)$) is prime (coprime) if and only if $\bar{T} := T/\text{ann}_T(M)$ is prime.

Corings and comodules

Fix a non-zero A -coring $(\mathcal{C}, \Delta, \varepsilon)$. With $\mathbb{M}^{\mathcal{C}}$ (${}^{\mathcal{C}}\mathbb{M}$) we denote the category of right (left) \mathcal{C} -comodules with the \mathcal{C} -colinear morphisms and by \mathcal{C}^r (\mathcal{C}^l) we mean the coring \mathcal{C} , considered as an object in $\mathbb{M}^{\mathcal{C}}$ (${}^{\mathcal{C}}\mathbb{M}$). For a right (left) \mathcal{C} -comodule M we denote with $E_M^{\mathcal{C}} := \text{End}^{\mathcal{C}}(M)^{op}$ (${}^{\mathcal{C}}E := {}^{\mathcal{C}}\text{End}(M)$) the ring of all \mathcal{C} -colinear endomorphisms of M with multiplication the opposite (usual) composition of maps and call an R -submodule $X \subseteq M$ *fully invariant*, iff $f(X) \subseteq X$ for every $f \in E_M^{\mathcal{C}}$ ($f \in {}^{\mathcal{C}}E$).

In module categories, monomorphisms are injective maps. In comodule categories this is not the case in general. In fact we have:

Remark 2.7. For any coring \mathcal{C} over a ground ring A , the module ${}_A\mathcal{C}$ is flat if and only if every monomorphism in $\mathbb{M}^{\mathcal{C}}$ is injective (e.g. [Abu03, Proposition 1.10]). In this case, $\mathbb{M}^{\mathcal{C}}$ is a Grothendieck category with kernels formed in the category of right A -modules and given a right \mathcal{C} -comodule M , the intersection $\bigcap_{\lambda \in \Lambda} M_{\lambda} \subseteq M$ of any family $\{M_{\lambda}\}_{\lambda \in \Lambda}$ of \mathcal{C} -subcomodules of M is again a \mathcal{C} -subcomodule.

Definition 2.8. Let ${}_A\mathcal{C}$ (\mathcal{C}_A) be flat. We call a non-zero right (left) \mathcal{C} -subcomodule M *simple*, iff M has no non-trivial \mathcal{C} -subcomodules;
semisimple, iff $M = \text{Soc}(M)$ where

$$\text{Soc}(M) := \bigoplus \{K \subseteq M \mid K \text{ is a simple } \mathcal{C}\text{-subcomodule}\}. \quad (1)$$

The right (left) \mathcal{C} -subcomodule $\text{Soc}(M) \subseteq M$ defined in (1) is called the *socle* of M . We call a non-zero right (left) \mathcal{C} -subcomodule $0 \neq K \subseteq M$ *essential* in M , and write $K \triangleleft_e M$, provided $K \cap \text{Soc}(M) \neq 0$.

Lemma 2.9. ([Abu03, Proposition 1.10]) *If A_A is injective (cogenerator) and N is a right A -module, then the canonical right \mathcal{C} -comodule $M := (N \otimes_A \mathcal{C}, id \otimes_A \Delta_{\mathcal{C}})$ is injective (cogenerator) in $\mathbb{M}^{\mathcal{C}}$. In particular, if A_A is injective (cogenerator) then $\mathcal{C} \simeq A \otimes_A \mathcal{C}$ is injective (cogenerator) in $\mathbb{M}^{\mathcal{C}}$.*

For an A -coring \mathcal{C} , the dual module ${}^*\mathcal{C} := \text{Hom}_{A-}(\mathcal{C}, A)$ ($\mathcal{C}^* := \text{Hom}_{-A}(\mathcal{C}, A)$) of left (right) A -linear maps from \mathcal{C} to A is a ring under the so called *convolution product*. We remark here that the multiplications used below are opposite to those in previous papers of the author, e.g. [Abu03], and are consistent with the ones in [BW03].

2.10. Dual rings of corings. Let $(\mathcal{C}, \Delta, \varepsilon)$ be an A -coring. Then ${}^*\mathcal{C} := \text{Hom}_{A-}(\mathcal{C}, A)$ (respectively $\mathcal{C}^* := \text{Hom}_{-A}(\mathcal{C}, A)$) is an A^{op} -ring with multiplication

$$(f *^l g)(c) = \sum f(c_1 g(c_2)) \quad (\text{respectively } (f *^r g)(c) = \sum g(f(c_1) c_2))$$

and unity ε . The coring \mathcal{C} is a $({}^*\mathcal{C}, \mathcal{C}^*)$ -bimodule through the left ${}^*\mathcal{C}$ -action (respectively the right \mathcal{C}^* -action):

$$f \rightharpoonup c := \sum c_1 f(c_2) \quad \text{for all } f \in {}^*\mathcal{C} \quad (\text{respectively } c \leftarrow g := \sum g(c_1) c_2 \quad \text{for all } g \in \mathcal{C}^*).$$

2.11. Let M be a right (left) \mathcal{C} -comodule. Then M is a left ${}^*\mathcal{C}$ -module (a right \mathcal{C}^* -module) under the left (right) action

$$f \rightharpoonup m := \sum m_{\langle 0 \rangle} f(m_{\langle 1 \rangle}) \text{ for all } f \in {}^*\mathcal{C} \quad (m \leftarrow g := \sum g(m_{\langle -1 \rangle}) m_{\langle 0 \rangle} \text{ for all } g \in \mathcal{C}^*).$$

Notice that M is a $({}^*\mathcal{C}, E_M^{\mathcal{C}})$ -bimodule (a $(\mathcal{C}^*, {}^{\mathcal{C}}_M E)$ -bimodule) in the canonical way. A right (left) \mathcal{C} -subcomodule $K \subseteq M$ is said to be *fully invariant*, provided K is a $({}^*\mathcal{C}, E_M^{\mathcal{C}})$ -subbimodule ($(\mathcal{C}^*, {}^{\mathcal{C}}_M E)$ -subbimodule) of M . Since $\mathbb{M}^{\mathcal{C}}$ (${}^{\mathcal{C}}\mathbb{M}$) has cokernels, we conclude that for any $f \in E_M^{\mathcal{C}}$ (any $g \in {}^{\mathcal{C}}_M E$), $Mf := f(M) \subseteq M$ ($gM := g(M) \subseteq M$) is a right (left) \mathcal{C} -subcomodule and that for any right ideal $I \triangleleft_r E_M^{\mathcal{C}}$ (left ideal $J \triangleleft_l {}^{\mathcal{C}}_M E$) we have a fully-invariant right (left) \mathcal{C} -subcomodule $MI \subseteq M$ ($JM \subseteq M$).

Proposition 2.12. ([Abu03, Theorems 2.9, 2.11]) *For any A -coring \mathcal{C} we have*

1. $\mathbb{M}^{\mathcal{C}} \simeq \sigma[\mathcal{C}^*_{\mathcal{C}^{op}}] \simeq \sigma[{}^*\mathcal{C}]$ if and only if ${}_A\mathcal{C}$ is locally projective.
2. ${}^{\mathcal{C}}\mathbb{M} \simeq \sigma[\mathcal{C}^*_{\mathcal{C}^{op}}] \simeq \sigma[\mathcal{C}^*]$ if and only if \mathcal{C}_A is locally projective.

Notation. Let M be a right \mathcal{C} -comodule. We denote with $\mathcal{C}(M)$ ($\mathcal{C}_{f.i.}(M)$) the class of (fully invariant) \mathcal{C} -subcomodules of M and with $\mathcal{I}_r(E_M^{\mathcal{C}})$ ($\mathcal{I}_{t.s.}(E_M^{\mathcal{C}})$) the class of right (two-sided) ideals of $E_M^{\mathcal{C}}$. For $\emptyset \neq K \subseteq M$, $\emptyset \neq I \subseteq E_M^{\mathcal{C}}$ set

$$\text{An}(K) := \{f \in E_M^{\mathcal{C}} \mid f(K) = 0\}, \quad \text{Ke}(I) := \bigcap \{\text{Ker}(f) \mid f \in I\}.$$

The following notions for right \mathcal{C} -comodules will be used in the sequel. The analogous notions for left \mathcal{C} -comodules can be defined analogously:

Definition 2.13. Let ${}_A\mathcal{C}$ be flat. We say a right \mathcal{C} -comodule M is

self-injective, iff for every \mathcal{C} -subcomodule $K \subseteq M$, every \mathcal{C} -colinear morphism $f \in \text{Hom}^{\mathcal{C}}(K, M)$ extends to a \mathcal{C} -colinear endomorphism $\tilde{f} \in \text{End}^{\mathcal{C}}(M)$;

semi-injective, iff for every monomorphism $0 \longrightarrow N \xrightarrow{h} M$ in $\mathbb{M}^{\mathcal{C}}$, where N is a factor \mathcal{C} -comodule of M , and every $f \in \text{Hom}^{\mathcal{C}}(N, M)$, $\exists \tilde{f} \in \text{End}^{\mathcal{C}}(M)$ such that $\tilde{f} \circ h = f$;

self-projective, iff for every \mathcal{C} -subcomodule $K \subseteq M$, and every $g \in \text{Hom}^{\mathcal{C}}(M, M/K)$, $\exists \tilde{g} \in \text{End}^{\mathcal{C}}(M)$ such that $\pi_K \circ \tilde{g} = g$;

self-cogenerator, iff M cogenerates all of its factor \mathcal{C} -comodules;

self-generator, iff M generates each of its \mathcal{C} -subcomodules;

coretractable, iff $\text{Hom}^{\mathcal{C}}(M/K, M) \neq 0$ for every proper \mathcal{C} -subcomodule $K \subsetneq M$;

retractable, iff $\text{Hom}^{\mathcal{C}}(M, K) \neq 0$ for every non-zero \mathcal{C} -subcomodule $0 \neq K \subseteq M$;

intrinsically injective, iff $\text{AnKe}(I) = I$ for every f.g. right ideal $I \triangleleft E_M^{\mathcal{C}}$.

The following result follows immediately from ([Wis91, 31.11, 31.12]) and Proposition 2.12:

Proposition 2.14. *Let ${}_A\mathcal{C}$ be locally projective, M be a non-zero right \mathcal{C} -comodule and consider the ring $E_M^{\mathcal{C}} := \text{End}^{\mathcal{C}}(M)^{op} = \text{End}({}^*\mathcal{C}M)^{op}$.*

1. *If M is Artinian and self-injective, then $E_M^{\mathcal{C}}$ is right Noetherian.*
2. *If M is Artinian, self-injective and self-projective, then $E_M^{\mathcal{C}}$ is right Artinian.*
3. *If M is semi-injective and satisfies the ascending chain condition for annihilator \mathcal{C} -subcomodules, then $E_M^{\mathcal{C}}$ is semiprimary.*

Annihilator conditions for comodules

Analogous to the *annihilator conditions* for modules (e.g. [Wis91, 28.1]), the following result gives some annihilator conditions for comodules.

2.15. Let ${}_A\mathcal{C}$ be flat, M be a right \mathcal{C} -comodule and consider the order-reversing mappings

$$\text{An}(-) : \mathcal{C}(M) \rightarrow \mathcal{I}_r(\mathbf{E}_M^{\mathcal{C}}) \text{ and } \text{Ke}(-) : \mathcal{I}_r(\mathbf{E}_M^{\mathcal{C}}) \rightarrow \mathcal{C}(M). \quad (2)$$

1. For every $K \in \mathcal{C}_{f.i.}(M)$ ($I \in \mathcal{I}_{t.s.}(\mathbf{E}_M^{\mathcal{C}})$), we have $\text{An}(K) \in \mathcal{I}_{t.s.}(\mathbf{E}_M^{\mathcal{C}})$ ($\text{Ke}(I) \in \mathcal{C}_{f.i.}(M)$). Moreover $\text{An}(-)$ and $\text{Ke}(-)$ induce bijections

$$\begin{aligned} \mathcal{A}(\mathbf{E}_M^{\mathcal{C}}) &:= \{\text{An}(K) \mid K \in \mathcal{C}(M)\} &\leftrightarrow & \mathcal{K}(M) &:= \{\text{Ke}(I) \mid I \in \mathcal{I}_r(\mathbf{E}_M^{\mathcal{C}})\}; \\ \mathcal{A}_{t.s.}(\mathbf{E}_M^{\mathcal{C}}) &:= \{\text{An}(K) \mid K \in \mathcal{C}_{f.i.}(M)\} &\leftrightarrow & \mathcal{K}_{f.i.}(M) &:= \{\text{Ke}(I) \mid I \in \mathcal{I}_{t.s.}(\mathbf{E}_M^{\mathcal{C}})\}. \end{aligned}$$

2. For any \mathcal{C} -subcomodule $K \subseteq M$ we have

$$\text{KeAn}(K) = K \text{ if and only if } M/K \text{ is } M\text{-cogenerated.}$$

3. If M is self-injective, then

- (a) $\text{An}(\bigcap_{i=1}^n K_i) = \sum_{i=1}^n \text{An}(K_i)$ for any finite set of \mathcal{C} -subcomodules $K_1, \dots, K_n \subseteq M$.
- (b) M is intrinsically injective.

Remarks 2.16. let ${}_A\mathcal{C}$ be flat and M be a right \mathcal{C} -comodule.

1. If M is self-injective (self-cogenerator), then every *fully invariant* \mathcal{C} -subcomodule of M is also self-injective (self-cogenerator).
2. If M is self-injective, then M is semi-injective. If M is self-generator (self-cogenerator), then it is obviously retractable (coretractable).
3. If M is self-cogenerator (M is intrinsically injective and $\mathbf{E}_M^{\mathcal{C}}$ is right Noetherian), then the mapping

$$\text{An}(-) : \mathcal{C}(M) \rightarrow \mathcal{I}_r(\mathbf{E}_M^{\mathcal{C}}) \text{ (Ke}(-) : \mathcal{I}_r(\mathbf{E}_M^{\mathcal{C}}) \rightarrow \mathcal{C}(M))$$

is injective.

4. Let M be self-injective. If $H \subsetneq K \subseteq M$ are \mathcal{C} -subcomodules with K coretractable and fully invariant in M , then $\text{An}(K) \subsetneq \text{An}(H)$: since M is self-injective and $K \subseteq M$ is fully invariant, we have a surjective morphism of R -algebras $\mathbf{E}_M^{\mathcal{C}} \rightarrow \mathbf{E}_K^{\mathcal{C}} \rightarrow 0$, $f \mapsto f|_K$, which induces a bijection $\text{An}(H)/\text{An}(K) \longleftrightarrow \text{An}_{\mathbf{E}_K^{\mathcal{C}}}(H) \simeq \text{Hom}^{\mathcal{C}}(K/H, K) \neq 0$.

3 E-prime (E-semiprime) Comodules

In this section we study and characterize non-zero comodules, for which the ring of colinear endomorphisms is prime (respectively semiprime, domain, reduced). Throughout, we assume \mathcal{C} is a non-zero A -coring with ${}_A\mathcal{C}$ flat, M is a non-zero right \mathcal{C} -comodule and $E_M^{\mathcal{C}} := \text{End}^{\mathcal{C}}(M)^{op}$ is the ring of \mathcal{C} -colinear endomorphisms of M with the opposite composition of maps. We remark that analogous results to those obtained in this section can be obtained for left \mathcal{C} -comodules, by symmetry.

Definition 3.1. We define a fully invariant non-zero \mathcal{C} -subcomodule $0 \neq K \subseteq M$ to be

- E-prime in M* , iff $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is prime;
- E-semiprime in M* , iff $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is semiprime;
- completely E-prime in M* , iff $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is completely prime;
- completely E-semiprime in M* , iff $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is completely semiprime.

Definition 3.2. We call the right \mathcal{C} -comodule M *E-prime* (respectively *E-semiprime*, *completely E-prime*, *completely E-semiprime*), provided M is E-prime in M (respectively E-semiprime in M , completely E-prime in M , completely E-semiprime in M), equivalently iff R -algebra $E_M^{\mathcal{C}}$ is prime (respectively semiprime, domain, reduced).

Notation. For the right \mathcal{C} -comodule M we denote with $\text{EP}(M)$ (resp. $\text{ESP}(M)$, $\text{CEP}(M)$, $\text{CESP}(M)$) the class of fully invariant \mathcal{C} -subcomodules of M whose annihilator in $E_M^{\mathcal{C}}$ is prime (resp. semiprime, completely prime, completely semiprime).

Example 3.3. Let $P \triangleleft E_M^{\mathcal{C}}$ be a proper two-sided ideal with $P = \text{AnKe}(P)$ (e.g. M intrinsically injective and $P_{E_M^{\mathcal{C}}}$ finitely generated) and consider the fully invariant \mathcal{C} -subcomodule $0 \neq K := \text{Ke}(P) \subseteq M$. Assume $P \triangleleft E_M^{\mathcal{C}}$ to be prime (respectively semiprime, completely prime, completely semiprime). Then $K \in \text{EP}(M)$ (respectively $K \in \text{ESP}(M)$, $K \in \text{CEP}(M)$, $K \in \text{CESP}(M)$). If moreover M is self-injective, then we have isomorphisms of R -algebras

$$E_K^{\mathcal{C}} \simeq E_M^{\mathcal{C}}/\text{An}(K) = E_M^{\mathcal{C}}/\text{AnKe}(P) = E_M^{\mathcal{C}}/P,$$

hence K is E-prime (respectively E-semiprime, completely E-prime, completely E-semiprime).

For the right \mathcal{C} -comodule M we have

$$\text{CEP}(M) \subseteq \text{EP}(M) \subseteq \text{ESP}(M) \text{ and } \text{CEP}(M) \subseteq \text{CESP}(M) \subseteq \text{ESP}(M). \quad (3)$$

Remark 3.4. The idea of Example 3.3 can be used to construct *counterexamples*, which show that the inclusions in (3) are *in general* strict.

The E-prime coradical

Definition 3.5. We define the *E-prime coradical* of the right \mathcal{C} -comodule M as

$$\text{EPcorad}(M) = \sum_{K \in \text{EP}(M)} K.$$

Proposition 3.6. *Let M be intrinsically injective. If $E_M^{\mathcal{C}}$ is right Noetherian, then*

$$\text{Prad}(E_M^{\mathcal{C}}) = \text{An}(\text{EPcorad}(M)). \quad (4)$$

If moreover M is self-cogenerator, then

$$\text{EPcorad}(M) = \text{Ke}(\text{Prad}(E_M^{\mathcal{C}})). \quad (5)$$

Proof. If $K \in \text{EP}(M)$, then $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is a prime ideal (by definition). On the otherhand, if $P \triangleleft E_M^{\mathcal{C}}$ is a prime ideal then $P = \text{AnKe}(P)$ (since $P_{E_M^{\mathcal{C}}}$ is finitely generated and M is intrinsically injective) and so $K := \text{Ke}(P) \in \text{EP}(M)$. It follows then that

$$\begin{aligned} \text{Prad}(E_M^{\mathcal{C}}) &= \bigcap_{P \in \text{Spec}(E_M^{\mathcal{C}})} P &= \bigcap_{P \in \text{Spec}(E_M^{\mathcal{C}})} \text{AnKe}(P) \\ &= \bigcap_{K \in \text{EP}(M)} \text{AnKeAn}(K) &= \bigcap_{K \in \text{EP}(M)} \text{An}(K) \\ &= \text{An}(\sum_{K \in \text{EP}(M)} K) &= \text{An}(\text{EPcorad}(M)). \end{aligned}$$

If moreover M is self-cogenerator, then

$$\text{EPcorad}(M) = \text{KeAn}(\text{EPcorad}(M)) = \text{Ke}(\text{Prad}(E_M^{\mathcal{C}})). \blacksquare$$

Corollary 3.7. *Let M be intrinsically injective self-cogenerator. If $E_M^{\mathcal{C}}$ is right Noetherian, then*

$$M = \text{EPcorad}(M) \Leftrightarrow M \text{ is } E\text{-semiprime}.$$

Proof. Under the assumptions and Proposition 3.6 we have: $M = \text{EPcorad}(M) \Rightarrow \text{Prad}(E_M^{\mathcal{C}}) = \text{An}(\text{EPcorad}(M)) = \text{An}(M) = 0$, i.e. $E_M^{\mathcal{C}}$ is semiprime; on the otherhand $E_M^{\mathcal{C}}$ semiprime $\Rightarrow \text{EPcorad}(M) = \text{Ke}(\text{Prad}(E_M^{\mathcal{C}})) = \text{Ke}(0) = M$. \blacksquare

Remark 3.8. Let ${}_A\mathcal{C}$ be locally projective and M be right \mathcal{C} -comodule. A sufficient condition for $E_M^{\mathcal{C}}$ to be right Noetherian, so that the results of Proposition 3.6 and Corollary 3.7 follow, is that M is Artinian and self-injective (see 2.14 (1)). We recall here also that in case A_A is Artinian, every finitely generated right \mathcal{C} -comodule has finite length by [Abu03, Corollary 2.25 (4)].

Proposition 3.9. *Let $\theta : L \rightarrow M$ be an isomorphism of right \mathcal{C} -comodules. Then we have bijections*

$$\text{EP}(L) \leftrightarrow \text{EP}(M), \quad \text{ESP}(L) \leftrightarrow \text{ESP}(M), \quad \text{CEP}(L) \leftrightarrow \text{CEP}(M), \quad \text{CESP}(L) \leftrightarrow \text{CESP}(M). \quad (6)$$

In particular

$$\theta(\text{EPcorad}(L)) = \text{EPcorad}(M). \quad (7)$$

If moreover L, M are self-injective, then we have bijections between the class of E -prime (respectively E -semiprime, completely E -prime, completely E -semiprime) \mathcal{C} -subcomodules of L and the class of E -prime (respectively E -semiprime, completely E -prime, completely E -semiprime) \mathcal{C} -subcomodules of M .

Proof. Since θ is an isomorphism in $\mathbb{M}^{\mathcal{C}}$, we have an isomorphism of R -algebras

$$\tilde{\theta} : E_M^{\mathcal{C}} \rightarrow E_L^{\mathcal{C}}, f \mapsto [\theta^{-1} \circ f \circ \theta].$$

The result follows then from the fact that for every fully invariant \mathcal{C} -subcomodule $0 \neq H \subseteq L$ (respectively $0 \neq K \subseteq M$), $\tilde{\theta}$ induces an isomorphism of R -algebras

$$E_M^{\mathcal{C}}/\text{An}(\theta(H)) \simeq E_L^{\mathcal{C}}/\text{An}(H) \quad (\text{respectively } E_M^{\mathcal{C}}/\text{An}(\theta^{-1}(K)) \simeq E_M^{\mathcal{C}}/\text{An}(K)). \blacksquare$$

Remark 3.10. Let L be a non-zero right \mathcal{C} -comodule and $\theta : L \rightarrow M$ be a \mathcal{C} -colinear map. If θ is not bijective, then it is NOT evident that we have the correspondences (6).

Despite Remark 3.10 we have

Proposition 3.11. *Let M be self-injective and $0 \neq L \subseteq M$ be a fully invariant non-zero \mathcal{C} -subcomodule. Then*

$$\begin{aligned} \mathcal{C}_{f.i.}(L) \cap \text{EP}(M) &= \text{EP}(L) & ; & \quad \mathcal{C}_{f.i.}(L) \cap \text{CEP}(M) &= \text{CEP}(L) \\ \mathcal{C}_{f.i.}(L) \cap \text{ESP}(M) &= \text{ESP}(L) & ; & \quad \mathcal{C}_{f.i.}(L) \cap \text{CESP}(M) &= \text{CESP}(L). \end{aligned}$$

Proof. Assume M to be self-injective (so that L is also self-injective). Let $0 \neq K \subseteq L$ be an arbitrary non-zero fully invariant \mathcal{C} -subcomodule (so that $K \subseteq M$ is also fully invariant). The result follows then directly from the definitions and the canonical isomorphisms of R -algebras

$$E_M^{\mathcal{C}}/\text{An}_{E_M^{\mathcal{C}}}(K) \simeq E_K^{\mathcal{C}} \simeq E_L^{\mathcal{C}}/\text{An}_{E_L^{\mathcal{C}}}(K). \blacksquare$$

Corollary 3.12. *Let M be self-injective and $0 \neq L \subseteq M$ be a non-zero fully invariant \mathcal{C} -subcomodule. Then $L \in \text{EP}(M)$ (respectively $L \in \text{ESP}(M)$, $L \in \text{CEP}(M)$, $L \in \text{CESP}(M)$) if and only if L is E -prime (respectively E -semiprime, completely E -prime, completely E -semiprime).*

Sufficient and necessary conditions

Given a fully invariant non-zero \mathcal{C} -subcomodule $K \subseteq M$, we give sufficient and necessary conditions for $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ to be prime (respectively semiprime, completely prime, completely semiprime). These generalize the conditions given in [YDZ90] for the dual algebras of a coalgebra over a base field to be prime (respectively semiprime, domain).

Proposition 3.13. *Let $0 \neq K \subseteq M$ be a non-zero fully invariant \mathcal{C} -subcomodule. A sufficient condition for K to be in $\text{EP}(M)$ is that $K = KfE_M^{\mathcal{C}} \forall f \in E_M^{\mathcal{C}} \setminus \text{An}(K)$, where the later is also necessary in case M is self-cogenerator (or M is self-injective and K is coretractable).*

Proof. Let $I, J \triangleleft E_M^{\mathcal{C}}$ with $IJ \subseteq \text{An}(K)$. Suppose $I \not\subseteq \text{An}(K)$ and pick some $f \in I \setminus \text{An}(K)$. By assumption $K = KfE_M^{\mathcal{C}}$ and it follows then that $KJ = (KfE_M^{\mathcal{C}})J \subseteq K(IJ) = 0$, i.e. $J \subseteq \text{An}(K)$.

On the otherhand, assume M is self-cogenerator (or M is self-injective and K is coretractable). Suppose there exists some $f \in E_M^{\mathcal{C}} \setminus \text{An}(K)$, such that $H := KfE_M^{\mathcal{C}} \subsetneq K \neq 0$. Then obviously $(E_M^{\mathcal{C}}fE_M^{\mathcal{C}})\text{An}(H) \subseteq \text{An}(K)$, whereas our assumptions and Remarks 2.16 (3) & (4) imply that $E_M^{\mathcal{C}}fE_M^{\mathcal{C}} \not\subseteq \text{An}(K)$ and $\text{An}(H) \not\subseteq \text{An}(K)$ (i.e. $\text{An}(K)$ is not prime). \blacksquare

Proposition 3.14. *Let $0 \neq K \subseteq M$ be a non-zero fully invariant \mathcal{C} -subcomodule. A sufficient condition for K to be in $\text{ESP}(M)$ is that $Kf = KfE_M^{\mathcal{C}}f \forall f \in E_M^{\mathcal{C}} \setminus \text{An}(K)$, where the later is also necessary in case M is self-cogenerator.*

Proof. Let $I^2 \subseteq \text{An}(K)$ for some $I \triangleleft E_M^{\mathcal{C}}$. Suppose $I \not\subseteq \text{An}(K)$ and pick some $f \in I \setminus \text{An}(K)$. Then $0 \neq Kf \neq KfE_M^{\mathcal{C}}f \subseteq KI^2 = 0$, a contradiction. So $I \subseteq \text{An}(K)$.

On the otherhand, assume that M is self-cogenerator. Suppose there exists some $f \in E_M^{\mathcal{C}} \setminus \text{An}(K)$ with $KfE_M^{\mathcal{C}}f \subsetneq Kf \neq 0$. By assumptions and Remark 2.16 (3), there exists some $g \in \text{An}(KfE_M^{\mathcal{C}}f) \setminus \text{An}(Kf)$ and it follows then that $J := E_M^{\mathcal{C}}(fg)E_M^{\mathcal{C}} \not\subseteq \text{An}(K)$ while $J^2 \subseteq \text{An}(K)$ (i.e. $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is not semiprime).■

Proposition 3.15. *Let $0 \neq K \subseteq M$ be a non-zero fully invariant \mathcal{C} -subcomodule. A sufficient condition for K to be in $\text{CEP}(M)$ is that $K = Kf \forall f \in E_M^{\mathcal{C}} \setminus \text{An}(K)$, where the later is also necessary in case M is self-cogenerator (or M is self-injective and K is coretractable).*

Proof. 1. Let $fg \in \text{An}(K)$ for some $f, g \in E_M^{\mathcal{C}}$ and suppose $f \notin \text{An}(K)$. The assumption $K = Kf$ implies then that $Kg = (Kf)g = K(fg) = 0$, i.e. $g \in \text{An}(K)$.

On the otherhand, assume M is self-cogenerator (or M is self-injective and K is coretractable). Suppose $Kf \subsetneq K \neq 0$ for some $f \in E_M^{\mathcal{C}} \setminus \text{An}(K)$. By assumptions and Remarks 2.16 (3) & (4) there exists some $g \in \text{An}(Kf) \setminus \text{An}(K)$ with $fg \in \text{An}(K)$ (i.e. $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is not completely prime).■

Proposition 3.16. *Let $0 \neq K \subseteq M$ be a non-zero fully invariant \mathcal{C} -subcomodule. A sufficient condition for K to be in $\text{CESP}(M)$ is that $Kf = Kf^2$ for every $f \in E_M^{\mathcal{C}} \setminus \text{An}(K)$, where the later is also necessary in case M is self-cogenerator.*

Proof. Let $f \in E_M^{\mathcal{C}}$ be such that $f^2 \in \text{An}(K)$. The assumption $K = Kf$ implies then that $Kf = Kf^2 = 0$, i.e. $f \in \text{An}(K)$. On the otherhand, assume M is self-cogenerator. Suppose that $Kf^2 \subsetneq Kf \neq 0$ for some $f \in E_M^{\mathcal{C}} \setminus \text{An}(K)$. By assumptions and Remark 2.16 (3), there exists some $g \in \text{An}(Kf^2) \setminus \text{An}(Kf)$. Set

$$h := \begin{cases} fgf, & \text{in case } fgf \notin \text{An}(K); \\ fg, & \text{otherwise.} \end{cases}$$

So $h^2 \in \text{An}(K)$ while $h \notin \text{An}(K)$ (i.e. $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is not completely semiprime).■

The proof of the following result can be obtained directly from the proofs of the previous four propositions by replacing K with M .

Theorem 3.17. 1. *M is (completely) E-prime, if $M = MfE_M^{\mathcal{C}}$ ($M = Mf$) for every $0 \neq f \in E_M^{\mathcal{C}}$. If M is coretractable, then M is (completely) E-prime if and only if $M = MfE_M^{\mathcal{C}}$ ($M = Mf$) for every $0 \neq f \in E_M^{\mathcal{C}}$.*

2. *M is (completely) E-semiprime, if $Mf = MfE_M^{\mathcal{C}}f$ ($Mf = Mf^2$) for every $0 \neq f \in E_M^{\mathcal{C}}$. If M is self-cogenerator, then M is (completely) E-semiprime if and only if $Mf = MfE_M^{\mathcal{C}}f$ ($Mf = Mf^2$) for every $0 \neq f \in E_M^{\mathcal{C}}$.*

E-Prime versus simple

In what follows we show that E-prime comodules generalize simple comodules.

Theorem 3.18. *A sufficient condition for $E_M^{\mathcal{C}}$ to be right simple (a division ring) is that M is simple, where the later is also necessary in case M is self-cogenerator.*

Proof. If M is simple, then $E_M^{\mathcal{C}} := \text{End}^{\mathcal{C}}(M)^{op}$ is a Division ring by Schur's Lemma.

On the otherhand, assume M to be self-cogenerator. Let $K \subseteq M$ be a \mathcal{C} -subcomodule and consider the right ideal $\text{An}(K) \triangleleft_r E_M^{\mathcal{C}}$. If $E_M^{\mathcal{C}}$ is right simple, then $\text{An}(K) = (0_{E_M^{\mathcal{C}}})$ so that $K = \text{KeAn}(K) = \text{Ke}(0_{E_M^{\mathcal{C}}}) = M$; or $\text{An}(K) = E_M^{\mathcal{C}}$ so that $K = \text{KeAn}(K) = \text{Ke}(E_M^{\mathcal{C}}) = (0_M)$. Consequently M is simple. ■

Theorem 3.19. *A sufficient condition for $E_M^{\mathcal{C}}$ to be simple, in case M is intrinsically injective and $E_M^{\mathcal{C}}$ is right Noetherian, is to assume that M has no non-trivial fully invariant \mathcal{C} -subcomodules, where the later is also necessary if M is self-cogenerator.*

Proof. The proof is similar to that of Theorem 3.18 replacing right ideals of $E_M^{\mathcal{C}}$ by two-sided ideals and arbitrary \mathcal{C} -subcomodules of M with fully invariant ones. ■

Notation. Consider the non-zero right \mathcal{C} -comodule M . With $\mathcal{S}(M)$ ($\mathcal{S}_{f.i.}(M)$) we denote the class of simple \mathcal{C} -subcomodules of M (fully invariant \mathcal{C} -subcomodules of M with no non-trivial fully invariant \mathcal{C} -subcomodules).

Corollary 3.20. *Let M be self-injective self-cogenerator and $0 \neq K \subseteq M$ be a fully invariant non-zero \mathcal{C} -subcomodule. Then*

1. $K \in \mathcal{S}(M) \Leftrightarrow \text{An}(K) \in \text{Max}_r(E_M^{\mathcal{C}})$;
2. If $E_M^{\mathcal{C}}$ is right Noetherian, then $K \in \mathcal{S}_{f.i.}(M) \Leftrightarrow \text{An}(K) \in \text{Max}(E_M^{\mathcal{C}})$.

Proof. Recall that, since M is self-injective self-cogenerator and $K \subseteq M$ is fully invariant, K is also self-injective self-cogenerator. The result follows then from Theorems 3.18 and 3.19 applied to the R -algebra $E_K^{\mathcal{C}} \simeq E_M^{\mathcal{C}}/\text{An}(K)$. ■

Lemma 3.21. *Let M be intrinsically injective self-cogenerator and assume $E_M^{\mathcal{C}}$ to be right Noetherian. Then the order reversing mappings (2) give a bijection*

$$\mathcal{S}(M) \longleftrightarrow \text{Max}_r(E_M^{\mathcal{C}}) \text{ and } \mathcal{S}_{f.i.}(M) \longleftrightarrow \text{Max}(E_M^{\mathcal{C}}). \quad (8)$$

Proof. Let $K \in \mathcal{S}(M)$ ($K \in \mathcal{S}_{f.i.}(M)$) and consider the proper right ideal $\text{An}(K) \subsetneq E_M^{\mathcal{C}}$. If $\text{An}(K) \subseteq I \subseteq E_M^{\mathcal{C}}$, for some right (two-sided) ideal $I \subseteq E_M^{\mathcal{C}}$, then $\text{Ke}(I) \subseteq \text{KeAn}(K) = K$ and it follows from the assumption $K \in \mathcal{S}(M)$ ($K \in \mathcal{S}_{f.i.}(M)$) that $\text{Ke}(I) = 0$ so that $I = \text{AnKe}(I) = E_M^{\mathcal{C}}$; or $\text{Ke}(I) = K$ so that $I = \text{AnKe}(I) = \text{An}(K)$. This means that $\text{An}(K) \in \text{Max}_r(E_M^{\mathcal{C}})$ ($\text{An}(K) \in \text{Max}(E_M^{\mathcal{C}})$).

On the otherhand, let $I \in \text{Max}_r(E_M^{\mathcal{C}})$ ($I \in \text{Max}(E_M^{\mathcal{C}})$) and consider the non-zero \mathcal{C} -subcomodule $0 \neq \text{Ke}(I) \subseteq M$. If $K \subseteq \text{Ke}(I)$ for some (fully invariant) \mathcal{C} -subcomodule $K \subseteq M$, then $I \subseteq \text{AnKe}(I) \subseteq \text{An}(K) \subseteq E_M^{\mathcal{C}}$ and it follows by the maximality of I that $\text{An}(K) = E_M^{\mathcal{C}}$ so that $K = \text{KeAn}(K) = 0$; or $\text{An}(K) = I$ so that $K = \text{KeAn}(K) = \text{Ke}(I)$. Consequently $\text{Ke}(I) \in \mathcal{S}(M)$ ($K \in \mathcal{S}_{f.i.}(M)$). Since M is intrinsically injective self-cogenerator, $\text{Ke}(-)$ and $\text{An}(-)$ are injective by 2.15 and we are done. ■

Lemma 3.22. *Let A be left perfect and ${}_A\mathcal{C}$ be locally projective.*

1. *The non-zero right \mathcal{C} -comodule contains a simple \mathcal{C} -subcomodule.*
2. *$\text{Soc}(M) \triangleleft_e M$ (an essential \mathcal{C} -subcomodule).*

Proof. Let ${}_A A$ be perfect and ${}_A\mathcal{C}$ be locally projective.

1. By [Abu03, Corollary 2.25] M satisfies the descending chain condition on finitely generated non-zero \mathcal{C} -subcomodules, which turn out to be finitely generated right A -modules, hence M contains a non-zero simple \mathcal{C} -subcomodule.
2. Let M be a non-zero right \mathcal{C} -comodule. For every \mathcal{C} -subcomodule $0 \neq K \subseteq M$ we have $K \cap \text{Soc}(M) = \text{Soc}(K) \neq 0$, by (1). ■

Proposition 3.23. *We have*

$$\text{Jac}(E_M^{\mathcal{C}}) = \text{An}(\text{Soc}(M)) \text{ and } \text{Soc}(M) = \text{Ke}(\text{Jac}(E_M^{\mathcal{C}})), \quad (9)$$

if any of the following conditions holds:

1. *M is intrinsically injective self-cogenerator with $E_M^{\mathcal{C}}$ right Noetherian;*
2. *${}_A\mathcal{C}$ is locally projective and M is Artinian self-injective self cogenerator;*
3. *A is left perfect, ${}_A\mathcal{C}$ is locally projective and M is self-injective self-cogenerator.*

Proof. 1. By Lemma 3.21 we have

$$\begin{aligned} \text{Jac}(E_M^{\mathcal{C}}) &= \bigcap \{Q \mid Q \triangleleft_r E_M^{\mathcal{C}} \text{ is a maximal right ideal}\} \\ &= \bigcap \{\text{AnKe}(Q) \mid Q \triangleleft_r E_M^{\mathcal{C}} \text{ is a maximal right ideal}\} \\ &= \bigcap \{\text{AnKe}(\text{An}(K)) \mid K \subseteq M \text{ is a simple } \mathcal{C}\text{-subcomodule}\} \\ &= \bigcap \{\text{An}(K) \mid K \subseteq M \text{ is a simple } \mathcal{C}\text{-subcomodule}\} \\ &= \text{An}(\sum \{K \mid K \subseteq M \text{ is a simple } \mathcal{C}\text{-subcomodule}\}) \\ &= \text{An}(\text{Soc}(M)). \end{aligned}$$

Since M is self-cogenerator, we have $\text{Soc}(M) = \text{KeAn}(\text{Soc}(M)) = \text{Ke}(\text{Jac}(E_M^{\mathcal{C}}))$.

2. Since M is Artinian and self-injective in $\mathbb{M}^{\mathcal{C}} = \sigma[{}_{*}\mathcal{C}]$, we conclude that $E_M^{\mathcal{C}} := \text{End}^{\mathcal{C}}(M)^{op} = \text{End}({}_{*}\mathcal{C}M)$ is right Noetherian by Proposition 2.14 (2). The result follows then by (1).
3. Since A is left perfect and ${}_A\mathcal{C}$ is locally projective, $\text{Soc}(M) \triangleleft_e M$ is an essential \mathcal{C} -subcomodule by Lemma 3.22 (2) and it follows then, since M is self-injective, that

$$\begin{aligned} \text{Jac}(E_M^{\mathcal{C}}) &= \text{Jac}(\text{End}({}_{*}\mathcal{C}M)^{op}) &= \text{Hom}_{*}\mathcal{C}(M/\text{Soc}(M), M) & \text{ ([Wis91, 22.1 (5))]} \\ &= \text{Hom}^{\mathcal{C}}(M/\text{Soc}(M), M) &\simeq \text{An}(\text{Soc}(M)). \end{aligned}$$

Since M is self-cogenerator, we have moreover

$$\text{Soc}(M) = \text{KeAn}(\text{Soc}(M)) = \text{Ke}(\text{Jac}(E_M^{\mathcal{C}})). \blacksquare$$

Corollary 3.24. *If any of the three conditions in Proposition 3.23 holds, then we have*

$$M \text{ is semisimple} \Leftrightarrow E_M^{\mathcal{C}} \text{ is semiprimitive.}$$

Proof. By assumptions and Proposition 3.23 we have $\text{Jac}(E_M^{\mathcal{C}}) = \text{An}(\text{Soc}(M))$ and $\text{Soc}(M) = \text{Ke}(\text{Jac}(E_M^{\mathcal{C}}))$. Hence, M semisimple $\Rightarrow \text{Jac}(E_M^{\mathcal{C}}) = \text{An}(\text{Soc}(M)) = \text{An}(M) = 0$, i.e. $E_M^{\mathcal{C}}$ is semiprimitive; on the otherhand $E_M^{\mathcal{C}}$ semiprimitive implies $\text{Soc}(M) = \text{Ke}(\text{Jac}(E_M^{\mathcal{C}})) = \text{Ke}(0) = M$, i.e. M is semisimple. ■

E-Prime versus irreducible

In what follows we clarify the relation between E-prime and irreducible comodules.

Remark 3.25. Let $\{K_\lambda\}_\Lambda$ be a family of non-zero fully invariant \mathcal{C} -subcomodules of M and consider the fully invariant \mathcal{C} -subcomodule $K := \sum_{\lambda \in \Lambda} K_\lambda \subseteq M$. If $K_\lambda \in \text{EP}(M)$ ($K_\lambda \in \text{CEP}(M)$) for every $\lambda \in \Lambda$, then $\text{An}(K) = \bigcap_{\lambda \in \Lambda} \text{An}(K_\lambda)$ is an intersection of (completely) prime ideals, hence a (completely) semiprime ideal, i.e. $K \in \text{ESP}(M)$ ($K \in \text{CESP}(M)$). If M is self-injective, then we conclude that an arbitrary sum of (completely) E-prime \mathcal{C} -subcomodules of M is in general (completely) E-semiprime.

Despite Remark 3.25 we have the following result (which is most interesting in case $K = M$):

Proposition 3.26. *Let $\{K_\lambda\}_\Lambda$ be a family of non-zero fully invariant \mathcal{C} -subcomodules of M , such that for any $\gamma, \delta \in \Lambda$ either $K_\gamma \subseteq K_\delta$ or $K_\delta \subseteq K_\gamma$, and consider the fully invariant \mathcal{C} -subcomodule $K := \sum_{\lambda \in \Lambda} K_\lambda = \bigcup_{\lambda \in \Lambda} K_\lambda \subseteq M$. If $K_\lambda \in \text{EP}(M)$ ($K_\lambda \in \text{CEP}(M)$) for every $\lambda \in \Lambda$, then $K \in \text{EP}(M)$ ($K \in \text{CEP}(M)$).*

Proof. Let $I, J \triangleleft E_M^{\mathcal{C}}$ be such that $IJ \subseteq \text{An}(K) = \bigcap_{\lambda \in \Lambda} \text{An}(K_\lambda)$ and suppose $I \not\subseteq \text{An}(K)$. Pick some $\lambda_0 \in \Lambda$ with $I \not\subseteq \text{An}(K_{\lambda_0})$. By assumption $\text{An}(K_{\lambda_0}) \triangleleft E_M^{\mathcal{C}}$ is prime and $IJ \subseteq \text{An}(K_{\lambda_0})$, so $J \subseteq \text{An}(K_{\lambda_0})$. We **claim** that $J \subseteq \bigcap_{\lambda \in \Lambda} \text{An}(K_\lambda)$: Let $\lambda \in \Lambda$ be arbitrary. If $K_\lambda \subseteq K_{\lambda_0}$, then $J \subseteq \text{An}(K_{\lambda_0}) \subseteq \text{An}(K_\lambda)$. On the other hand, if $K_{\lambda_0} \subseteq K_\lambda$ and $J \not\subseteq \text{An}(K_\lambda)$, then the primeness of $\text{An}(K_\lambda)$ implies that $I \subseteq \text{An}(K_\lambda) \subseteq \text{An}(K_{\lambda_0})$, a contradiction. So $J \subseteq \bigcap_{\lambda \in \Lambda} \text{An}(K_\lambda) = \text{An}(K)$. Consequently $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is a prime ideal, i.e. $K \in \text{EP}(M)$.

In case $\text{An}(K_\lambda) \triangleleft E_M^{\mathcal{C}}$ is completely prime for every $\lambda \in \Lambda$, then replacing ideals in the argument above with elements yields that $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is a completely prime ideal, i.e. $K \in \text{CEP}(M)$. ■

Remark 3.27. If M is self-injective and the subcomodule K_λ in Proposition 3.26 are (completely) E-prime, then K is (completely) E-prime (recall that we have in this case an isomorphism of algebras $E_M^{\mathcal{C}}/\text{Ann}(K_\lambda) \simeq E_{K_\lambda}^{\mathcal{C}}$).

Proposition 3.28. *Let M be self-cogenerator and $K \in \text{EP}(M)$. Then K admits no decomposition as an internal direct sum of non-trivial fully invariant \mathcal{C} -subcomodules.*

Proof. Let $K \subseteq M$ be a fully invariant \mathcal{C} -subcomodule with $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ a prime ideal and suppose $K = K_{\lambda_0} \oplus \sum_{\lambda \neq \lambda_0} K_{\lambda}$ to be a decomposition of K as an internal direct sum of non-trivial fully invariant \mathcal{C} -subcomodules. Consider the two-sided ideals $I := \text{An}(K_{\lambda_0})$, $J := \text{An}(\sum_{\lambda \neq \lambda_0} K_{\lambda})$ of $E_M^{\mathcal{C}}$, so that $IJ \subseteq \text{An}(K)$. If $J \subseteq \text{An}(K)$, then $K_{\lambda_0} \subseteq K = \text{KeAn}(K) \subseteq \text{Ke}(J) = \sum_{\lambda \neq \lambda_0} K_{\lambda}$ (a contradiction). Since $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is prime, $I \subseteq \text{An}(K)$ and we conclude that $K = \text{KeAn}(K) \subseteq \text{Ke}(I) = \text{KeAn}(K_{\lambda_0}) = K_{\lambda_0}$ (a contradiction). ■

Definition 3.29. We call the non-zero right \mathcal{C} -comodule M *irreducible*, iff M has a *unique* simple \mathcal{C} -subcomodule that is contained in every \mathcal{C} -subcomodule of M (equivalently, iff the intersection of all non-zero \mathcal{C} -subcomodules of M is again non-zero).

The following result clarifies, under suitable conditions, the relation between E-prime and irreducible comodules.

Theorem 3.30. *Assume ${}_A\mathcal{C}$ to be locally projective, M to be self-injective self-cogenerator and $\text{End}^{\mathcal{C}}(M)$ to be commutative. If M is E-prime, then M is irreducible.*

Proof. If $\text{End}^{\mathcal{C}}(M = \text{End}(*_{\mathcal{C}}M)$ is commutative, then under the assumptions on M , [Wis91, 48.16] yield that M is a direct sum of irreducible fully invariant \mathcal{C} -subcomodules. The results follows then by Proposition 3.28. ■

4 Fully Coprime (fully cosemiprime) comodules

As before, \mathcal{C} is a non-zero A -coring with ${}_A\mathcal{C}$ flat, M is a non-zero right \mathcal{C} -comodule and $E_M^{\mathcal{C}} := \text{End}^{\mathcal{C}}(M)^{op}$ is the ring of \mathcal{C} -colinear endomorphisms of M with the opposite composition of maps.

4.1. For R -submodules $X, Y \subseteq M$, set

$$(X :_M^{\mathcal{C}} Y) := \bigcap \{f^{-1}(Y) \mid f \in \text{End}^{\mathcal{C}}(M) \text{ and } f(X) = 0\}.$$

If $Y \subseteq M$ is a right \mathcal{C} -subcomodule, then $f^{-1}(Y) \subseteq M$ is a \mathcal{C} -subcomodule for each $f \in E_M^{\mathcal{C}}$, being the kernel of the \mathcal{C} -colinear map $\pi_Y \circ f : M \rightarrow M/Y$, and it follows then that $(X :_M^{\mathcal{C}} Y) \subseteq M$ is a right \mathcal{C} -subcomodule, being the intersection of right \mathcal{C} -subcomodules of M . If $X \subseteq M$ is fully invariant, i.e. $f(X) \subseteq X$ for every $f \in E_M^{\mathcal{C}}$, then $(X :_M^{\mathcal{C}} Y) \subseteq M$ is clearly fully invariant. If $X, Y \subseteq M$ are right \mathcal{C} -subcomodules, then the right \mathcal{C} -subcomodule $(X :_M^{\mathcal{C}} Y)$ is called the *internal coproduct* of X and Y in the category $\mathbb{M}^{\mathcal{C}}$ of right \mathcal{C} -comodules. If \mathcal{C}_A is flat, then the internal coproduct of \mathcal{C} -subcomodules of left \mathcal{C} -comodules can be defined analogously.

Remark 4.2. The *internal coproduct* of submodules of a given module over a ring was first introduced by Bican et. al. [BJKN80] to present the notion of *coprime modules*. The definition was modified in [RRW05], where arbitrary submodules are replaced by the fully invariant ones. To avoid any possible confusion, we refer to coprime modules in the sense of [RRW05] as *fully coprime modules* and transfer that terminology to *fully coprime comodules*.

Definition 4.3. A fully invariant \mathcal{C} -subcomodule $0 \neq K \subseteq M$ will be called

fully M -coprime, iff for any two fully invariant \mathcal{C} -subcomodules $X, Y \subseteq M$ with $K \subseteq (X :_M^{\mathcal{C}} Y)$, we have $K \subseteq X$ or $K \subseteq Y$;

fully M -cosemiprime, iff for any fully invariant \mathcal{C} -subcomodule $X \subseteq M$ with $K \subseteq (X :_M^{\mathcal{C}} X)$, we have $K \subseteq X$.

We call M *fully coprime* (*fully cosemiprime*), iff M is fully M -coprime (fully M -cosemiprime).

The fully coprime coradical

Definition 4.4. We define the *fully coprime spectrum* of M as

$$\text{CPSpec}(M) := \{K \mid 0 \neq K \subseteq M \text{ is a fully } M\text{-coprime } \mathcal{C}\text{-subcomodule}\}.$$

We define the *fully coprime coradical* of M as

$$\text{CPcorad}(M) = \sum_{K \in \text{CPSpec}(M)} K.$$

Moreover, we set

$$\text{CSP}(M) := \{K \mid 0 \neq K \subseteq M \text{ is an fully } M\text{-cosemiprime } \mathcal{C}\text{-subcomodule}\}.$$

The fully coprime spectra (fully coprime coradicals) of comodules are invariant under isomorphisms of comodules:

Proposition 4.5. *Let $\theta : L \rightarrow M$ be an isomorphism of \mathcal{C} -comodules. Then we have bijections*

$$\text{CPSpec}(L) \longleftrightarrow \text{CPSpec}(M) \text{ and } \text{CSP}(L) \longleftrightarrow \text{CSP}(M).$$

In particular

$$\theta(\text{CPcorad}(L)) = \text{CPcorad}(M). \quad (10)$$

Proof. Let $\theta : L \rightarrow M$ be an isomorphism of right \mathcal{C} -comodules. Let $0 \neq H \subseteq L$ be a fully invariant \mathcal{C} -subcomodule that is fully L -coprime and consider the fully invariant \mathcal{C} -subcomodule $0 \neq \theta(H) \subseteq M$. Let $X, Y \subseteq M$ be two fully invariant \mathcal{C} -subcomodules with $\theta(H) \subseteq (X :_M^{\mathcal{C}} Y)$. Then $\theta^{-1}(X), \theta^{-1}(Y) \subseteq L$ are two fully invariant \mathcal{C} -subcomodules and $H \subseteq (\theta^{-1}(X) :_L^{\mathcal{C}} \theta^{-1}(Y))$. By assumption H is fully L -coprime and we conclude that $H \subseteq \theta^{-1}(X)$ so that $\theta(H) \subseteq X$; or $H \subseteq \theta^{-1}(Y)$ so that $\theta(H) \subseteq Y$. Consequently $\theta(H)$ is fully M -coprime. Analogously one can show that for any fully invariant fully M -coprime \mathcal{C} -subcomodule $0 \neq K \subseteq M$, the fully invariant \mathcal{C} -subcomodule $0 \neq \theta^{-1}(K) \subseteq L$ is fully L -coprime.

Repeating the proof above with $Y = X$, one can prove that for any fully L -cosemiprime (fully M -cosemiprime) fully invariant \mathcal{C} -subcomodule $0 \neq H \subseteq L$ (resp. $0 \neq K \subseteq M$), the fully invariant \mathcal{C} -subcomodule $0 \neq \theta(H) \subseteq M$ ($0 \neq \theta^{-1}(K) \subseteq L$) is fully M -cosemiprime (fully L -cosemiprime). ■

Remark 4.6. Let L be a non-zero right \mathcal{C} -comodules and $\theta : L \rightarrow M$ be a \mathcal{C} -colinear map. If θ is not bijective, then it is NOT evident that for $K \in \text{CPSpec}(L)$ (respectively $K \in \text{CSP}(L)$) we have $\theta(K) \subseteq \text{CPSpec}(M)$ (respectively $\theta(K) \in \text{CSP}(M)$).

Despite Remark 4.6 we have

Proposition 4.7. *Let $0 \neq L \subseteq M$ be a non-zero fully invariant \mathcal{C} -subcomodule. Then we have*

$$\mathcal{M}_{f.i.}(L) \cap \text{CPSpec}(M) \subseteq \text{CPSpec}(L) \text{ and } \mathcal{M}_{f.i.}(L) \cap \text{CSP}(M) \subseteq \text{CSP}(L), \quad (11)$$

with equality in case M is self injective.

Proof. Let $0 \neq H \subseteq L$ be a fully invariant \mathcal{C} -subcomodule and assume H to be fully M -coprime (fully M -cosemiprime). Suppose $H \subseteq (X :_L^{\mathcal{C}} Y)$ for two (equal) fully invariant \mathcal{C} -subcomodules $X, Y \subseteq L$. Since $L \subseteq M$ is a fully invariant \mathcal{C} -subcomodule, it follows that X, Y are also fully invariant \mathcal{C} -subcomodules of M and moreover $(X :_L^{\mathcal{C}} Y) \subseteq (X :_M^{\mathcal{C}} Y)$. By assumption H is fully M -coprime (fully M -cosemiprime), and so the inclusions $H \subseteq (X :_L^{\mathcal{C}} Y) \subseteq (X :_M^{\mathcal{C}} Y)$ imply $H \subseteq X$ or $H \subseteq Y$. Consequently H is fully L -coprime (fully L -cosemiprime). Hence the inclusions in (11) hold.

Assume now that M is self-injective. Let $0 \neq H \subseteq L$ to be an fully L -coprime (fully L -cosemiprime) \mathcal{C} -subcomodule. Suppose $X, Y \subseteq M$ are two (equal) fully invariant \mathcal{C} -subcomodules with $H \subseteq (X :_M^{\mathcal{C}} Y)$ and consider the fully invariant \mathcal{C} -subcomodules $X \cap L, Y \cap L \subseteq L$. Since M is self-injective, the embedding $\iota : L/X \cap L \hookrightarrow M/X$ induces a surjective set map

$$\Phi : \text{Hom}^{\mathcal{C}}(M/X, M) \rightarrow \text{Hom}^{\mathcal{C}}(L/X \cap L, M), f \mapsto f|_{L/X \cap L}.$$

Since $L \subseteq M$ is fully invariant, Φ induces a surjective set map

$$\Psi : \text{An}_{\text{E}_M^{\mathcal{C}}}(X) \rightarrow \text{An}_{\text{E}_L^{\mathcal{C}}}(X \cap L), g \mapsto g|_L, \quad (12)$$

which implies that $H \subseteq (X \cap L :_L^{\mathcal{C}} Y \cap L)$. By assumption H is fully L -coprime, hence $H \subseteq X \cap L$ so that $H \subseteq X$; or $H \subseteq Y \cap L$ so that $H \subseteq Y$. Hence H is fully M -coprime (fully M -cosemiprime). Consequently the inclusions in (11) become equality. ■

Remark 4.8. Let $0 \neq L \subseteq M$ be a non-zero fully invariant \mathcal{C} -subcomodule. By Proposition 4.7, a sufficient condition for L to be fully coprime (fully cosemiprime) is that L is fully M -coprime (fully M -cosemiprime), where the later is also necessary in case M is self-injective.

Lemma 4.9. *Let $X, Y \subseteq M$ be any R -submodules. Then*

$$(X :_M^{\mathcal{C}} Y) \subseteq \text{Ke}(\text{An}(X) \circ^{op} \text{An}(Y)), \quad (13)$$

with equality in case M is self-cogenerator and $Y \subseteq M$ is a \mathcal{C} -subcomodule.

Proof. Let $m \in (X :_{\mathcal{C}}^M Y)$ be arbitrary. Then for all $f \in \text{An}(X)$ we have $f(m) = y$ for some $y \in Y$ and so for each $g \in \text{An}(Y)$ we get

$$(f \circ^{op} g)(m) = (g \circ f)(m) = g(f(m)) = g(y) = 0,$$

i.e. $(X :_{\mathcal{C}}^M Y) \subseteq \text{Ke}(\text{An}(X) \circ^{op} \text{An}(Y))$.

Assume now that M is self-cogenerator and that $Y \subseteq M$ is a \mathcal{C} -subcomodule (so that $\text{KeAn}(Y) = Y$ by 2.15 (2)). If $m \in \text{Ke}(\text{An}(X) \circ^{op} \text{An}(Y))$ and $f \in \text{An}(X)$ are arbitrary, then by our choice

$$g(f(m)) = (f \circ^{op} g)(m) = 0 \text{ for all } g \in \text{An}(Y),$$

so $f(m) \in \text{KeAn}(Y) = Y$, i.e. $m \in (X :_{\mathcal{C}}^M Y)$. Hence, $(X :_{\mathcal{C}}^M Y) = \text{Ke}(\text{An}(X) \circ^{op} \text{An}(Y))$. ■

Proposition 4.10. *Let M be self-cogenerator. Then*

$$\text{EP}(M) \subseteq \text{CPSpec}(M) \text{ and } \text{ESP}(M) \subseteq \text{CSP}(M)$$

with equality, if M is intrinsically injective self-cogenerator (whence $\text{EPcorad}(M) = \text{CPcorad}(M)$).

Proof. Assume M to be self-cogenerator. Let $0 \neq K \subseteq M$ be a fully invariant \mathcal{C} -subcomodule that is E-prime (E-semiprime) in M , and suppose $X, Y \subseteq M$ are two (equal) fully invariant \mathcal{C} -subcomodules with $K \subseteq (X :_{\mathcal{C}}^M Y)$. Then we have by Lemma 4.9 (1)

$$\text{An}(X) \circ^{op} \text{An}(Y) \subseteq \text{AnKe}(\text{An}(X) \circ^{op} \text{An}(Y)) \subseteq \text{An}(X :_{\mathcal{C}}^M Y) \subseteq \text{An}(K).$$

By assumption $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is prime (semiprime), hence $\text{An}(X) \subseteq \text{An}(K)$, so that $K = \text{KeAn}(K) \subseteq \text{KeAn}(X) = X$; or $\text{An}(Y) \subseteq \text{An}(K)$ so that $K = \text{KeAn}(K) \subseteq \text{KeAn}(Y) = Y$. Consequently K is fully M -coprime (fully M -cosemiprime).

Assume now that M is intrinsically injective self-cogenerator. Let $0 \neq K \subseteq M$ be an fully M -coprime (fully M -cosemiprime) \mathcal{C} -subcomodule and consider the proper two-sided ideal $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$. Suppose $I, J \triangleleft E_M^{\mathcal{C}}$ are two (equal) ideals with $I \circ^{op} J \subseteq \text{An}(K)$ and $I_{E_M^{\mathcal{C}}}, J_{E_M^{\mathcal{C}}}$ are *finitely generated*. Consider the fully invariant \mathcal{C} -subcomodules $X := \text{Ke}(I)$, $Y := \text{Ke}(J)$ of M . Since M is self-cogenerator, it follows by Lemma 4.9 that

$$K = \text{KeAn}(K) \subseteq \text{Ke}(I \circ^{op} J) = \text{Ke}(\text{An}(X) \circ^{op} \text{An}(Y)) = (X :_{\mathcal{C}}^M Y).$$

Since K is fully M -coprime (fully M -cosemiprime), we conclude that $K \subseteq X$ so that $I = \text{AnKe}(I) = \text{An}(X) \subseteq \text{An}(K)$; or $K \subseteq Y$ so that $J = \text{AnKe}(J) = \text{An}(Y) \subseteq \text{An}(K)$. Consequently $\text{An}(K) \triangleleft E_M^{\mathcal{C}}$ is prime (semiprime), i.e. K is E-prime (E-semiprime) in M . ■

Remark 4.11. It follows from Proposition 4.10 that a sufficient condition for M to be fully coprime (fully cosemiprime), in case M is self-cogenerator, is that M is E-prime (E-semiprime), where the later is also necessary in case M is intrinsically injective self-cogenerator.

As a direct consequence of Propositions 3.6, 4.10 we have

Proposition 4.12. *Let M be intrinsically injective self-cogenerator and $E_M^{\mathcal{C}}$ be right Noetherian. Then*

$$\text{Prad}(E_M^{\mathcal{C}}) = \text{An}(\text{CPcorad}(M)) \text{ and } \text{CPcorad}(M) = \text{Ke}(\text{Prad}(E_M^{\mathcal{C}})). \quad (14)$$

Using Proposition 4.12, a similar proof to that of Corollary 3.7 yields:

Corollary 4.13. *Let M be intrinsically injective self-cogenerator and $E_M^{\mathcal{C}}$ be right Noetherian. Then*

$$M \text{ is fully cosemiprime} \Leftrightarrow M = \text{CPcorad}(M).$$

Corollary 4.14. *Let ${}_A\mathcal{C}$ be locally projective and M be self injective self-cogenerator. If M is Artinian (e.g. A is right Artinian and M is finitely generated), then*

1. $\text{Prad}(E_M^{\mathcal{C}}) = \text{An}(\text{CPcorad}(M))$ and $\text{CPcorad}(M) = \text{Ke}(\text{Prad}(E_M^{\mathcal{C}}))$.
2. M is fully cosemiprime $\Leftrightarrow M = \text{CPcorad}(M)$.

Comodules with rings of colinear endomorphisms right Artinian

Under the *strong assumption* $E_M^{\mathcal{C}} := \text{End}^{\mathcal{C}}(M)^{op}$ is right Artinian, several primeness and coprimeness properties of the non-zero right \mathcal{C} -comodule M coincide and become, in case ${}_A\mathcal{C}$ locally projective, equivalent to M being simple as a $({}^*\mathcal{C}, E_M^{\mathcal{C}})$ -bimodule. This follows from the fact that right Artinian prime rings are simple.

If M has no non-trivial fully invariant \mathcal{C} -subcomodules, then it is obviously fully coprime. The following result gives a partial converse:

Theorem 4.15. *Let M be intrinsically injective self-cogenerator and assume $E_M^{\mathcal{C}}$ to be right Artinian. Then the following are equivalent:*

1. M is E-prime (i.e. $E_M^{\mathcal{C}}$ is a prime ring);
2. $E_M^{\mathcal{C}}$ is simple;
3. M has no non-trivial fully invariant \mathcal{C} -subcomodules;
4. M is fully coprime.

Proof. Let M be intrinsically injective self-cogenerator and assume $E_M^{\mathcal{C}}$ to be right Artinian.

- (1) \Rightarrow (2) : Right Artinian prime rings are simple (e.g. [Wis91, 4.5 (2)]).
- (2) \Rightarrow (3) : Since M is self-cogenerator, this follows by Theorem 3.19.
- (3) \Rightarrow (4) : Trivial.
- (4) \Rightarrow (1) : Since M be intrinsically injective self-cogenerator, this follows by 4.11. ■

Proposition 4.16. *Let ${}_A\mathcal{C}$ be locally projective and M be self-injective self-cogenerator. If any of the following additional conditions is satisfied, then M is fully coprime if and only if M is simple as a $({}^*\mathcal{C}, E_M^{\mathcal{C}})$ -bimodule:*

1. M has finite length; or
2. A is right Artinian and M_A is finitely generated; or
3. M is Artinian and self-projective.

Proof. By Theorem 4.15, it suffices to show that $E_M^{\mathcal{C}} = \text{End}({}_* \mathcal{C}M)^{op}$ is right Artinian under each of the additional conditions.

1. By assumption M is self-injective and Artinian (semi-injective and Noetherian) and it follows then by Proposition 2.14 that $E_M^{\mathcal{C}}$ is right Noetherian (semiprimary). Applying Hopkins Theorem (e.g. [Wis91, 31.4]), we conclude that $E_M^{\mathcal{C}}$ is right Artinian.
2. If A is right Artinian and ${}_A \mathcal{C}$ is locally projective, then every finitely generated right \mathcal{C} -comodule has finite length by [Abu03, Corollary 2.25].
3. Since M is Artinian, self-injective and self-projective, $E_M^{\mathcal{C}}$ is right Artinian by Proposition 2.14 (2). ■

Fully coprimeness versus irreducibility

In what follows we clarify, under suitable conditions, the relation between fully coprime and irreducible comodules:

Proposition 4.17. *Let $\{K_\lambda\}_\Lambda$ be a family of non-zero fully invariant \mathcal{C} -subcomodules of M , such that for any $\gamma, \delta \in \Lambda$ either $K_\gamma \subseteq K_\delta$ or $K_\delta \subseteq K_\gamma$, and consider the fully invariant \mathcal{C} -subcomodule $K := \sum_{\lambda \in \Lambda} K_\lambda = \bigcup_{\lambda \in \Lambda} K_\lambda \subseteq M$. If $K_\lambda \in \text{CPSpec}(M)$ for all $\lambda \in \Lambda$, then $K \in \text{CPSpec}(M)$.*

Proof. Let $X, Y \subseteq M$ be any fully invariant \mathcal{C} -subcomodules with $K \subseteq (X :_M^{\mathcal{C}} Y)$ and suppose $K \not\subseteq X$. We **claim** that $K \subseteq Y$.

Since $K \not\subseteq X$, there exists some $\lambda_0 \in \Lambda$ with $K_{\lambda_0} \not\subseteq X$. Since $K_{\lambda_0} \subseteq (X :_M^{\mathcal{C}} Y)$, it follows from the assumption $K_{\lambda_0} \in \text{CPSpec}(M)$ that $K_{\lambda_0} \subseteq Y$. Let $\lambda \in \Lambda$ be arbitrary. If $K_\lambda \subseteq K_{\lambda_0}$, then $K_\lambda \subseteq Y$. If otherwise $K_{\lambda_0} \subseteq K_\lambda$, then the inclusion $K_\lambda \subseteq (X :_M^{\mathcal{C}} Y)$ implies $K_\lambda \subseteq Y$ (since $K_\lambda \subseteq X$ would imply $K_{\lambda_0} \subseteq X$, a contradiction). So $K := \bigcup_{\lambda \in \Lambda} K_\lambda \subseteq Y$. ■

Corollary 4.18. *Let $M = \sum_{\lambda \in \Lambda} M_\lambda$, where $\{M_\lambda\}_\Lambda$ is a family of non-zero fully invariant \mathcal{C} -subcomodules of M such that for any $\gamma, \delta \in \Lambda$ either $M_\gamma \subseteq M_\delta$ or $M_\delta \subseteq M_\gamma$. If $M_\lambda \in \text{CPSpec}(M)$ for each $\lambda \in \Lambda$, then M is fully coprime.*

Proposition 4.19. *Let $0 \neq K \subseteq M$ be a non-zero fully invariant \mathcal{C} -subcomodule. If $K \in \text{CPSpec}(M)$, then K has no decomposition as an internal direct sum of non-trivial fully invariant \mathcal{C} -subcomodules.*

Proof. Let $K \in \text{CPSpec}(M)$ and suppose $K := K_{\lambda_0} \oplus \sum_{\lambda \neq \lambda_0} K_\lambda$, an internal direct sum of non-trivial fully invariant \mathcal{C} -subcomodules. Then $K \subseteq (K_{\lambda_0} :_M^{\mathcal{C}} \sum_{\lambda \neq \lambda_0} K_\lambda)$ and it follows that $K \subseteq K_{\lambda_0}$ or $K \subseteq \sum_{\lambda \neq \lambda_0} K_\lambda$ (contradiction). ■

Corollary 4.20. *If M is fully coprime, then M has no decomposition as an internal direct sum of non-trivial fully invariant \mathcal{C} -subcomodules.*

As a direct consequence of Corollary 4.20 we get a restatement of Theorem 3.30:

Theorem 4.21. *Let ${}_A \mathcal{C}$ locally projective and M be self-injective self-cogenerator with $\text{End}^{\mathcal{C}}(M)$ commutative. If M is fully coprime, then M is irreducible.*

5 Primeness and Coprimeness Conditions for Corings

Throughout this section $(\mathcal{C}, \Delta, \varepsilon)$ is a non-zero coring. We consider in what follows several coprimeness (cosemiprimeness) and primeness (semiprimeness) properties of \mathcal{C} , considered as an object in the category $\mathbb{M}^{\mathcal{C}}$ of right \mathcal{C} -comodules, denoted by \mathcal{C}^r , as well as an object in the category ${}^{\mathcal{C}}\mathbb{M}$ of left \mathcal{C} -comodules, denoted by \mathcal{C}^l . In particular, we clarify the relation between these properties and the simplicity (semisimplicity) of \mathcal{C} . Several results in this section can be obtained directly from the corresponding ones in the previous sections. Moreover, we state many of these in the case A is a QF ring, as in this case \mathcal{C} is an injective cogenerator in both the categories of right and left \mathcal{C} -comodules by Lemma 2.9.

5.1. (e.g. [BW03, 17.8.]) We have an isomorphism of R -algebras

$$\phi_r : \mathcal{C}^* \rightarrow \text{End}^{\mathcal{C}}(\mathcal{C})^{op}, \quad f \mapsto [c \mapsto c \leftarrow f := \sum f(c_1)c_2]$$

with inverse map $\psi_r : g \mapsto \varepsilon \circ g$, and there is a ring morphism $\iota_r : A \rightarrow (\mathcal{C}^*)^{op}$, $a \mapsto \varepsilon(a-)$.

Similarly, we have an isomorphism of R -algebras

$$\phi_l : {}^*\mathcal{C} \rightarrow {}^{\mathcal{C}}\text{End}(\mathcal{C}), \quad f \mapsto [c \mapsto f \rightarrow c := \sum c_1 f(c_2)]$$

with inverse map $\psi_l : g \mapsto \varepsilon \circ g$, and there is a ring morphism $\iota_l : A \rightarrow ({}^*\mathcal{C})^{op}$, $a \mapsto \varepsilon(-a)$.

Definition 5.2. 1. We call a right (left) A -submodule $K \subseteq \mathcal{C}$ a *right (left) \mathcal{C} -coideal*, iff K is a right (left) \mathcal{C} -subcomodule of \mathcal{C} with structure map the restriction of $\Delta_{\mathcal{C}}$ to K .

2. We call an (A, A) -subbimodule $B \subseteq \mathcal{C}$ a *\mathcal{C} -bicoideal*, iff B is a \mathcal{C} -subbicomodule of \mathcal{C} with structure map the restriction of $\Delta_{\mathcal{C}}$ to B ;

3. We call an (A, A) -subbimodule $\mathcal{D} \subseteq \mathcal{C}$ an *A -subcoring*, iff \mathcal{D} is an A -coring with structure maps the restrictions of $\Delta_{\mathcal{C}}$ and $\varepsilon_{\mathcal{C}}$ to \mathcal{D} .

Notation. With $\mathcal{R}(\mathcal{C})$ ($\mathcal{R}_{f.i.}(\mathcal{C})$) we denote the class of (fully invariant) right \mathcal{C} -coideals and with $\mathcal{I}_r(\mathcal{C}^*)$ ($\mathcal{I}_{t.s.}(\mathcal{C}^*)$) the class of right (two-sided) ideals of \mathcal{C}^* . Analogously, we denote with $\mathcal{L}(\mathcal{C})$ ($\mathcal{L}_{f.i.}(\mathcal{C})$) the class of (fully invariant) left \mathcal{C} -coideals and with $\mathcal{I}_l({}^*\mathcal{C})$ ($\mathcal{I}_{t.s.}({}^*\mathcal{C})$) the class of left (two-sided) ideals of ${}^*\mathcal{C}$. With $\mathcal{B}(\mathcal{C})$ we denote the class of \mathcal{C} -bicoideals and for each $B \in \mathcal{B}(\mathcal{C})$ we write B^r (B^l) to indicate that we consider B as an object in the category of right (left) \mathcal{C} -comodules.

Remarks 5.3. For $\emptyset \neq I \subseteq \mathcal{C}^*$ ($\emptyset \neq I \subseteq {}^*\mathcal{C}$) and $\emptyset \neq K \subseteq \mathcal{C}$, set

$$I^{\perp(\mathcal{C})} := \bigcap_{f \in I} \{c \in \mathcal{C} \mid f(c) = 0\}$$

and

$$K^{\perp(\mathcal{C})} := \{f \in {}^*\mathcal{C} \mid f(K) = 0\}; \quad K^{\perp(\mathcal{C}^*)} := \{f \in \mathcal{C}^* \mid f(K) = 0\}.$$

1. If ${}_A\mathcal{C}$ is flat, then a right A -submodule $K \subseteq \mathcal{C}$ is a right \mathcal{C} -coideal, iff $\Delta(K) \subseteq K \otimes_A \mathcal{C}$.
 If \mathcal{C}_A is flat, then a left A -submodule $K \subseteq \mathcal{C}$ is a left \mathcal{C} -coideal, iff $\Delta(K) \subseteq \mathcal{C} \otimes_A K$.
 If ${}_A\mathcal{C}$ and \mathcal{C}_A are flat, then an A -subbimodule $B \subseteq \mathcal{C}$ is a \mathcal{C} -bicoideal, iff $\Delta(B) \subseteq (B \otimes_A \mathcal{C}) \cap (\mathcal{C} \otimes_A B)$.
 If ${}_A\mathcal{C}$ and \mathcal{C}_A are flat, then an A -subbimodule $\mathcal{D} \subseteq \mathcal{C}$ is a subcoring, iff $\Delta(\mathcal{D}) \subseteq \mathcal{D} \otimes_A \mathcal{D}$.

2. Every A -subcoring $\mathcal{D} \subseteq \mathcal{C}$ is a \mathcal{C} -bicoideal in the canonical way.

If $B \subseteq \mathcal{C}$ is a \mathcal{C} -bicoideal that is pure as a left and as a right A -submodule, then we have by [BW03, 40.16]:

$$\Delta(B) \subseteq (B \otimes_A \mathcal{C}) \cap (\mathcal{C} \otimes_A B) = B \otimes_A B,$$

i.e. $B \subseteq \mathcal{C}$ is an A -subcoring.

3. If \mathcal{C}_A (respectively ${}_A\mathcal{C}$) is locally projective, then $\mathcal{R}_{f.i.}(\mathcal{C}) = \mathcal{B}(\mathcal{C})$ (respectively $\mathcal{L}_{f.i.}(\mathcal{C}) = \mathcal{B}(\mathcal{C})$): if $B \subseteq \mathcal{C}$ is a fully invariant right (left) \mathcal{C} -coideal, then $B \subseteq \mathcal{C}$ is a right \mathcal{C}^* -submodule (left ${}^*\mathcal{C}$ -submodule) and it follows by Proposition 2.12 that $B \subseteq \mathcal{C}$ is a \mathcal{C} -subbicomodule with structure map the restriction of $\Delta_{\mathcal{C}}$ to B , i.e. B is a \mathcal{C} -bicoideal.
4. Let \mathcal{C}_A (${}_A\mathcal{C}$) be locally projective. If $P \triangleleft \mathcal{C}^*$ ($P \triangleleft {}^*\mathcal{C}$) is a two-sided ideal, then the fully invariant right (left) \mathcal{C} -coideal $B := \text{ann}_{\mathcal{C}}(P) \subseteq \mathcal{C}$ is a \mathcal{C} -bicoideal.
5. If ${}_A\mathcal{C}$ is locally projective and $I \triangleleft_r {}^*\mathcal{C}$ is a right ideal, then the left ${}^*\mathcal{C}$ -submodule $I^{\perp(\mathcal{C})} \subseteq \mathcal{C}$ is a right \mathcal{C} -coideal.
 If \mathcal{C}_A is locally projective and $I \triangleleft_l \mathcal{C}^*$ is a left ideal, then the right \mathcal{C}^* -submodule $I^{\perp(\mathcal{C})} \subseteq \mathcal{C}$ is a left \mathcal{C} -coideal.
6. If $K \subseteq \mathcal{C}$ is a (fully invariant) right \mathcal{C} -coideal, then $K^{\perp(\mathcal{C}^*)} = \text{ann}_{\mathcal{C}^*}(K) \simeq \text{An}_{\mathbb{E}_{\mathcal{C}}}^{\mathcal{C}}(K)$; in particular $K^{\perp(\mathcal{C}^*)} \subseteq \mathcal{C}^*$ is a right (two-sided) ideal.
 If $K \subseteq \mathcal{C}$ is a (fully invariant) left \mathcal{C} -coideal, then $K^{\perp({}^*\mathcal{C})} = \text{ann}_{{}^*\mathcal{C}}(K) \simeq \text{An}_{\mathbb{E}}^{\mathcal{C}}(K)$; in particular $K^{\perp({}^*\mathcal{C})} \subseteq {}^*\mathcal{C}$ is a left (two-sided) ideal.

7. If A_A is an injective cogenerator and ${}_A\mathcal{C}$ is flat, then for every right ideal $I \triangleleft_r \mathcal{C}^*$ we have $\text{ann}_{\mathcal{C}}(I) = I^{\perp(\mathcal{C})}$: Write $I = \bigcup_{\lambda \in \Lambda} I_{\lambda}$, where $I_{\lambda} \triangleleft_r \mathcal{C}^*$ is a finitely generated right ideal for each $\lambda \in \Lambda$. If $\text{ann}_{\mathcal{C}}(I_{\lambda_0}) \subsetneq I_{\lambda_0}^{\perp(\mathcal{C})}$ for some $\lambda_0 \in \Lambda$, then $\text{Hom}_A(\mathcal{C}/\text{ann}_{\mathcal{C}}(I_{\lambda_0}), A) \not\subseteq \text{Hom}_A(\mathcal{C}/I_{\lambda_0}^{\perp(\mathcal{C})}, A)$ (since A_A is a cogenerator). Since A_A is injective, \mathcal{C} is injective in $\mathbb{M}^{\mathcal{C}}$ by Lemma 2.9 and it follows by 2.15 (3-b) and the remarks above that $I_{\lambda_0} = \text{ann}_{\mathcal{C}^*}(\text{ann}_{\mathcal{C}}(I_{\lambda_0})) = (\text{ann}_{\mathcal{C}}(I_{\lambda_0}))^{\perp(\mathcal{C}^*)} \not\subseteq I_{\lambda_0}^{\perp(\mathcal{C})\perp(\mathcal{C}^*)}$ (a contradiction). So $\text{ann}_{\mathcal{C}}(I_{\lambda}) = I_{\lambda}^{\perp(\mathcal{C})}$ for each $\lambda \in \Lambda$ and we get

$$\text{ann}_{\mathcal{C}}(I) = \bigcap_{\lambda \in \Lambda} \text{ann}_{\mathcal{C}}(I_{\lambda}) = \bigcap_{\lambda \in \Lambda} I_{\lambda}^{\perp(\mathcal{C})} = \left(\bigcup_{\lambda \in \Lambda} I_{\lambda} \right)^{\perp(\mathcal{C})} = I^{\perp(\mathcal{C})}. \blacksquare$$

Sufficient and necessary conditions

The following result gives sufficient and necessary conditions for the dual rings of the non-zero coring \mathcal{C} to be prime (respectively semiprime, domain, reduced) generalizing results of [YDZ90] for coalgebras over base fields. Several results follow directly from previous sections recalling the isomorphism of rings $\mathcal{C}^* \simeq \text{End}^{\mathcal{C}}(\mathcal{C})^{op}$. Analogous statements can be formulated for ${}^*\mathcal{C} \simeq {}^{\mathcal{C}}\text{End}(\mathcal{C})$.

Theorem 3.17 yields directly.

Theorem 5.4. *Let ${}_A\mathcal{C}$ be flat.*

1. \mathcal{C}^* is prime (domain), if $\mathcal{C} = \mathcal{C}f\mathcal{C}^*$ ($\mathcal{C} = \mathcal{C}f$) for all $0 \neq f \in \mathcal{C}^*$. If \mathcal{C} is coretractable in $\mathbb{M}^{\mathcal{C}}$, then \mathcal{C}^* is prime (domain) if and only if $\mathcal{C} = \mathcal{C}f\mathcal{C}^*$ ($\mathcal{C} = \mathcal{C}f$) $\forall 0 \neq f \in \mathcal{C}^*$.
2. \mathcal{C}^* is semiprime (reduced), if $\mathcal{C}f = \mathcal{C}f\mathcal{C}^*f$ ($\mathcal{C}f = \mathcal{C}f^2$) for all $0 \neq f \in \mathcal{C}^*$. If \mathcal{C} is self-cogenerator in $\mathbb{M}^{\mathcal{C}}$, then \mathcal{C}^* is semiprime (reduced) if and only if $\mathcal{C}f = \mathcal{C}f\mathcal{C}^*f$ ($\mathcal{C}f = \mathcal{C}f^2$) $\forall 0 \neq f \in \mathcal{C}^*$.

Proposition 5.5. *Let ${}_A\mathcal{C}$ and \mathcal{C}_A be flat.*

1. Let \mathcal{C} be coretractable in $\mathbb{M}^{\mathcal{C}}$ and $\mathcal{C}_{\mathcal{C}^*}$ satisfy condition (**). If \mathcal{C}^* is prime (domain), then ${}^*\mathcal{C}$ is prime (domain).
2. Let \mathcal{C} be coretractable in $\mathbb{M}^{\mathcal{C}}$, ${}^{\mathcal{C}}\mathbb{M}$ and $\mathcal{C}_{\mathcal{C}^*, {}^*\mathcal{C}}$ satisfy condition (**). Then \mathcal{C}^* is prime (domain) if and only if ${}^*\mathcal{C}$ is prime (domain).
3. Let \mathcal{C} be coretractable in $\mathbb{M}^{\mathcal{C}}$ and ${}_A\mathcal{C}$ be locally projective. If \mathcal{C}^* is prime, then ${}^*\mathcal{C}$ is prime.
4. Let \mathcal{C} be coretractable in $\mathbb{M}^{\mathcal{C}}$, ${}^{\mathcal{C}}\mathbb{M}$ and ${}_A\mathcal{C}$, \mathcal{C}_A be locally projective. Then \mathcal{C}^* is prime if and only if ${}^*\mathcal{C}$ is prime.

Proof. 1. Let \mathcal{C}^* be prime (domain). If ${}^*\mathcal{C}$ were not prime (not a domain), then there exists by an analogous statement of Theorem 5.4 some $0 \neq f \in {}^*\mathcal{C}$ with ${}^*\mathcal{C}f\mathcal{C} \subsetneq \mathcal{C}$ ($f\mathcal{C} \subsetneq \mathcal{C}$). By assumption $\mathcal{C}_{\mathcal{C}^*}$ satisfies condition (**) and so there exists some $0 \neq h \in \mathcal{C}^*$ such that $({}^*\mathcal{C}f\mathcal{C})h = 0$ ($(f\mathcal{C})h = 0$). But this implies $\mathcal{C} \neq \mathcal{C}h\mathcal{C}^*$ ($\mathcal{C} \neq \mathcal{C}h$): otherwise $f\mathcal{C} = f(\mathcal{C}h\mathcal{C}^*) = ((f\mathcal{C})h)\mathcal{C}^* = 0$ ($f\mathcal{C} = f(\mathcal{C}h) = (f\mathcal{C})h = 0$), which implies $f = 0$, a contradiction. Since \mathcal{C} is coretractable in $\mathbb{M}^{\mathcal{C}}$, Theorem 5.4 (1) implies that \mathcal{C}^* is not prime (not a domain), which contradicts our assumptions.

2. Follows from (1) by symmetry.

3. The proof is similar to that of (1) recalling that, in case ${}_A\mathcal{C}$ locally projective, for any $f \in {}^*\mathcal{C}$, the left ${}^*\mathcal{C}$ -submodule ${}^*\mathcal{C}f\mathcal{C} \subseteq \mathcal{C}$ is a right \mathcal{C} -subcomodule.

4. Follows from (3) by symmetry. ■

E-Prime versus simple

In what follows we show that E-prime corings generalize simple corings. The results are obtained by direct application of the corresponding results in the Section 3.

As a direct consequence of Theorems 3.18 and 3.19 we get

Theorem 5.6. *Let A be a QF ring and assume ${}_A\mathcal{C}$ to be (locally) projective.*

1. \mathcal{C}^r is simple if and only if \mathcal{C}^* is right simple.
2. If \mathcal{C}^* is simple, then \mathcal{C} is simple (as a $({}^*\mathcal{C}, \mathcal{C}^*)$ -bimodule).
3. Let \mathcal{C}^* be right Noetherian. Then \mathcal{C}^* is simple if and only if \mathcal{C} is simple (as a $({}^*\mathcal{C}, \mathcal{C}^*)$ -bimodule).

Corollary 5.7. *Let A be a QF ring, ${}_A\mathcal{C}$, \mathcal{C}_A be locally projective, ${}^*\mathcal{C}$ be left Noetherian and \mathcal{C}^* be right Noetherian. Then*

$$\mathcal{C}^* \text{ is simple} \Leftrightarrow \mathcal{C} \text{ is simple (as a } ({}^*\mathcal{C}, \mathcal{C}^*)\text{-bimodule)} \Leftrightarrow {}^*\mathcal{C} \text{ is simple.}$$

Proposition 5.8. *Let A be a QF ring. If ${}_A\mathcal{C}$ is (locally) projective, then we have*

$$\text{Jac}(\mathcal{C}^*) = \text{ann}_{\mathcal{C}^*}(\text{Soc}(\mathcal{C}^r)) = \text{Soc}(\mathcal{C}^r)^{\perp(\mathcal{C}^*)} \text{ and } \text{Soc}(\mathcal{C}^r) = \text{Jac}(\mathcal{C}^*)^{\perp(\mathcal{C})}.$$

Proof. The result in (1) follows from Proposition 3.23 (3) recalling the isomorphisms of R -algebras $\mathcal{C}^* \simeq \text{End}^{\mathcal{C}}(\mathcal{C})^{op}$ and Remarks 5.3 (6) & (7). ■

Corollary 5.9. *Let A be a QF ring.*

1. If ${}_A\mathcal{C}$ is (locally) projective, then \mathcal{C} is right semisimple if and only if \mathcal{C}^* is semiprimitive.
2. If ${}_A\mathcal{C}$ and \mathcal{C}_A are (locally) projective, then \mathcal{C}^* is semiprimitive if and only if ${}^*\mathcal{C}$ is semiprimitive.

The wedge product

The *wedge product* of subspaces of a given coalgebra C over a base field was already defined and investigated in [Swe69, Section 9]. In [NT01], the wedge product of subcoalgebras was used to define *fully coprime coalgebras*.

Definition 5.10. We define the *wedge product* of a right A -submodule $K \subseteq \mathcal{C}$ and a left A -submodule $L \subseteq \mathcal{C}$ as

$$K \wedge L := \Delta^{-1}(\text{Im}(K \otimes_A \mathcal{C}) + \text{Im}(\mathcal{C} \otimes_A L)) = \text{Ker}((\pi_K \otimes \pi_L) \circ \Delta : \mathcal{C} \longrightarrow \mathcal{C}/K \otimes_A \mathcal{C}/L).$$

Remark 5.11. ([Swe69, Proposition 9.0.0.]) Let C be a coalgebra over a base field and $K, L \subseteq C$ be any subspaces. Then $K \wedge L = (K^{\perp(\mathcal{C}^*)} * L^{\perp(\mathcal{C}^*)})^{\perp(\mathcal{C})}$. If moreover K is a left C -coideal and L is a right C -coideal, then $K \wedge L \subseteq C$ is a subcoalgebra.

Lemma 5.12. (See [Abu03, Corollary 2.9.]) *Let $K, L \subseteq \mathcal{C}$ be A -subbimodules.*

1. *Consider the canonical A -bilinear map*

$$\kappa_l : K^{\perp(*\mathcal{C})} \otimes_A L^{\perp(*\mathcal{C})} \rightarrow {}^*(\mathcal{C} \otimes_A \mathcal{C}), [f \otimes_A g \mapsto (c \otimes_A c') = g(cf(c'))].$$

If A is right Noetherian, \mathcal{C}_A is flat and $L^{\perp(\mathcal{C})\perp} \subseteq \mathcal{C}$ is pure as a right A -module, then*

$$(\kappa_l(K^{\perp(*\mathcal{C})} \otimes_A L^{\perp(*\mathcal{C})}))^{\perp(\mathcal{C} \otimes_A \mathcal{C})} = L^{\perp(*\mathcal{C})\perp} \otimes_A \mathcal{C} + \mathcal{C} \otimes_A K^{\perp(*\mathcal{C})\perp}. \quad (15)$$

2. *Consider the canonical A -bilinear map*

$$\kappa_r : L^{\perp(\mathcal{C}^*)} \otimes_A K^{\perp(\mathcal{C}^*)} \rightarrow (\mathcal{C} \otimes_A \mathcal{C})^*, [g \otimes_A f \mapsto (c' \otimes_A c) = g(f(c')c)].$$

If A is left Noetherian, ${}_A\mathcal{C}$ is flat and $L^{\perp(\mathcal{C}^)\perp} \subseteq \mathcal{C}$ is pure as a left A -module, then*

$$(\kappa_r(L \otimes_A K))^{\perp(\mathcal{C} \otimes_A \mathcal{C})} = K^{\perp(\mathcal{C}^*)\perp} \otimes_A \mathcal{C} + \mathcal{C} \otimes_A L^{\perp(\mathcal{C}^*)\perp}. \quad (16)$$

Definition 5.13. For R -submodules $K, L \subseteq \mathcal{C}$ we set

$$\begin{aligned} (K :_{\mathcal{C}^r} L) &:= \bigcap \{f^{-1}(Y) \mid f \in \text{End}^{\mathcal{C}}(\mathcal{C})^{op} \text{ and } f(K) = 0\} \\ &= \bigcap \{c \in \mathcal{C} \mid c \leftarrow f \in L \text{ for all } f \in \text{ann}_{\mathcal{C}^*}(K)\}. \end{aligned}$$

and

$$\begin{aligned} (K :_{\mathcal{C}^l} L) &:= \bigcap \{f^{-1}(L) \mid f \in {}^{\mathcal{C}}\text{End}(\mathcal{C}) \text{ and } f(K) = 0\} \\ &= \bigcap \{c \in \mathcal{C} \mid f \rightarrow c \in L \text{ for all } f \in \text{ann}_{*\mathcal{C}}(K)\}. \end{aligned}$$

If $K, L \subseteq \mathcal{C}$ are right (left) \mathcal{C} -coideals, then we call $(K :_{\mathcal{C}^r} L)$ ($(K :_{\mathcal{C}^l} L)$) the *internal coproduct* of X and Y in $\mathbb{M}^{\mathcal{C}}$ (in ${}^{\mathcal{C}}\mathbb{M}$).

Lemma 5.14. *Let $K, L \subseteq \mathcal{C}$ be \mathcal{C} -bicoideals.*

1. *If ${}_A\mathcal{C}$ is flat and \mathcal{C} is self-cogenerator in $\mathbb{M}^{\mathcal{C}}$, then*

$$(K :_{\mathcal{C}^r} L) = \text{ann}_{\mathcal{C}}(\text{ann}_{\mathcal{C}^*}(K) *^r \text{ann}_{\mathcal{C}^*}(L)).$$

2. *If \mathcal{C}_A is flat and \mathcal{C} is self-cogenerator in ${}^{\mathcal{C}}\mathbb{M}$, then*

$$(K :_{\mathcal{C}^l} L) = \text{ann}_{\mathcal{C}}(\text{ann}_{*\mathcal{C}}(K) *^l \text{ann}_{*\mathcal{C}}(L)).$$

Proof. The proof of (1) is analogous to that of Lemma 4.9, while (2) follows by symmetry. ■

The following result clarifies the relation between the *wedge product* and the *internal coproduct* of right (left) \mathcal{C} -coideals under suitable purity conditions:

Proposition 5.15. *Let A be a QF ring, $(\mathcal{C}, \Delta, \varepsilon)$ be an A -coring and $K, L \subseteq \mathcal{C}$ be A -subbimodules.*

1. *Let ${}_A\mathcal{C}$ be flat and K, L be right \mathcal{C} -coideals. If ${}_A L \subseteq {}_A \mathcal{C}$ is pure, then $(K :_{\mathcal{C}^r} L) = K \wedge L$.*

2. Let \mathcal{C}_A be flat and K, L be left \mathcal{C} -coideals. If $K_A \subseteq \mathcal{C}_A$ is pure, then $(K :_{\mathcal{C}^l} L) = K \wedge L$.
3. Let ${}_A\mathcal{C}, \mathcal{C}_A$ be flat and $K, L \subseteq \mathcal{C}$ be \mathcal{C} -bicoideals. If ${}_A K \subseteq {}_A\mathcal{C}$ and $L_A \subseteq \mathcal{C}_A$ are pure, then

$$(K :_{\mathcal{C}^r} L) = K \wedge L = (K :_{\mathcal{C}^l} L). \quad (17)$$

Proof. 1. Assume ${}_A\mathcal{C}$ to be flat and consider the map

$$\kappa_r : L^{\perp(\mathcal{C}^*)} \otimes_A K^{\perp(\mathcal{C}^*)} \rightarrow (\mathcal{C} \otimes_A \mathcal{C})^*, [g \otimes_A f \mapsto (c' \otimes_A c) = g(f(c')c)].$$

Then we have

$$\begin{aligned} (K :_{\mathcal{C}^r} L) &= (\text{ann}_{\mathcal{C}^*}(K) *^r \text{ann}_{\mathcal{C}^*}(L))^{\perp} && \text{(Lemma 5.14)} \\ &= (K^{\perp(\mathcal{C}^*)} *^r L^{\perp(\mathcal{C}^*)})^{\perp} && (X, Y \subseteq \mathcal{C} \text{ are right coideals)} \\ &= ((\Delta^* \circ \kappa_r)(L^{\perp(\mathcal{C}^*)} \otimes_A K^{\perp(\mathcal{C}^*)})^{\perp}) \\ &= \Delta^{-1}((\kappa_r(L^{\perp(\mathcal{C}^*)} \otimes_A K^{\perp(\mathcal{C}^*)}))^{\perp}) && \text{([Abu05, Proposition 1.10 (3-c)])} \\ &= \Delta^{-1}(K^{\perp(\mathcal{C}^*)\perp} \otimes_A \mathcal{C} + \mathcal{C} \otimes_A L^{\perp(\mathcal{C}^*)\perp}) && \text{([Abu05, Corollary 2.9])} \\ &= \Delta^{-1}(K \otimes_A \mathcal{C} + \mathcal{C} \otimes_A L) && A \text{ is cogenerator} \\ &= K \wedge L. \end{aligned}$$

2. This follows from (1) by symmetry.

3. This is a combination of (1) and (2). ■

Fully coprime (fully cosemiprime) corings

In addition to the notions of *right (left) fully coprime* and *right (left) fully cosemiprime bicoideals*, considered as right (left) comodules in the canonical way, we present the notion of a *fully coprime (fully cosemiprime) bicoideal*.

Definition 5.16. Let $(\mathcal{C}, \Delta, \varepsilon)$ be a non-zero A -coring and assume ${}_A\mathcal{C}, \mathcal{C}_A$ to be flat. Let $0 \neq B \subseteq \mathcal{C}$ be a \mathcal{C} -bicomodule and consider the right \mathcal{C} -comodule B^r and the left \mathcal{C} -comodule B^l . We call B :

fully \mathcal{C} -coprime (fully \mathcal{C} -cosemiprime), iff both B^r and B^l are fully \mathcal{C} -coprime (fully \mathcal{C} -cosemiprime);

fully coprime (fully cosemiprime), iff both B^r and B^l are fully coprime (fully cosemiprime).

The fully coprime coradical

The *prime spectra* and the associated *prime radicals* for rings play an important role in the study of structure of rings. Dually, we define the *fully coprime spectra* and the *fully coprime coradicals* for corings.

Definition 5.17. Let $(\mathcal{C}, \Delta, \varepsilon)$ be a non-zero ring and assume ${}_A\mathcal{C}$ to be flat. We define the *fully coprime spectrum* of \mathcal{C}^r as

$$\text{CPSpec}(\mathcal{C}^r) := \{0 \neq B \mid B^r \subseteq \mathcal{C}^r \text{ is a fully } \mathcal{C}\text{-coprime}\}$$

and the *fully coprime coradical* of \mathcal{C}^r as

$$\text{CPcorad}(\mathcal{C}^r) := \sum_{B \in \text{CPSpec}(\mathcal{C}^r)} B.$$

Moreover, we set

$$\text{CSP}(\mathcal{C}^r) := \{0 \neq B \mid B^r \subseteq \mathcal{C}^r \text{ is a fully } \mathcal{C}\text{-cosemiprime}\}.$$

In case \mathcal{C}_A is flat, one defines analogously $\text{CPSpec}(\mathcal{C}^l)$, $\text{CPcorad}(\mathcal{C}^l)$ and $\text{CSP}(\mathcal{C}^l)$.

As a direct consequence of Remark 4.11 we get:

Theorem 5.18. *Let A be a QF ring and ${}_A\mathcal{C}$ be flat. Then \mathcal{C}^* is prime (semiprime) if and only if \mathcal{C}^r is fully coprime (fully cosemiprime).*

The following result shows that fully coprime spectrum (fully coprime coradical) of corings is invariant under isomorphisms of corings. The proof is analogous to that of Proposition 4.5.

Proposition 5.19. *Let $\theta : \mathcal{C} \rightarrow \mathcal{D}$ be an isomorphism of A -corings and assume ${}_A\mathcal{C}$, ${}_A\mathcal{D}$ to be flat. Then we have bijections*

$$\text{CPSpec}(\mathcal{C}^r) \longleftrightarrow \text{CPSpec}(\mathcal{D}^r) \text{ and } \text{CSP}(\mathcal{C}^r) \longleftrightarrow \text{CSP}(\mathcal{D}^r).$$

In particular, $\theta(\text{CPcorad}(\mathcal{C}^r)) = \text{CPcorad}(\mathcal{D}^r)$.

Remark 5.20. If $\theta : \mathcal{C} \rightarrow \mathcal{D}$ is a morphism of A -corings, then it is NOT evident that θ maps fully \mathcal{C} -coprime (fully \mathcal{C} -cosemiprime) \mathcal{C} -bicoideals into fully \mathcal{D} -coprime (fully \mathcal{D} -cosemiprime) \mathcal{D} -bicoideals, contrary to what was mentioned in [NT01, Theorem 2.4(i)].

The following example, given by Chen Hui-Xiang in his review of [NT01] (Zbl 1012.16041), shows moreover that a homomorphic image of a fully coprime coalgebra need not be fully coprime:

Example 5.21. Let $A := M_n(F)$ be the algebra of all $n \times n$ matrices over a field F , $B := T_n(F)$ be the subalgebra of upper-triangular $n \times n$ matrices over F where $n > 1$. Consider the dual coalgebras A^*, B^* . The embedding of F -algebras $\iota : B \hookrightarrow A$ induces a surjective map of F -coalgebras $A^* \xrightarrow{\iota^*} B^* \longrightarrow 0$. However, A is prime while B is not, i.e. A^* is a fully coprime F -coalgebra, while B^* is not (see Theorem 5.18).

As a direct consequence of Proposition 4.12 we have

Proposition 5.22. *Let A be a QF ring. If ${}_A\mathcal{C}$ is flat and \mathcal{C}^* is right Noetherian, then*

$$\text{Prad}(\mathcal{C}^*) = \text{CPcorad}(\mathcal{C}^r)^{\perp(\mathcal{C}^*)} \text{ and } \text{CPcorad}(\mathcal{C}^r) = \text{Prad}(\mathcal{C}^*)^{\perp(\mathcal{C})}.$$

Making use of Proposition 5.22, a similar proof to that of Corollary 3.7 yields:

Corollary 5.23. *Let A be a QF ring. If ${}_A\mathcal{C}$ is flat and \mathcal{C}^* is Noetherian, then*

$$\mathcal{C}^r \text{ is fully cosemiprime} \Leftrightarrow \mathcal{C} = \text{CPcorad}(\mathcal{C}^r).$$

Corollary 5.24. *Let A be a QF ring. If ${}_A\mathcal{C}$ is (locally) projective and \mathcal{C}^r is Artinian (e.g. \mathcal{C}_A is finitely generated), then*

1. $\text{Prad}(\mathcal{C}^*) = \text{CPcorad}(\mathcal{C}^r)^{\perp(\mathcal{C}^*)}$ and $\text{CPcorad}(\mathcal{C}^r) = \text{Prad}(\mathcal{C}^*)^{\perp(\mathcal{C})}$.
2. \mathcal{C}^r is fully cosemiprime if and only if $\mathcal{C} = \text{CPcorad}(\mathcal{C}^r)$.

Corings with Artinian dual rings

For corings over QF ground rings several primeness and coprimeness properties become equivalent. As a direct consequence Theorems 4.15, 5.18 and [FR, Theorem 2.9, Corollary 2.10] we get the following characterizations of fully coprime locally projective corings over QF ground rings:

Theorem 5.25. *Let A be a QF ring and ${}_A\mathcal{C}$, \mathcal{C}_A be projective and assume \mathcal{C}^* is right Artinian and ${}^*\mathcal{C}$ is left Artinian. Then the following statements are equivalent:*

1. \mathcal{C}^* (or ${}^*\mathcal{C}$) is prime;
2. $\mathcal{C}_{\mathcal{C}^*}$ (or ${}^*\mathcal{C}_{\mathcal{C}}$) is diprime;
3. $\mathcal{C}_{\mathcal{C}^*}$ (or ${}^*\mathcal{C}_{\mathcal{C}}$) is prime;
4. \mathcal{C}^* (or ${}^*\mathcal{C}$) is simple Artinian;
5. $\mathcal{C}_{\mathcal{C}^*}$ (or ${}^*\mathcal{C}_{\mathcal{C}}$) is strongly prime;
6. \mathcal{C}^r (or \mathcal{C}^l) is fully coprime;
7. \mathcal{C} has non-trivial fully invariant right (left) \mathcal{C} -coideals;
8. \mathcal{C} is simple.

As a direct consequence of Theorem 4.21 we get

Theorem 5.26. *Let C be a locally projective cocommutative R -coalgebra and assume C to be self-injective self-cogenerator in \mathbb{M}^C . If C is fully coprime, then C is irreducible.*

Examples and Counterexamples

In what follows we give some examples of *fully coprime* corings (coalgebras) over arbitrary (commutative) ground rings. An important class of fully coprime path coalgebras over fields is considered by Prof. Jara et. al. in [JMR]. For other examples of fully coprime coalgebras over fields, the reader is referred to [NT01].

We begin with a counterexample to a conjecture in [NT01], communicated to the author by Ch. Lomp, which shows that the converse of Theorem 5.26 is not true in general:

Counterexample 5.27. Let C be a \mathbb{C} -vector space spanned by g and an infinite family of elements $\{x_\lambda\}_\Lambda$ where Λ is a non-empty set. Define a coalgebra structure on C by

$$\begin{aligned} \Delta(g) &= g \otimes g, & \varepsilon(g) &= 1; \\ \Delta(x_\lambda) &= g \otimes x_\lambda + x_\lambda \otimes g, & \varepsilon(x_\lambda) &= 0. \end{aligned} \tag{18}$$

Then C is a cocommutative coalgebra with unique simple (1-dimensional) subcoalgebra $C_0 = \mathbb{C}g$. Let $V(\Lambda)$ be the \mathbb{C} -vector space of families $\{b_\lambda\}_\Lambda$, where $b_\lambda \in \mathbb{C}$ and consider the trivial extension

$$\mathbb{C} \ltimes V(\Lambda) = \left\{ \begin{pmatrix} a & w \\ 0 & a \end{pmatrix} \mid a \in \mathbb{C} \text{ and } w \in V(\Lambda) \right\}, \tag{19}$$

which is a ring under the ordinary matrix multiplication and addition. Then there exists a ring isomorphism

$$C^* \simeq \mathbb{C} \times V(\Lambda), f \mapsto \begin{pmatrix} f(g) & (f(x_\lambda))_\Lambda \\ 0 & f(g) \end{pmatrix} \text{ for all } f \in C^*. \quad (20)$$

Since

$$\text{Jac}(C^*) \simeq \text{Jac}(\mathbb{C} \times V(\Lambda)) = \begin{pmatrix} 0 & V(\Lambda) \\ 0 & 0 \end{pmatrix}, \quad (21)$$

we have $(\text{Jac}(C^*))^2 = 0$, which means that C^* is not semiprime. So C is an infinite dimensional irreducible cocommutative coalgebra, which is not fully coprime (even not fully cosemiprime).

5.28. (The comatrix coring [EG-T03]) Let A, B be R -algebras, Q a (B, A) -bimodule and assume Q_A to be finitely generated projective with dual basis $\{(e_i, \pi_i)\}_{i=1}^n \subset Q \times Q^*$. By [EG-T03], $\mathcal{C} := Q^* \otimes Q$ is an A -coring (called the *comatrix coring*) with coproduct and counit given by

$$\Delta_{\mathcal{C}}(f \otimes_B q) := \sum_{i=1}^n (f \otimes_B e_i) \otimes_A (\pi_i \otimes_B q) \text{ and } \varepsilon_{\mathcal{C}}(f \otimes_B q) := f(q).$$

Notice that we have R -algebra isomorphisms

$$\mathcal{C}^* := \text{Hom}_{-A}(Q^* \otimes_B Q, A) \simeq \text{Hom}_{-B}(Q^*, \text{Hom}_{-A}(Q, A)) = \text{End}_{-B}(Q^*);$$

and

$${}^*\mathcal{C} := \text{Hom}_{A-}(Q^* \otimes_B Q, A) \simeq \text{Hom}_{B-}(Q, \text{Hom}_{A-}(Q^*, A))^{op} \simeq \text{End}_{B-}(Q)^{op}.$$

Example 5.29. Consider the (A, A) -bimodule $Q = A^n$ and the corresponding comatrix A -coring $\mathcal{C} := Q^* \otimes_A Q$ (called also the *matrix coalgebra* in case $A = R$, a commutative ring). Then we have isomorphisms of rings

$$\mathcal{C}^* \simeq \text{End}_{-A}((A^n)^*) \simeq \text{End}_{-A}((A^*)^n) \simeq \mathbb{M}_n(\text{End}_{-A}(A^*)) \simeq \mathbb{M}_n(\text{End}_{-A}(A)) \simeq \mathbb{M}_n(A),$$

and

$${}^*\mathcal{C} \simeq \text{End}_{A-}(A^n)^{op} \simeq \mathbb{M}_n(\text{End}_{A-}(A))^{op} \simeq \mathbb{M}_n(A^{op})^{op}.$$

Let A be prime. Then $\mathcal{C}^* \simeq \mathbb{M}_n(A)$ and ${}^*\mathcal{C} \simeq \mathbb{M}_n(A^{op})^{op}$ are prime (e.g. [AF74, Proposition 13.2]). If moreover A_A (${}_A A$) is a cogenerator, then \mathcal{C}^r (\mathcal{C}^l) is self-cogenerator and it follows by Remark 4.11 that \mathcal{C}^r is fully coprime (\mathcal{C}^l is fully coprime).

Example 5.30. Let $A \rightarrow B$ be a ring homomorphism and assume B_A to finitely generated and projective. Then the A -comatrix coring $\mathcal{C} := B^* \otimes_B B \simeq B^*$, is called the *dual A -coring* of the A -ring B as its coring structure can also be obtained from the the A -ring structure of B (see [Swe75, 3.7.]). If B is a prime ring, then ${}^*\mathcal{C} := {}^*(B^*) \simeq B$ is prime. If moreover, B_A^* is flat and self-cogenerator, then it follows by analogy to Remark 4.11 that ${}^l\mathcal{C}$ is fully coprime.

Example 5.31. Let R be Noetherian, A a non-zero R -algebra for which the *finite dual* $A^\circ \subset R^A$ is a pure submodule (e.g. R is a Dedekind domain) and assume ${}_R A^\circ$ to be a self-cogenerator. By [AG-TW00], A° is an R -coalgebra. If the R -algebra $A^{\circ*}$ is prime, then A° is a fully coprime R -coalgebra. If A is a reflexive R -algebra (i.e. $A \simeq A^{\circ*}$ canonically), then A is prime if and only if A° is fully coprime.

Example 5.32. Let A be a prime R -algebra and assume ${}_R A$ to be finitely generated projective. Then $C := A^*$ is an R -coalgebra (with no assumption on the commutative ground ring R) and $C^* := A^{**} \simeq A$. If ${}_R A^*$ is self-cogenerator (e.g. ${}_R R$ is a cogenerator), then A is a prime R -algebra if and only if C is a fully coprime R -coalgebra.

Example 5.33. Let R be an integral domain and $C := R[x]$ be the R -coalgebra with coproduct and counit defined on the generators by

$$\Delta(x^n) := \sum_{j=0}^n x^j \otimes_R x^{n-j} \text{ and } \varepsilon(x^n) := \delta_{n,0} \text{ for all } n \geq 0.$$

Then $C^* \simeq R[[x]]$, the power series ring, is an integral domain. If moreover, ${}_R C$ is self-cogenerator (e.g. ${}_R R$ is a cogenerator), then C is fully coprime.

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References

- [Abu05] J.Y. Abuhlail, *On the linear weak topology and dual pairings over rings*, Topol. Appl. **149(1-3)**, 161-175 (2005).
- [Abu03] J.Y. Abuhlail, *Rational modules for corings*, Comm. Alg. **31(12)**, 5793-5840 (2003).
- [AF74] F. Anderson and K. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics: **Vol. 13**, Springer Verlag (1974).
- [AG-TW00] J.Y. Abuhlail, J. Gómez-Torrecillas and R. Wisbauer, *Dual coalgebras of algebras over commutative rings*, J. Pure Appl. Algebra **153(2)**, 107-120 (2000).
- [Ann] S. Annin, *Attached primes over noncommutative rings*, **preprint**.
- [BJKN80] L. Bican, P. Jambor, T. Kepka, P. Nĕmec, *Prime and coprime modules*, Fund. Math. **107(1)** 33-45 (1980).
- [BW03] T. Brzeziński and R. Wisbauer, *Corings and Comodules*, Lond. Math. Soc. Lec. Not. Ser. **309**, Cambridge University Press (2003).

- [DNR2001] S. Dăscălescu, C. Năstăsescu and Ş. Raianu, *Hopf Algebras*, Marcel Dekker (2001).
- [EG-T03] L. El Kaoutit and J. Gomez-Torrecillas, *Comatrix corings: Galois corings, descent theory, and a structure theorem for cosemisimple corings*, Math. Z., **244**, 887-906 (2003).
- [FR] M. Ferrero and V. Rodrigues, *On prime and semiprime modules and comodules*, **to appear** in J. Algebra Appl.
- [HV05] A. Haghany and R. Vedadi, *Endoprime modules*, Acta Math. Hungar. **106(1-2)**, 89-99 (2005).
- [JMR] P. Jara, L.M. Morino and J.F. Ruiz, *Prime Path Coalgebras*, preprint.
- [Joh53] R. Johnson, *Representations of prime rings*, Trans. Amer. Math. Soc. **74**, 351-357 (1953).
- [Lom05] C. Lomp, *Prime elements in partially ordered groupoids applied to modules and Hopf Algebra actions*, J. Algebra Appl. **4(1)**, 77-98 (2005).
- [NT01] R. Nekooei and L. Torkzadeh, *Topology on coalgebras*, Bull. Iran. Math. Soc. **27(2)**, 45-63 (2001).
- [RRW05] F. Raggi, J. Ríos Montes and R. Wisbauer, *Coprime preradicals and modules*, J. Pur. App. Alg. **200**, 51-69 (2005).
- [Swe69] M. Sweedler, *Hopf Algebras*, New York: Benjamin, (1969).
- [Swe75] M. Sweedler, *The predual theorem to the Jacobson-Bourbaki theorem*, Trans. Amer. Math. Soc. **213**, 391-406 (1975).
- [vdBW01] J.E. van den Berg and R. Wisbauer, *Duprime and dusemiprime modules*, J. Pure Appl. Algebra **165(3)**, 337-356 (2001).
- [Wis96] R. Wisbauer, *Modules and Algebras : Bimodule Structure and Group Action on Algebras*, Pitman Monographs and Surveys in Pure and Applied Mathematics **81**, Addison Wesley Longman Limited (1996).
- [Wis91] R. Wisbauer, *Foundations of Module and Ring Theory, a Handbook for Study and Research*, Algebra, Logic and Applications **3**, Philadelphia etc.: Gordon and Breach Science Publishers (1991).
- [YDZ90] Yonghua Xu, Di Ming Lu and Hong Xin Zhu, *A necessary and sufficient conditions for dual algebras of coalgebras to be prime*, Zhong Li et al. (editors), Proceedings of the Asian mathematical conference 1990, Hong Kong, World Scientific, 502-510 (1992).
- [Z-H76] B. Zimmermann-Huisgen, *Pure submodules of direct products of free modules*, Math. Ann. **224**, 233-245 (1976).