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ABSTRACT

In this paper, we study the Bayesian estimation of functions of parameters of some power series distributions. These estimators are better than the classical minimum variance unbiased estimators (MVUE) as given by Patil and Joshi (1970), in the sense that these increase the range of the estimation and also have simpler forms.

Key Words : Generalized power series distribution, squared error loss function, weighted squared error loss function, likelihood function, minimum variance unbiased estimation, Bayesian estimation.

AMS Subject Classification 2000: 62F10, 62F15

1. Introduction

Patil (1962) defined a generalized power series distribution (GPSD) if its probability mass function (pmf) is given by

$$P(X = x) = \frac{a_x \ \theta^x}{g(\theta)}, \qquad x \in S \tag{1.1}$$

where $g(\theta)$ is a generating function i.e.,

$$g(\theta) = \sum_{x \in s} a_x \theta^x , \qquad \theta \geq 0, a_x \geq 0$$

so that $g(\theta)$ is positive, finite and differentiable and S is a non-empty countable sub-set of non-negative integers.

It can be easily seen that proper choice of S and $g(\theta)$, the GPSD model (1.1) reduces to the binomial, negative binomial, Poisson and logarithmic series distributions. When $g(\theta) = \theta$, the GPSD model (1.1) coincides with the class of distributions considered by Roy and Mitra (1957). In a series of papers, Patil (1957, 1959, 1961, 1962) has investigated some structural properties and statistical problems associated with GPSD. Patil (1962a) obtained the maximum likelihood estimation of GPSD whereas Patil (1962b) obtained estimation by two moment method for GPSD. Patil (1963) and Patil and Joshi (1970) studied properties associated with minimum variance unbiased estimation (MVUE) for power series distributions.

In this paper we study the Bayesian estimation of GPSD for proper choice of S and $g(\theta)$. These estimators are better than the classical minimum variance unbiased estimators (MVUE) as given by Patil (1963), and Patil and Joshi (1970) in the sense that these increase the range of the estimation and also have simpler forms. In particular, we derive the Bayesian estimator of $\phi(\theta) = \theta^r, r \in (-\infty, \infty)$. Note that the range of estimation is increased as we have taken $r \in (-\infty, \infty)$.

For Bayesain estimation of discrete distributions we refer, among others, to Irony (1982) and Howlader and Balasooriya (2003).

2. Some Preliminaries

Let $X_1, X_2, \cdots X_N$ denote a random sample of size N from a given pmf, then

$$T_N = \sum_{i=1}^{N} X_i . \tag{2.1}$$

we shall use the following result as given Abranowitz and Stegun (1964)

$$\Gamma(x)=\int_0^\infty u^{x-1}e^{-u}du\ , \tag{2.2}$$

$$\Gamma(x)\ b^{-x}=\int_0^\infty u^{x-1}e^{-bu}du\ , \tag{2.3}$$

$$\frac{\Gamma(b-a)\ \Gamma(a)\ M(a,b;z)}{\Gamma(b)}=\int_0^1 t^{a-1}(1-t)^{b-a-1}e^{zt}dt\ , \tag{2.4}$$

where $M(a,b;z)$ is the confluent hypergeometric function and has a series representation given by

$$M(a,b;z)=\sum_{j=0}^{\infty}\frac{(a)_j\,z^j}{(b)_j\ j!},\quad \text{where } (a)_0=1 \text{ and} \tag{2.5}$$

$$(a)_n=a(a+1)(a+2)\cdots(a+n-1)\,. \tag{2.6}$$

3. Bayesian Estimation

Let $X_1,X_2,\cdots,X_N$ be a random sample of size N from (1.1). The likelihood function of the random sample denoted by $L(\theta)$ is given by

$$\begin{aligned} L(\theta)&=\prod_{i=1}^{N}\frac{a_{x_i}\theta^{x_i}}{g(\theta)}\\ &=\theta^{t_N}\left(g(\theta)\right)^{-N}\prod_{i=1}^{N}a_{x_i}\end{aligned}$$

$$\text{i.e. } L(\theta)\propto\theta^{t_N}\left(g(\theta)\right)^{-N} \tag{3.1}$$

where t_N is an observed value of T_N defined by (2.1). The posterior pdf of θ, corresponding to prior $h(\theta)$ is given by

$$\pi(\theta \mid t_N) = \frac{L(\theta)\ h(\theta)}{\int_0^\infty L(\theta)\ h(\theta)\ d\theta}. \tag{3.2}$$

Under the Squared Error Loss Function (SELF) given by $L\left(\phi(\theta), d\right) = \left(\phi(\theta) - d\right)^2$, where $\phi(\theta)$ is a function of θ and d is a decision, the Bayes Estimate $\hat{\phi}_\beta$ of $\phi(\theta)$ is given by

$$\hat{\phi}_\beta = \int_0^\infty \phi(\theta)\pi\left(\theta/t_N\right) d\theta. \tag{3.3}$$

Similarly, under the Weighted Squared Error Loss Function (WSELF) given by $L\left(\phi(\theta), d\right) = w(\theta)\left(\phi(\theta) - d\right)^2$, the Bayes Estimate $\hat{\phi}_w$ of $\phi(\theta)$ is given by

$$\hat{\phi}_w = \frac{\int_0^\infty w(\theta)\phi(\theta)\pi(\theta/t_N)d\theta}{\int_0^\infty w(\theta)\pi(\theta/t_N)d\theta}. \tag{3.4}$$

We consider two different forms of $w(\theta)$ as given below:

(i) Let $w(\theta) = \theta^{-2}$. The Bayes Estimate $\hat{\phi}_M$ of $\phi(\theta)$ known as the Minimum Expected Loss (MEL) estimate is given by

$$\hat{\phi}_M = \frac{\int_0^\infty \theta^{-2}\phi(\theta)\pi(\theta \mid t_N)\ d\theta}{\int_0^\infty \theta^{-2}\pi(\theta \mid t_N)\ d\theta}. \tag{3.5}$$

This loss function was used by Tammala and Sath (1978) for estimating the reliability of certain life time distributions and by Zellner (1979) for estimating functions of parameters of some econometric models.

(ii) Let $w(\theta) = \theta^{-2}e^{-a\theta}, a > 0$. The Bayes Estimate $\hat{\phi}_E$ of $\phi(\theta)$ known as the Exponentially

Weighted Maximum Expected Loss (EWMEL) estimate and is given by

$$\hat{\phi}_E = \frac{\int_0^\infty \theta^{-2} e^{-a\theta} \phi(\theta) \pi(\theta \mid t_N) d\theta}{\int_0^\infty \theta^{-2} e^{-a\theta} \pi(\theta \mid t_N) d\theta}. \tag{3.6}$$

Now we shall consider some special cases of the p.m.f given by (1.1) and obtain the corresponding Bayesian estimate in each case.

4. Poisson Distribution

A random variable X is said to have Poisson distribution if its p.m.f is given by

$$P_\theta(x) = \begin{cases} \dfrac{\theta^x e^{-\theta}}{x\,!}, & x \in S, \quad \theta > 0, \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

where $S = \{0,1,2,\cdots\}$. It is a special case of (1.1) when $a_x = \dfrac{1}{x!}$ and $g(\theta) = e^{-\theta}$.

In this case the likelihood function $L(\theta)$ is given by

$$L(\theta) \propto (\theta)^{t_N} e^{-\theta N}. \tag{4.2}$$

With the gamma prior for θ given by

$$h(\theta) = \frac{\alpha^\beta}{\Gamma\beta} e^{-\alpha\theta} \theta^{\beta-1}, \quad \alpha, \beta, \theta > 0, \tag{4.3}$$

the posterior pdf of θ is given by

$$\begin{aligned} \pi(\theta \mid t_N) &= \frac{L(\theta) h(\theta)}{\int_0^\infty L(\theta) h(\theta) d\theta} \\ &= \frac{(N+\alpha)^{t_N+\beta}}{\Gamma(t_N+\beta)} (\theta)^{(t_N+\beta)-1} e^{-(N+\alpha)\theta}. \end{aligned} \tag{4.4}$$

Under the Squared Error Loss Function (SELF) given by $L\left(\phi(\theta), d\right) = \left(\phi(\theta) - d\right)^2$ the Bayes Estimate $\hat{\phi}_B$ of $\phi(\theta)$ is given by

$$\begin{aligned}\hat{\phi}_\beta^r &= \int_0^\infty \phi(\theta)\pi(\theta \mid t_N)\,d\theta \\ &= \frac{\Gamma(t_N + \beta + r)}{\Gamma(t_N + \beta)\,(N + \alpha)^r}.\end{aligned} \tag{4.5}$$

Note that if $r = 1$, then we get the Bayes estimator of θ as

$$\theta^* = \frac{\Gamma(t_N + \beta + 1)}{\Gamma(t_N + \beta)\,(N + \alpha)} = \frac{t_N + \beta}{(N + \alpha)} = \frac{\Sigma x_i + \beta}{N + \alpha} \tag{4.6}$$

which is also identical to maximum likelihood estimator (MLE) of θ if $\alpha = \beta = 0$. Similarly under WSELF, when $w(\theta) = \theta^{-2}$, the MEL Estimate of $\phi(\theta) = \theta^r$ is given by

$$\begin{aligned}\hat{\theta}_M^r &= \frac{\int_0^\infty \theta^{-2}.\theta^r \dfrac{(N + \alpha)^{t_N + \beta}}{\Gamma(t_N + \beta)} \cdot (\theta)^{(t_N + \beta) - 1} e^{-(N + \alpha)\theta}\,d\theta}{\int_0^\infty \theta^{-2} \dfrac{(N + \alpha)^{t_N + \beta}}{\Gamma(t_N + \beta)} (\theta)^{(t_N + \beta) - 1} e^{-(N + \alpha)\theta}\,d\theta} \\ &= \frac{\Gamma(t_N + \beta + r - 2)}{\Gamma(t_N + \beta - 2)\,(N + \alpha)}.\end{aligned} \tag{4.7}$$

Finally under WSELF $w(\theta) = \theta^{-2} e^{-a\theta},\ a > 0,$ the EWMEL estimate of $\phi(\theta) = \theta^r$ is given by

$$\begin{aligned}\hat{\theta}_E{}^r &= \frac{\int_0^\infty \theta^{-2} e^{-a\theta} \theta^r \dfrac{(N + \alpha)^{t_N + \beta}}{\Gamma(t_N + \beta)}\ \theta^{t_N + \beta - 1} e^{-(N + \alpha)\theta}\,d\theta}{\int_0^\infty \theta^{-2} e^{-a\theta} \dfrac{(N + \alpha)^{t_N + \beta}}{\Gamma(t_N + \beta)}\ \theta^{t_N + \beta} e^{-(N + \alpha)\theta}\,d\theta} \\ &= \frac{\Gamma(t_N + \beta + r - 2)}{\Gamma(t_N + \beta - 2)\,(N + \alpha + a)^r}\end{aligned} \tag{4.8}$$

If $r = 1$, we get Bayes estimator of θ for (4.7) and 4.8). Bayes estimator of θ^r, in each case can be obtained by replacing t_N by T_N, in each equations of Bayes estimator.

Note that the MVUE of θ^r , exists as long as $z \geq r$ and is zero (0) if $z < r$ which is a serious limitation of the MVUE. The Bayes estimates, on the other hand, as given above, are free from such restrictions on t_N and r, as long as $r \geq 1$. This is another advantage of Bayesian estimation over the MVUE. However, if $r < 0$, Bayes estimates are zero (0), if $t_N + \beta < -r$ in (4.5) and $t_N + \beta - 2 < -r$ in (4.7) and (4.8) respectively.

5.1 Negative Binomial Distribution

A discrete random variable X is said to have Negative Binomial Distribution if its pmf is given by

$$P_\theta(x) = \binom{n+x-1}{x} \theta^x (1-\theta)^n \tag{5.1}$$

which is a special case of GPSD (1.1) whenever

$$a(x) = \binom{n+x-1}{x} \quad , \qquad \mathrm{g}(\theta) = (1-\theta)^{-n}, \; x \in S = \{0,1,\cdots,\infty\}, 0 < \theta < 1.$$

The likelihood function $L(\theta)$, in this case, is given by

$$L(\theta) \;\alpha\; (\theta)^{t_N} (1-\theta)^{nN}. \tag{5.2}$$

Since $0 < \theta < 1$, we take two different prior distributions given below

$$h_1(\theta) = \frac{\theta^{p-1}(1-\theta)^{q-1}}{B(p,q)}, \; 0 < \theta < 1, \; p,q > 0 \tag{5.3}$$

where $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ and

$$h_2(\theta) = \frac{e^{-b\,\theta}\theta^{p-1}(1-\theta)^{q-1}}{B(p,q)\; M(p, p+1; -b)}, \text{ if } \; 0 < \theta < 1, \; p,q > 0 \tag{5.4}$$

with $M(a,b,z)$ given by (2.4).

The posterior p.d.f of θ, corresponding to $h_1(\theta)$ is given by

$$\pi_1(\theta \mid t_N) = \frac{L(\theta)h_1(\theta)d\theta}{\int_0^1 L(\theta)h_1(\theta)d\theta} = \frac{(\theta)^{(t_N+p)-1}(1-\theta)^{(nN+1)-1}}{B(t_N+p, nN+q)} . \tag{5.5}$$

The posterior p.d.f of θ, corresponding to $h_2(\theta)$ is given by

$$\begin{aligned} \pi_2(\theta \mid t_N) &= \frac{L(\theta)h_2(\theta)d\theta}{\int_0^1 L(\theta)h_2(\theta)d\theta} \\ &= \frac{(\theta)^{(t_N+p)-1}(1-\theta)^{(nN+q)-1}e^{-b\theta}}{B(t_N+p, nN+q)\, M(t_N+p, p+q+t_N+nN; -b)} . \end{aligned} \tag{5.6}$$

Both $h_1(\theta)$ and $h_2(\theta)$ are natural conjugate prior density. The prior density $h_2(\theta)$ is known as the generalized beta density considered by Holla (1968) and Bhattacharya (1968). Under SELF, the Bayes Estimate of θ^r, corresponding to posterior density (5.5) is given by

$$\begin{aligned} \hat{\theta}^r_{1B} &= \int_0^1 \theta^r \pi_1(\theta / t_N)\, d\theta \\ &= \frac{1}{B(t_N+p,\ nN+q)} \int_0^1 \theta^r (\theta)^{(t_N+p)-1}(1-\theta)^{(nN+q)-1} d\theta \\ &= \frac{B(t_N+p+r,\ nN+q)}{B(t_N+p,\ nN+q)} . \end{aligned} \tag{5.7}$$

Similarly under the WSELF, when $w(\theta) = \theta^{-2}$, the MEL Estimate of θ^r, corresponding to posterior density (5.5) is given by

$$\begin{aligned} \hat{\theta}^r_{1M} &= \frac{\int_0^1 \theta^r \theta^{-2} \frac{\theta^{(t_N+p)-1}(1-\theta)^{(nN+q)-1}}{B(t_N+p, nN+q)} d\theta}{\int_0^1 \theta^{-2} \frac{\theta^{(t_N+p)-1}(1-\theta)^{(nN+q)-1}}{B(t_N+p, nN+q)} d\theta} \\ &= \frac{B(t_N+p+r-2, nN+q)}{B(t_N+p-2, nN+q)} . \end{aligned} \tag{5.8}$$

Finally under the WSELF, when $w(\theta) = \theta^{-2}e^{-a\theta}, a > 0$, the EWMEL Estimate of θ^r, corresponding to Posterior density (5.5) is given by

$$\hat{\theta}_{1E}^{r}=\frac{\int_0^1 \theta^{-2}e^{-a\theta}\theta^{r}\,\frac{\theta^{(t_N+p)-1}(1-\theta)^{(nN+q)-1}d\theta}{B(t_N+p,nN+q)}}{\int_0^1 \theta^{-2}e^{-a\theta}\,\frac{\theta^{(t_N+p)-1}(1-\theta)^{(nN+q)-1}d\theta}{B(t_N+p,nN+q)}}$$

$$=\frac{B(t_N+p+r-2,nN+q)\;M(t_N+p+r-2,\,p+q+r-2+t_N+nN,-a)}{B(t_N+p-2,nN+q)M(t_N+p-2,\,p+q+t_N+nN-2,-a)} \qquad (5.9)$$

Also under SELF, the Bayes estimate of θ^{r}, corresponding to posterior density (5.6) is given by

$$\hat{\theta}_{2B}=\int_0^1\frac{\theta^{r}(\theta)^{(t_N+p)-1}(1-\theta)^{(nN+q)-1}e^{-b\theta}\cdot d\theta}{B(t_N+p,nN+q)\;M(t_N+p,\,p+q+t_N+nN,-b)}$$
$$=\frac{B(t_N+p+r,nN+q)\;M(t_N+p+r,\,nN+p+q+t_N+r,-b)}{B(t_N+p,nN+q)\;M(t_N+p,\,p+q+t_N+nN,-b)}. \qquad (5.10)$$

Similarly under the WSELF, when $w(\theta)=\theta^{-2}$, the MEL estimate of $\phi(\theta)=\theta^{r}$ is given by

$$\hat{\theta}_{2B}^{r}=\frac{\int_0^1\theta^{-2}\cdot\theta^{r}\theta^{(t_N+p)-1}(1-\theta)^{(nN+q)-1}e^{-b\theta}d\theta}{\int_0^1\theta^{-2}\cdot\theta^{(t_N+p)-1}(1-\theta)^{(nN+q)-1}e^{-b\theta}d\theta} \qquad (5.11)$$
$$=\frac{B(t_N+p+r-2,nN+q)\;M(t_N+p+r-2,\,nN+p+q+t_N+r-2,-b)}{B(t_N+p-2,nN+q)\;M(t_N+p-2,\,nN+p+q+t_N-2,-b)}.$$

Finally, under the WSELF $w(\theta)=\theta^{-2}e^{-a\theta}, a>0$, the EWEL estimate of θ^{r} corresponding to posterior density (5.6) is given by

$$\hat{\theta}_{2E}^{r}=\frac{\int_0^1\theta^{-2}e^{-a\theta}\cdot\theta^{r}\theta^{(t_N+p)-1}(1-\theta)^{(nN+q)-1}e^{-b\theta}d\theta}{\int_0^1\theta^{-2}e^{-a\theta}\cdot\theta^{(t_N+p)-1}(1-\theta)^{(nN+q)-1}e^{-b\theta}d\theta} \qquad (5.12)$$
$$=\frac{B(t_N+p+r-2,nN+q)\;M(t_N+p+r-2,\,p+q+nN+t_N+r-2,-(a+b))}{B(t_N+p-2,nN+q)\;M(t_N+p+r-2,\,p+q+nN+t_N+r-2,-(a+b))}.$$

Note that the MVUE of θ^{r} is zero (0) if $z<r$ which is a serious limitation of

MVUE. The Bayes estimates, on the other hand, are 0, if $r < 0$ such that $t_N + p < -r,\ t_N + p - 2 < -r,\ p + q + t_N + nN < -r,\ p + q + t_N + nN - 2 < -r,$ depending upon the various loss functions. For $n = 1$, we get the estimate for the geometric series distribution.

6. Logarithmic Series Distribution

A random X is said to follow logarithmic series distribution if its pmf is given by

$$P(X = x) = \frac{1}{x}\frac{\theta^x}{\left(-\log(1-\theta)\right)}\ , \quad x = 1, 2, \cdots, \tag{6.1}$$

which is a special case of generalized power series (1.1) whenever $a(x) = 1/x$ and $g(\theta) = -\log(1-\theta)$. In this case the likelihood function $L(\theta)$ is given by

$$L(\theta)\alpha \ \ \theta^{t_N}\left[-\log(1-\theta)\right]^{-N}. \tag{6.2}$$

With the prior density given by

$$h(\theta) = \begin{cases} \dfrac{(\lambda+1)^{N+1}(1-\theta)^{\lambda}\left(-\ln(1-\theta)\right)^{N}}{\Gamma(N+1)}, & (0 < \theta < 1, \ \ \lambda > 0 \\ 0, & \text{otherwise} \end{cases} \tag{6.3}$$

the posterior density would be

$$\pi(\theta \mid t_N) = \frac{L(\theta)h(\theta)\ d\theta}{\int_0^{\infty} L(\theta)h(\theta)\ d\theta} = \frac{\theta^{t_N}(1-\theta)^{\lambda}}{B(t_N+1, \lambda+1)}. \tag{6.4}$$

Under SELF, the Bayes Estimate of θ^r, denoted by

$$\hat{\theta}_B^r = \int_0^1 \theta^r \cdot \pi(\theta / t_N)d\theta = \frac{B(t_N + r + 1,\ k+1)}{B(t_N + 1,\ k+1)} \tag{6.5}$$

Similarly under the WSELF, when $w(\theta) = \theta^{-2}$, the MEL estimate of $\phi(\theta) = \theta^r$ is given by

$$\hat{\theta}_M^r = \frac{\int_0^1 \frac{\theta^{-2} \cdot \theta^r \cdot \theta^{t_N} (1-\theta)^{\lambda} d\theta}{B(t_N+1, \lambda+1)}}{\int_0^1 \frac{\theta^{-2} \cdot \theta^{t_N} (1-\theta)^{\lambda} d\theta}{B(t_N+1, \lambda+1)}} = \frac{B(t_N+r-1, \lambda+1)}{B(t_N-1, \lambda+1)} \tag{6.6}$$

Finally under the WSELF, when $w(\theta) = \theta^{-2} e^{-a\theta}, a > 0$ the EWMEL estimate of $\phi(\theta) = \theta^r$ is given by

$$\hat{\theta}_E^r = \frac{\int_0^1 \frac{\theta^{-2} \cdot e^{-2\theta} \cdot \theta^r \cdot \theta^{t_N} (1-\theta)^{\lambda} d\theta}{B(t_N+1, \lambda+1)}}{\int_0^1 \frac{\theta^{-2} e^{-a\theta} \theta^{t_N} (1-\theta)^{\lambda} d\theta}{B(t_N+1, \lambda+1)}} = \frac{B(t_N+r-1, \lambda+1)\, M(t_N+r-1, t_N+r+\lambda, -a)}{B(t_N-1, \lambda+1)\, M(t_N-1, t_N+\lambda, -a)}. \tag{6.7}$$

Note that the MVUE of θ^r is 0 if $z < r$. The Bayes estimates, on the other hand, are 0, if $r < 0$ such that $t_N + 1 < -r, t_N - 1 < -r, t_N + k < -r$, depending upon the various hypergeometric functions $M(t_N + r - 1, t_N + r + k, -a)$.

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