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Skewness and Kurtosis for Bivariate Distributions

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Abstract The coefficients of kurtosis and skewness are easily calculated for bivariate elliptical distribution which includes bivariate normal as a special case. But for many bivariate distributions these are challenging. A set of alternative formulae is developed to derive the coefficients of kurtosis and skewness for any bivariate distribution. These are the second and the third order moments of standardized distance better known as Mahalanobis distance. The proposed method works well if the product moments have closed forms. Ideas are illustrated with examples.

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Key Words and Phrases: bivariate distribution, standardized moments, Mahalanobis distance, product moments, kurtosis, skewness

1. Introduction

There are no satisfactory measures for skewness and kurtosis for bivariate or multivariate distributions. In a series of papers, Mardia (1970, 1974, 1975) defined and discussed the properties of measures of kurtosis and skewness based on Mahalanobis distance. The coefficients of kurtosis and skewness are second and third order moments of standardized distance better known as Mahalanobis distance. The moments can be referred to as standardized moments, Mahalanobis moments or Mardia moments. Interested readers may go through Kotz, Balakrishnan and Johnson (2000) for an excellent discussion on multivariate skewness and kurtosis.

For some distributions, it is easy to derive the distribution of the Mahalanobis distance and calculate moments but for others these are difficult. Kotz, Balakrishnan and Johnson (2000) have provided the kurtosis of the Marshall-Olkin bivariate exponential distribution. In this paper we provide an alternative method to calculate Mahalanobis moments for bivariate distributions in terms of product moments of the components of a bivariate vector. Product moments (also called raw product moments or product moments around zero) of order a and b

for two random variables X_1 and X_2 are defined by $\mu'(a,b) = E\left(X_1^a X_2^b\right)$ while the centered product moments (sometimes called central product moments, corrected product moments or central mixed moments) are defined by

$$\mu(a,b) = E\left[\left(X_1 - E(X_1)\right)^a \left(X_2 - E(X_2)\right)^b\right]. \tag{1.1}$$

Interested readers may go through Johnson, Kotz and Kemp (1993, 46) or Johnson, Kotz and Balakrishnan (1997, 3). Evidently $\mu'(a,0) = E\left(X_1^a\right)$ is the a-th moment of X_1, and $\mu'(0,b) = E\left(X_2^b\right)$ is the b-th moment of X_2. In case X_1 and X_2 are independent, then $\mu'(a,b) = E\left(X_1^a\right)E\left(X_2^b\right) = \mu'(a,0)\mu'(0,b)$ and $\mu(a,b) = \mu(a,0)\mu(0,b)$. The correlation coefficient $\rho(-1 < \rho < 1)$ between X_1 and X_2 is denoted by

$$\rho_{X_1,X_2} = \frac{\mu(1,1)}{\sqrt{\mu(2,0)\mu(0,2)}}. \tag{1.2}$$

Note that $\mu(2,0) = E\left(X_1 - E(X_1)\right)^2 = \sigma_{20}$ which is popularly denoted by σ_1^2 while the central product moment, $\mu(1,1) = E\left[\left(X_1 - E(X_1)\right)\left(X_2 - E(X_2)\right)\right]$ denoted popularly by σ_{12}, is, in fact, the covariance between X_1 and X_2.

As it is difficult to derive the distribution of Mahalanobis distance for many distributions and calculate moments, we derive Mahalanobis moments in terms of centered product moments. Mahalanobis moments for a bivariate normal distribution and bivariate *t*-distribution are calculated. It is observed that general formulae for bivariate elliptical distribution which includes bivariate normal and *t*-distributions as special cases is easy to obtain. An example of bivariate chisquare distribution is considered for which the proposed method developed in Section 2 seems to be appropriate. In what follows we will rather use $X_1 = X$ and $X_2 = Y$ to avoid all confusion of a trivial nature, and define $\mu(a,b) = E\left[(X - \xi)^a (Y - \theta)^b\right]$ where $\xi = E(X), \theta = E(Y)$.

2. Kurtosis and Skewness in Terms of Product Moments

For a bivariate random vector $W = (X, Y)'$, with mean vector $\mu = (\xi, \theta)'$ and covariance matrix

$$Cov(W) = E(W-\mu)(W-\mu)' = \Sigma = \begin{pmatrix} \mu(2,0) & \mu(1,1) \\ \mu(1,1) & \mu(0,2) \end{pmatrix},$$

the standardized distance is defined by

$$\begin{aligned} Q &= (W-\mu)'\Sigma^{-1}(W-\mu) \\ &= ((X-\xi)\ \ (Y-\theta))\Sigma^{-1}((X-\xi)\ \ (Y-\theta))'. \end{aligned} \tag{2.1}$$

The quantity Q is also known to be generalized distance or Mahalanobis distance. The coefficients of kurtosis and skewness are $\beta_2 = E(Q^2)$ and $\beta_3 = E(Q^3)$ respectively (Kotz, Balakrishnan and Johnson, 2000, 77). For a bivariate random vector W with $E(W) = \mu$ and $Cov(W) = \Sigma$, we define standardized moments or Mahalanobis moments by $\beta_i = E(Q^i),\ i = 1, 2, \cdots$ where $Q = (W-\mu)'\Sigma^{-1}(W-\mu) = \|\Sigma^{-1/2}(W-\mu)\|^2$.

In fact regardless of the distribution of the variable in question the first standardized moment is the dimension of the random variable since

$$\beta_1 = E(trW'\Omega^{-1}W) = E(tr\Omega^{-1}WW') = tr[\Omega^{-1}E(WW')] = tr(\Omega^{-1}\Omega) = tr(I_p) = p. \tag{2.2}$$

Kotz, Nadarajah and Mitov (2003) presented an elegant technique for product moments of multivariate random vectors in terms of cumulative distribution function or survival function. It appears that if the cumulative distribution function or the survival function has a closed form, the Nadarajah and Mitov (2003) technique works well. For Marshall-Olkin bivariate exponential distribution with survival function

$$P(X \ge x, Y \ge y) = \begin{cases} e^{-x-(1+\lambda)y} & ,0 \le x \le y \\ e^{-y-(1+\lambda)x} & ,0 \le y \le x \end{cases}$$

where $\lambda > 0$, Nadarajah and Mitov (2003) calculated raw product moment of general order from which it is possible to calculate skewness and kurtosis of the distribution. Kotz, Balakrishnan and Johnson (2000, 82) mentioned that the coefficient of kurtosis of the distribution is given by $\beta_2 = 2(1+\rho)^{-3}(3\rho^4 + 9\rho^3 + 15\rho^2 + 12\rho + 4)$ where the correlation coefficient ρ is given by $(\lambda+2)\rho = \lambda$. They also mentioned that in case $\rho = 0$, the components X and Y become independent, in which case $\beta_2 = 8$ (which is the same as that of the bivariate normal distribution). Interested readers may go through Kotz, Nadarajah and Mitov (2003) for a useful formula for product moments for univariate distributions.

We derive standardized moments in terms of centered product moments just to demonstrate the potential of an alternative way.

Theorem 2.1 Let $\mu(a,b)$ be centered product moments between X and Y. Then

$$(i)\ \left[\mu(2,0)\mu(0,2)-\mu^2(1,1)\right]^2 E(Q^2)$$
$$\begin{aligned}&=\mu(4,0)\mu^2(0,2)+\mu(0,4)\mu^2(2,0)\\&+4\mu^2(1,1)\mu(2,2)+2\mu(2,0)\mu(0,2)\mu(2,2)\\&-4\mu(0,2)\mu(1,1)\mu(3,1)-4\mu(2,0)\mu(1,1)\mu(1,3),\end{aligned}$$

$$(ii)\left[\mu(2,0)\mu(0,2)-\mu^2(1,1)\right]^3 E(Q^3)$$
$$\begin{aligned}&=\mu(6,0)\mu^3(0,2)+\mu^3(2,0)\mu(0,6)\\&-6\mu^2(0,2)\mu(1,1)\mu(5,1)-6\mu^2(2,0)\mu(1,1)\mu(1,5)\\&+12\mu(0,2)\mu^2(1,1)\mu(4,2)+12\mu(2,0)\mu^2(1,1)\mu(2,4)\\&+3\mu(2,0)\mu^2(0,2)\mu(4,2)+3\mu^2(2,0)\mu(0,2)\mu(2,4)\\&-8\mu^3(1,1)\mu(3,3)-12\mu(2,0)\mu(0,2)\mu(1,1)\mu(3,3).\end{aligned}$$

Proof. From (2.1) we have

$$Q=(X-\xi\ \ Y-\theta)\begin{pmatrix}\mu(2,0) & \mu(1,1)\\ \mu(1,1) & \mu(0,2)\end{pmatrix}^{-1}\begin{pmatrix}X-\xi\\ Y-\theta\end{pmatrix},$$

which can be simplified as

$$\begin{aligned}&[\mu(2,0)\mu(0,2)-\mu^2(1,1)]Q\\&=\mu(0,2)(X-\xi)^2-2\mu(1,1)(X-\xi)(Y-\theta)+\mu(2,0)(Y-\theta)^2.\end{aligned}\tag{2.4}$$

By taking expected values in both sides of the above identity we have

$$\begin{aligned}&[\mu(2,0)\mu(0,2)-\mu^2(1,1)]E(Q)\\&=\mu(2,0)\mu(0,2)-2\mu^2(1,1)+\mu(0,2)\mu(2,0)\\&=\mu(2,0)\mu(0,2)-2\rho^2\mu(2,0)\mu(0,2)+\mu(0,2)\mu(2,0)\\&=2\mu(2,0)\mu(0,2)(1-\rho^2),\end{aligned}$$

i.e. $E(Q)=2$

which is generally true (see 2.2). By squaring both sides of (2.4) we have

$$
\begin{aligned}
&\left[\mu(2,0)\mu(0,2)-\mu^2(1,1)\right]^2 Q^2\\
&=\mu^2(0,2)(X-\xi)^4+4\mu^2(1,1)(X-\xi)^2(Y-\theta)^2+\mu^2(2,0)(Y-\theta)^4-4\mu(0,2)\mu(1,1)(X-\xi)^3(Y-\theta)\\
&+2\mu(2,0)\mu(0,2)(X-\xi)^2(Y-\theta)^2-4\mu(2,0)\mu(1,1)(X-\xi)(Y-\theta)^3.
\end{aligned}
$$

Then the result in (i) follows by taking expected values in both sides of the above identity. By cubing both sides of (2.4) we have:

$$
\begin{aligned}
&\left[\mu(2,0)\mu(0,2)-\mu^2(1,1)\right]^3 Q^3\\
&=\mu^3(0,2)(X-\xi)^6-8\mu^3(1,1)(X-\xi)^3(Y-\theta)^3+\mu^3(2,0)(Y-\theta)^6\\
&-6\mu^2(0,2)\mu(1,1)(X-\xi)^5(Y-\theta)+12\mu(0,2)\mu^2(1,1)(X-\xi)^4(Y-\theta)^2\\
&+3\mu(2,0)\mu^2(0,2)(X-\xi)^4(Y-\theta)^2+3\mu^2(2,0)\mu(0,2)(X-\xi)^2(Y-\theta)^4\\
&+12\mu(2,0)\mu^2(1,1)(X-\xi)^2(Y-\theta)^2-6\mu^2(2,0)\mu(1,1)(X-\xi)(Y-\theta)^5\\
&-12\mu(2,0)\mu(0,2)\mu(1,1)X-\xi)^3(Y-\theta)^3.
\end{aligned}
$$

Part (ii) follows by taking expected values of the above identity.

Corollary 2.1 Let $\mu(a,b)$ be the centered product moment and $(\mu(2,0)\mu(0,2))^{-1/2}\mu(1,1)=\rho$ be the correlation coefficient between X and Y . Then

$$
\begin{aligned}
&(i)\left[\mu(2,0)\mu(0,2)(1-\rho^2)\right]^2 E(Q^2)\\
&\quad=\mu(4,0)\mu^2(0,2)+\mu(0,4)\mu^2(2,0)+(4\rho^2+2)\mu(2,0)\mu(0,2)\mu(2,2)\\
&\quad-4\rho\left(\mu(2,0)\mu(0,2)\right)^{1/2}[\mu(0,2)\mu(3,1)+\mu(2,0)\mu(1,3)],
\end{aligned}
$$

$$
\begin{aligned}
&(ii)\left[\mu(2,0)\mu(0,2)(1-\rho^2)\right]^3 E(Q^3)\\
&\quad=\mu(6,0)\mu^3(0,2)+\mu^3(2,0)\mu(0,6)-\left(\mu(2,0)\mu(0,2)\right)^{3/2}\mu(3,3)4\rho(2\rho^2+3)\\
&\quad-6\left(\mu(2,0)\mu(0,2)\right)^{1/2}\rho[\mu^2(0,2)\mu(5,1)+\mu^2(2,0)\mu(1,5)]\\
&\quad+3\mu(2,0)\mu(0,2)(4\rho^2+1)\left[\mu(0,2)\mu(4,2)+\mu(2,0)\mu(2,4)\right].
\end{aligned}
$$

Corollary 2.2 Let X and Y have a bivariate distribution with $E(X^aY^b)=E(X^bY^a)$ and the correlation between them is $\rho_{XY}=\rho$. Then

$$
\begin{aligned}
&(i)\mu^2(0,2)(1-\rho^2)^2E(Q^2)=2\mu(4,0)+(4\rho^2+2)\mu(2,2)-8\rho\mu(3,1),\\
&(ii)\mu^3(0,2)(1-\rho^2)^3E(Q^3)=2\mu(6,0)-(8\rho^3+12\rho)\mu(3,3)-12\rho\mu(5,1)+(24\rho^2+6)\mu(4,2).
\end{aligned}
$$

Corollary 2.3 Let X and Y have a bivariate distribution. If X and Y are independent, then

$$(i)\,E(Q^2) = 2 + \frac{\mu(4,0)}{\mu^2(2,0)} + \frac{\mu(0,4)}{\mu^2(0,2)},$$

$$(ii)\,E(Q^3) = \frac{\mu(6,0)}{\mu^3(2,0)} + \frac{\mu(0,6)}{\mu^3(0,2)} + 3\left(\frac{\mu(4,0)}{\mu^2(2,0)} + \frac{\mu(0,4)}{\mu^2(0,2)}\right).$$

Corollary 2.4 Let X and Y have a bivariate distribution. If X and Y are independently and identically distributed, then

$$(i)\,E(Q^2) = 2\left(1 + \frac{\mu(4,0)}{\mu^2(2,0)}\right),$$

$$(ii)\,E(Q^3) = 2\left(\frac{\mu(6,0)}{\mu^3(2,0)} + 3\frac{\mu(4,0)}{\mu^2(2,0)}\right).$$

3. Centered Product Moments of the Bivariate Normal Distribution

The pdf of the bivariate normal distribution is given by

$$f_1(x,y) = \frac{(1-\rho^2)^{-1/2}}{2\pi\sigma_1\sigma_2}\,\exp\left(\frac{-q(x,y)}{2}\right), \qquad (3.1)$$

where

$$(1-\rho^2)q(x,y) = \left(\frac{x-\xi}{\sigma_1}\right)^2 + \left(\frac{y-\theta}{\sigma_2}\right)^2 - \frac{2\rho(x-\xi)(y-\theta)}{\sigma_1\sigma_2}.$$

The following theorem is due to Kendal and Stuart (1969, 91).

Theorem 3.1 The centered product moments $\mu(a,b) = E\left[(X-\xi)^a(Y-\theta)^b\right]$ of the bivariate normal distribution with pdf in (3.1) are given by

$$\mu(a,b) = \sigma_1^a\sigma_2^b\lambda(a,b) \text{ where}$$

$$\lambda(a,b) = (a+b-1)\rho\lambda(a-1,b-1) + (a-1)(b-1)(1-\rho^2)\lambda(a-2,b-2),$$

$$\lambda(2a,2b) = \frac{(2a)!(2b)!}{2^{a+b}}\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!},$$

$$\lambda(2a+1,2b+1) = \frac{(2a+1)!(2b+1)!}{2^{a+b}}\rho\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!},$$

$$\lambda(2a,2b+1) = \lambda(2a+1,2b) = 0.$$

The above can be rewritten as

$$\mu(a,b) = (a+b-1)\rho\sigma_1\sigma_2\mu(a-1,b-1) + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,b-2),$$

$$\mu(2a,2b)=\sigma_1^{2a}\sigma_2^{2b}\,\frac{(2a)!(2b)!}{2^{a+b}}\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!},$$

$$\mu(2a+1,2b+1)=\sigma_1^{2a+1}\sigma_2^{2b+1}\,\frac{(2a+1)!(2b+1)!}{2^{a+b}}\rho\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!},$$

$$\mu(2a,2b+1)=\mu(2a+1,2b)=0.$$

Product moments that are needed for standardized moments are provided below:

$$\mu(2,0)=\sigma_1^2,$$
$$\mu(0,2)=\sigma_2^2,$$
$$\mu(4,0)=3\sigma_1^4,$$
$$\mu(0,4)=3\sigma_2^4,$$
$$\mu(6,0)=15\sigma_1^6,$$
$$\mu(0,6)=3\sigma_1^6,$$

$$\mu(1,1)=\rho\sigma_1\sigma_2,$$
$$\mu(2,2)=(1+2\rho^2)\sigma_1^2\sigma_2^2,$$
$$\mu(1,3)=3\rho\sigma_1\sigma_2^3,$$
$$\mu(3,1)=3\rho\sigma_1^3\sigma_2,$$

$$\mu(2,4)=3(1+4\rho^2)\sigma_1^2\sigma_2^4,$$
$$\mu(3,3)=3\rho(3+2\rho^2)\sigma_1^3\sigma_2^3,$$
$$\mu(4,2)=3(1+4\rho^2)\sigma_1^4\sigma_2^2,$$
$$\mu(1,5)=15\rho\sigma_1\sigma_2^5$$
$$\mu(5,1)=15\rho\sigma_1^5\sigma_2.$$

4. Some Examples

(i) Bivariate Normal Distribution

Let us represent the bivariate normal distribution with pdf in (3.1) by,

$$W=\begin{pmatrix}X\\Y\end{pmatrix}\sim N_2(\mu,\Sigma),\ \mu=\begin{pmatrix}\xi\\\theta\end{pmatrix},\Sigma=\begin{pmatrix}\mu(2,0)&\mu(1,1)\\\mu(1,1)&\mu(0,2)\end{pmatrix}.$$

It is known that for a p-variate normal distribution, $W\sim N_p(\mu,\Sigma)$, the standardized distance $Q=(W-\mu)'\Sigma^{-1}(W-\mu)\sim\chi_p^2$ so that $\beta_1=E(Q)=p$, $\beta_2=E(Q^2)=p(p+2)$ and

$\beta_3 = E(Q^3) = p(p+2)(p+4)$. That is for the univariate normal distribution, $\beta_1 = 1, \beta_2 = 3, \beta_3 = 15$ and for the bivariate normal distribution,

$$\beta_1 = 2, \beta_2 = 8, \beta_3 = 48. \tag{4.1}$$

We derive the kurtosis and skewness for bivariate normal and bivariate *t*-distribution by the method developed in Section 2. It may be mentioned that the kurtosis and skeness for bivariate elliptical distributions are easily obtained, but these are not easy for other distributions as it is difficult to derive the distribution of standardized distance. As an example we outline kurtosis and skewness for bivariate chi-square distribution in Theorem 4.5.

Theorem 4.1 The second and the third order standardized moments of bivariate normal distribution are given by $\beta_2 = 8, \beta_3 = 48$.

Proof. By the use of moments from Section 3, it follows from Theorem 2.1 (i) or preferably Corollary 2.1(i) that

$$\begin{aligned}
&\left[\sigma_1^2\sigma_2^2(1-\rho^2)\right]^2 E(Q^2) \\
&= (3\sigma_1^4)(\sigma_2^2)^2 + (3\sigma_2^4)(\sigma_1^2)^2 + 4(\rho^2\sigma_1^2\sigma_2^2)[\sigma_1^2\sigma_2^2(1+2\rho^2)] \\
&- 4(\sigma_2^2)(\rho\sigma_1\sigma_2)(3\sigma_1^3\sigma_2\rho) - 4\sigma_1^2(\rho\sigma_1\sigma_2)(3\sigma_1\sigma_2^3\rho), \\
&+ 2\sigma_1^2\sigma_2^2[\sigma_1^2\sigma_2^2(1+2\rho^2)]
\end{aligned}$$

so that

$$\begin{aligned}
(1-\rho^2)^2 E(Q^2) &= 3 + 3 + 4\rho^2(1+2\rho^2) + 2(1+2\rho^2) - 4\rho(3\rho) - 4\rho(3\rho) \\
&= 8(1-2\rho^2+\rho^4).
\end{aligned}$$

Similarly by plugging in the moments from Section 3, it follows from Theorem 2.1 (ii) or preferably Corollary 2.1(ii) that

$$\begin{aligned}
&\left[\sigma_1^2\sigma_2^2(1-\rho^2)\right]^3 E(Q^3) \\
&= (15\sigma_1^6)(\sigma_2^2)^3 + (\sigma_1^2)^3(15\sigma_2^6) \\
&-8(\rho^3\sigma_1^3\sigma_2^3)[3\sigma_1^3\sigma_2^3\rho(3+2\rho^2)] - 12\sigma_1^2\sigma_2^2(\rho\sigma_1\sigma_2)[3\sigma_1^3\sigma_2^3\rho(3+2\rho^2)] \\
&-6\sigma_2^4(\rho\sigma_1\sigma_2)[15\sigma_1^5\sigma_2\rho] - 6\sigma_1^4(\rho\sigma_1\sigma_2)(15\sigma_1\sigma_2^5\rho) \\
&+12\sigma_2^2(\rho^2\sigma_1^2\sigma_2^2)[3\sigma_1^4\sigma_2^2(1+4\rho^2)] + 12\sigma_1^2(\rho^2\sigma_1^2\sigma_2^2)[3\sigma_1^2\sigma_2^4(1+4\rho^2)] \\
&+3\sigma_1^2(\sigma_2^2)^2[3\sigma_1^4\sigma_2^2(1+4\rho^2)] + 3(\sigma_1^2)^2(\sigma_2^2)[3\sigma_1^2\sigma_2^4(1+4\rho^2)]
\end{aligned}$$

so that

$$\begin{aligned}
&(1-\rho^2)^3 E(Q^3)\\
&=15+15-8\rho^3[3\rho(3+2\rho^2)]-12\rho[3\rho(3+2\rho^2)]\\
&-6\rho(15\rho)-6\rho(15\rho)+12\rho^2[3(1+4\rho^2)]+12\rho^2[3(1+4\rho^2)]\\
&+3[3(1+4\rho^2)]+3[3(1+4\rho^2)]\\
&=48(1-3\rho^2+3\rho^4-\rho^6).
\end{aligned}$$

(ii) **Bivariate T-Distribution**

Let $X' = (X_1, X_2)$ be the bivariate t-random vector with probability density function

$$f_2(x) = (2\pi)^{-1} |\Sigma|^{-1/2} \left(1 + (x-\theta)'(\nu\Sigma)^{-1}(x-\theta)\right)^{-\nu/2-1} \tag{4.1}$$

where $\theta' = (\theta_1, \theta_2)$ is an unknown vector of location parameters and Σ is the 2×2 unknown positive definite matrix of scale parameters while the scalar ν is assumed to be a known positive constant (Anderson, 2003, 289). For recent update on t-distributions see Kotz and Nadarajah (2005) and Kibria (2006) and the references therein.

The following theorem, due to Joarder (2006a), is needed to calculate kurtosis and skewness of the above bivariate *t*-distribution given by (4.1),

Theorem 4.2 The centered product moments of the bivariate *t*-distribution with pdf in (4.1) are given by

$$\mu(a,b;\nu) = (a+b-1)\rho\sigma_1\sigma_2\mu(a-1,b-1)\gamma_2 + (a-1)(b-1)(1-\rho^2)\sigma_1^2\sigma_2^2\mu(a-2,b-2)\gamma_4,$$

$$\mu(2a,2b;\nu) = \sigma_1^{2a}\sigma_2^{2b}\frac{(2a)!(2b)!}{2^{a+b}}\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j)!}\gamma_{2a+2b},$$

$$\mu(2a+1,2b+1;\nu) = \sigma_1^{2a+1}\sigma_2^{2b+1}\frac{(2a+1)!(2b+1)!}{2^{a+b}}\rho\sum_{j=0}^{\min(a,b)}\frac{(2\rho)^{2j}}{(a-j)!(b-j)!(2j+1)!}\gamma_{2a+2b+2},$$

$$\mu(2a,2b+1;\nu) = \mu(2a+1,2b;\nu) = 0$$

where $\gamma_a = \dfrac{(\nu/2)^{a/2}\Gamma(\nu/2-a/2)}{\Gamma(\nu/2)}, \quad \nu > a$

By the use of the above moments in Theorem 2.1 or preferably in Corollary 2.1, we have the following kurtosis and skewness of the bivariate *t*-distribution having pdf in (4.1):

$$\beta_2 = 8\frac{\nu-2}{\nu-4},\ \nu > 4,$$

$$\beta_3 = 48\frac{(\nu-2)^2}{(\nu-4)(\nu-6)},\ \nu > 6.$$

(iii) **Bivariate Elliptical Distribution**

The kurtosis and skewness are calculated for p-variate elliptical distribution. With $p = 2$, the results boil down to bivariate elliptical distribution. Consider the multivariate elliptical distribution with pdf

$$f(x) = g((x-\mu)'\Sigma^{-1}(x-\mu)), \tag{4.2}$$

where x is a p-dmensional column vector with mean $E(X) = \mu$ and the covariance matrix $Cov(X) = p^{-1}E(R^2)\Sigma$ where $R^2 = Z'Z$ and $Z = \Sigma^{-1/2}(X-\mu)$. Then we have the following theorem (cf. Anderson, 2003, 103):

Theorem 4.3 Let X have the multivariate elliptical distribution with pdf in (4.2). Then the kurtosis and skewness of the distribution are given by

$$\beta_2 = E(Q^2) = p^2\frac{E(R^4)}{E^2(R^2)}, \text{ and}$$
$$\beta_3 = E(Q^3) = p^3\frac{E(R^6)}{E^3(R^2)}$$

respectively, where $R^2 = Z'Z$ and $Z = \Sigma^{-1/2}(X-\mu)$.

Proof. The covariance matrix of the elliptical distribution is given by $Cov(X) = p^{-1}E(R^2)\Sigma$ so that the standardized distance is given by $Q = (X-\mu)'(p^{-1}E(R^2)\Sigma)^{-1}(X-\mu)$. Then the theorem is obvious by virtue of

$$Q = \frac{pR^2}{E(R^2)},\ Q^2 = \frac{p^2R^4}{E^2(R^2)},\ Q^3 = \frac{p^3R^6}{E^3(R^2)}.$$

Note that if the form of $g(.)$ is known, the kurtosis and skewness can be calculated by the pdf of R given by

$$h(r) = \frac{2\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1}\, g(r^2),\ 0 < r.$$

It is well known that for the multivariate normal distribution $R^2 \sim \chi^2_p$, and for the multivariate *t*-distribution with pdf

$$f_1(x)=\frac{\Gamma((\nu+p)/2)}{\Gamma(\nu/2)(\nu\pi)^{p/2}}|\Sigma|^{-1/2}\left(1+(x-\theta)'(\nu\Sigma)^{-1}(x-\theta)\right)^{-\nu/2-1},\ \nu>2,$$

we have $p^{-1}R^2 \sim F(p,\nu)$.

(iv) Bivariate Chisquare Distribution

The results in this section are due to Joarder (2006b).

Theorem 4.4 The random variables U and V are said to have a correlated bivariate chisquare distribution each with m degrees of freedom, if its probability density function is given by

$$f_3(u,v)=\frac{(uv)^{m/2-1}\ e^{\frac{-(u+v)}{2(1-\rho^2)}}}{2^m\sqrt{\pi}\ \Gamma(m/2)\left(1-\rho^2\right)^{m/2}}\sum_{k=0}^{\infty}\left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^k\frac{\Gamma\left((k+1)/2\right)}{k!\Gamma\left((k+m)/2\right)},$$

$m=N-1>2$, $-1<\rho<1$.

Theorem 4.5 For $m>0$, the second and the third standardized moments of the bivariate chisquare distribution with pdf in Theorem 4.4 are given by

$$(i)m(1-\rho^4)^2E(Q^2)=8m(1-\rho^4)^2+8(3+2\rho^2-11\rho^4+4\rho^6+2\rho^8),$$
$$(ii)m^2(1+\rho^2)^3E(Q^3)=48(1+\rho^2)^3m^2+16(18\rho^6+126\rho^4+135\rho^2+37)m$$
$$+192(5+16\rho^2+18\rho^4+2\rho^6).$$

In case $\rho=0$, the pdf of the joint probability distribution in Theorem 4.4, would be that of the product of two independent chisquare random variables i.e.,

$$f(u,v)=\frac{\left(uv\right)^{m/2-1}e^{-(u+v)/2}}{2^m\ \Gamma^2(m/2)},\ u>0, v>0.$$

which is the product of the density function of two independent chisquare variables $U\sim\chi^2_m$ and $V\sim\chi^2_m$.

Corollary 4.1 For independent bivariate chisquare distribution with pdf given above we have

$$(i)E(Q^2)=8+(24/m),$$
$$(ii)E(Q^3)=48+(592/m)+(960/m^2).$$

These moments coincide, as expected, to that of the bivariate normal distribution as m tends to infinity (See Theorem 4.1).

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References

1. Anderson, T.W. (2003). *An Introduction to Multivariate Statistical Analysis*. John Wiley, New York.

2. Joarder, A.H. (2006a). Moments of the bivariate *t*-distribution. Technical Report No. 346, Depratment of Mathematical Sciences, King Fahd University of Petroleum & Minerals, Saudi Arabia.

3. Joarder, A.H. (2006b). Mahalanobis moments of the bivariate chisquare distribution. Technical Report No. 350, Depratment of Mathematical Sciences, King Fahd University of Petroleum & Minerals, Saudi Arabia.

4. Johnson, N.L.; Kotz, S. and Balakrishnan, N. (1997). *Discrete Multivariate Distributions*. John Wiley.

5. Johnson, N.L.; Kotz, S. and Kemp, A.W. (1993). *Univariate Discrete Distributions*. John Wiley.

6. Kendal, M.G. and Stuart, A. (1969). *The Advanced Theory of Statistics.* Charles Griffin and Co. Ltd, London.

7. Kibria, B. M. G. (2006). The Matric t Distribution and its Applications in Predictive Inference. *Journal of Multivariate Analysis*, 97, (3), 785-795.

8. Kotz, S.; Balakrishnan, N. and Johnson, N.L. (2000). *Continuous Multivariate Distributions*. John Wiley and Sons, New York.

9. Kotz, S. ; Nadarajah, S. and Mitov, K.V. (2003). A useful formula for product moments, Student, 4(4), 239-244.

10. Kotz, S. and Nadarajah (2005). *Multivariate t Distributions and Their Applications,* Cambridge University Press.

11. Mardia, K.V. (1970a). A translation family of bivariate distributions and Frechet's bounds. *Sankhya*, 32A, 119-121.

12. Mardia, K.V. (1970b). *Families of Bivariate Distributions*. London: Griffin.

13. Mardia, K.V. (1970c). Measures of multivariate skewness and Kurtosis with applications. *Biometrika,* 57, 59-530.

14. Mardia, K.V. (1974). Applications of some measures of multivariate skewnewss and kurtosis for testing normality and robustness studies. *Sankhya*, 36B, 115-128.

15. Mardia, K.V. (1975). Assessment of multinormality and the robustness of the Hotelling's T^2 test. *Applied Statistics*, 24, 163-171.

16. Nadarajah, S. and Mitov, K. (2003). Product moments of multivariate random vectors.

Commuincations in Statistics – Theory and Methods, 32(1), 47-60.

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