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Abstract

In this paper we consider the semilinear viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0,$$

in a bounded domain, and establish a general decay estimate for weak solutions. This result generalizes and improves earlier ones in the literature.

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1 Introduction

In this paper we consider the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int\limits_0^t g(t-\tau)\Delta u(x,\tau)d\tau = 0, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \quad x \in \partial\Omega\ , \quad t \geq 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function defined on $\mathbb{R}^+$. Cavalcanti *et al.* [5] studied (1.1) in the presence of a localized damping cooperating with the dissipation induced by the viscoelastic term. Under the condition

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t),\ t \geq 0,$$

with $||g||_{L^1((0,\infty))}$ small enough, they obtained an exponential rate of decay. Berrimi *et al.* [2] improved Cavalcanti's result by showing that the viscoelastic dissipation alone is enough to stabilize the system. To achieve their goal, Berrimi e*t al.* introduced a different functional, which allowed them to weaken the conditions on g as well as on the localized damping. This result has been later extended to a situation, where a source is competing with the viscoelastic dissipation, by Berrimi *et al.* [2]. Cavalcanti *et al.* [6] considered

$$u_{tt} - k_0 \Delta u + \int_0^t div[a(x)g(t-\tau)\nabla u(\tau)]d\tau + b(x)h(u_t) + f(u) = 0,$$

under similar conditions on the relaxation function g and $a(x) + b(x) \geq \delta > 0$ and improved the result in [5]. They established an exponential stability when g is decaying exponentially and h is linear and a polynomial stability when g is decaying polynomially and h is nonlinear. Though both results in [1] and [6] improve the earlier one in [5], the approaches are different. Another problem, where the damping induced by the viscosity is acting on the domain and a part of the boundary, was also discussed by Cavalcanti *et al.* [5]. An xistence and uniform decay rate results were established. A related problem, in a bounded domain, of the form

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma \Delta u_t = 0, \tag{1.2}$$

for $\rho > 0$, was also studied by Cavalcanti *et al.* [4]. A global existence result for $\gamma \geq 0$, as well as an exponential decay for $\gamma > 0$, has been established. This last result has been extended to a situation, where a source term is competing with the strong mechanism damping and the one induced by the viscosity, by Messaoudi and Tatar [8]. In their work, Messaoudi and Tatar combined the well depth method with the perturbation techniques to show that solutions with positive, but small, initial energy exist globally and decay to the rest state exponentially. Furthermore, Messaoudi and Tatar [11], [12] considered (1.2), for $\gamma = 0$, and established exponential and polynomial decay results in the absence, as well as in the presence, of a source term. We also mention the work of Kawashima and Shibata [7], in which a global existence and exponential stability of small solutions to a nonlinear viscoelastic problem has been established.

For nonexistence, Messaoudi [9] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + au_t|u_t|^m = b|u|^\gamma u, \quad \text{in } \Omega \times (0,\infty)$$

and showed, under suitable conditions on g, that solutions with negative energy blow up in finite time if $\gamma > m$ and continue to exist if $m \geq \gamma$. This blow-up result has been pushed to some situations, where the initial energy is positive, by Messaoudi [10]. A similar result has been also obtained by Wu [13] using a different method.

In the present work we generalize our earlier decay result to solutions of (1.1). Precisely, we show that the solution energy decays at a similar rate of decay of the relaxation function, which is not necessarily decaying in a polynomial or exponential fashion. In fact, our result allows a larger class of relaxation functions. The paper is organized as follows. In Section 2, we present some notations and material needed for our work and state a global existence theorem, which can be proved following exactly the arguments of [5]. Section 3 contains the statement and the proof of our main result.

2 Preliminaries

In this section we present some material needed in the proof of our main result. Also, for the sake of completeness we state, without a proof, the global existence result of [5] and [6]. We use the standard Lebesgue space $L^p(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms.

For the relaxation function g we assume
(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l > 0.$$

(G2) There exists a differentiable function ξ satisfying

$$g'(t) \leq -\xi(t)g(t), \ t \geq 0,$$

$$\left|\frac{\xi'(t)}{\xi(t)}\right| \leq k, \quad \xi(t) > 0, \quad \xi'(t) \leq 0, \quad \forall t > 0.$$

Remark 2.1. There are many functions satisfying (G1) and (G2). Examples of such functions are

$$\begin{aligned} g(t) &= a(1+t)^\nu, \quad \nu < -1 \\ g(t) &= ae^{-b(t+1)^p}, \quad 0 < p \leq 1. \end{aligned}$$

for a and b to be chosen properly.
Remark 2.2. Since ξ is nonincreasing then $\xi(t) \leq \xi(0) = M$
Remark 2.3 Condition (G1) is necessary to guarantee the hyperbolicity of the system (1.1).
Proposition *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that* g *satisfies* $(G1)$*. Then problem* (1.1) *has a unique global solution*

$$u \in C\left([0, \ \infty); H_0^1(\Omega)\right), \quad u_t \in C\left([0, \ \infty); L^2(\Omega)\right). \tag{2.1}$$

We introduce the "modified" energy functional

$$\mathcal{E}(t) := \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)||\nabla u(t)||_2^2 + \frac{1}{2}||u_t||_2^2 + \frac{1}{2}(g \circ \nabla u)(t), \tag{2.2}$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau)||v(t) - v(\tau)||_2^2 d\tau. \tag{2.3}$$

3 Decay of solutions

In this section we state and prove our main result. For this purpose we set

$$F(t) := \mathcal{E}(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t), \tag{3.1}$$

where ε_1 and ε_2 are positive constants and

$$\begin{aligned} \Psi(t) \ &: = \xi(t) \int_\Omega u u_t dx \qquad (3.2) \\ \chi(t) \ &: = -\xi(t) \int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau)) d\tau dx. \end{aligned}$$

Lemma 3.1 *If u is a solution of* (1.1), *then the "modified" energy satisfies*

$$\mathcal{E}'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2} g(t) ||\nabla u(t)||^2 \leq \frac{1}{2}(g' \circ \nabla u)(t) \leq 0. \tag{3.3}$$

Proof. By multiplying equation (1.1) by u_t and integrating over Ω, using integration by parts, hypotheses (G1) and (G2) and some manipulations as in [9], we obtain (3.3) for regular solutions. This inequality remains valid for weak solutions by a simple density argument.
Lemma 3.2. *For $u \in H_0^1(\Omega)$,* we have

$$\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau)) d\tau \right)^2 dx \leq (1-l) C_p^2 (g \circ \nabla u)(t),$$

where C_p is the Poincaré constant.
Proof.

$$\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau)) d\tau \right)^2 dx = \int_\Omega \left(\int_0^t \sqrt{g(t-\tau)} \sqrt{g(t-\tau)} (u(t)-u(\tau)) d\tau \right)^2 dx.$$

By applying Cauchy-Schwarz inequality and Poincaré's inequality, we easily see that

$$\begin{aligned} & \int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau)) d\tau \right)^2 dx \\ \leq \ & \int_\Omega \left(\int_0^t g(t-\tau) d\tau \right) \left(\int_0^t g(t-\tau)(u(t)-u(\tau))^2 d\tau \right) dx \leq (1-l) C_p^2 (g \circ \nabla u)(t). \end{aligned}$$

Lemma 3.3. *For ε_1 and ε_2 small enough, we have*

$$\alpha_1 F(t) \leq \mathcal{E}(t) \leq \alpha_2 F(t) \tag{3.4}$$

holds for two positive constants α_1 and α_2.

Proof. Straightforward computations, using Lemma 3.2, lead to

$$\begin{aligned} F(t) &\leq \mathcal{E}(t) + (\varepsilon_1/2)\,\xi(t) \int_\Omega |u_t|^2 dx + (\varepsilon_1/2)\,\xi(t) \int_\Omega |u|^2\, dx \\ &+ (\varepsilon_2/2)\,\xi(t) \int_\Omega |u_t|^2 dx + (\varepsilon_2/2)\,\xi(t) \int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau \right)^2 dx \\ &\leq \mathcal{E}(t) + [(\varepsilon_1+\varepsilon_2)/2]\, M \int_\Omega |u_t|^2 dx + (\varepsilon_1/2)\, C_p^2 M \int_\Omega |\nabla u|^2\, dx \\ &+ (\varepsilon_2/2)\, C_p^2 M (1-l)(g\circ \nabla u)(t) \leq \alpha_2 \mathcal{E}(t). \end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned} F(t) &\geq \mathcal{E}(t) - (\varepsilon_1/2)\,\xi(t) \int_\Omega |u_t|^2 dx - (\varepsilon_1/2)\,\xi(t) C_p^2 \int_\Omega |\nabla u|^2\, dx \\ &- (\varepsilon_2/2)\,\xi(t) \int_\Omega |u_t|^2 dx - (\varepsilon_2/2)\,\xi(t) C_p^2 (1-l)(g\circ \nabla u)(t) \\ &\geq + \left[\tfrac{1}{2} - M\,(\varepsilon_1+\varepsilon_2)\,/2 \right] \int_\Omega |u_t|^2 dx + [\tfrac{1}{2} l - M\,(\varepsilon_1/2)\, C_p^2] \int_\Omega |\nabla u|^2\, dx \\ &+ [\tfrac{1}{2} - M\,(\varepsilon_2/2)\, C_p^2 (1-l)](g\circ \nabla u)(t) \geq \alpha_1 \mathcal{E}(t), \end{aligned} \tag{3.6}$$

for ε_1 and ε_1 small enough.

Lemma 3.4 *Under the assumptions (G1) and (G2), the functional*

$$\Psi(t) := \xi(t) \int_\Omega u u_t dx$$

satisfies, along the solution of (1.1),

$$\Psi'(t) \leq \left[1 + \frac{k^2 C_p^2}{l} \right] \xi(t) \int_\Omega u_t^2 dx - \frac{l}{4} \xi(t) \int_\Omega |\nabla u|^2 dx + \frac{(1-l)}{2l} \xi(t) (g\circ \nabla u)(t). \tag{3.7}$$

Proof.

By using equation (1.1), we easily see that

$$\begin{aligned} \Psi'(t) &= \xi(t) \int_\Omega (u u_{tt} + u_t^2) dx + \xi'(t) \int_\Omega u u_t dx \\ &= \xi(t) \int_\Omega u_t^2 dx - \xi(t) \int_\Omega |\nabla u|^2 dx \\ &\quad + \xi(t) \int_\Omega \nabla u(t). \int_0^t g(t-\tau) \nabla u(\tau) d\tau dx + \xi'(t) \int_\Omega u u_t dx. \end{aligned} \tag{3.8}$$

We now estimate the third term in the RHS of (3.8) as follows:

$$\int_\Omega \nabla u(t). \int_0^t g(t-\tau)\nabla u(\tau) d\tau dx \leq \frac{1}{2} \int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2} \int_\Omega \left(\int_0^t g(t-\tau) |\nabla u(\tau)| d\tau \right)^2 dx$$

$$\leq \frac{1}{2} \int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2} \int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau \right)^2 dx. \tag{3.9}$$

We then use Lemma 3.2, Young's inequality, and the fact that
$\int_0^t g(\tau)d\tau \leq \int_0^\infty g(\tau)d\tau = 1-l,$ to obtain, for any $\eta > 0$,

$$\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|)d\tau \right)^2 dx \tag{3.10}$$

$$\leq \int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)|d\tau \right)^2 dx + \int_\Omega \left(\int_0^t g(t-\tau)|\nabla u(t)|d\tau \right)^2 dx$$
$$+2\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)|d\tau \right) \left(\int_0^t g(t-\tau)|\nabla u(t)|d\tau \right) dx$$
$$\leq (1+\eta)\int_\Omega \left(\int_0^t g(t-\tau)|\nabla u(t)|d\tau \right)^2 dx + (1+\tfrac{1}{\eta})\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau) - \nabla u(t)|d\tau \right)^2 dx$$
$$\leq (1+\tfrac{1}{\eta})(1-l)(g \circ \nabla u)(t) + (1+\eta)(1-l)^2 \int_\Omega |\nabla u(t)|^2 dx.$$

By combining (3.8)-(3.10) and using

$$\int_\Omega u u_t dx \leq \alpha C_p^2 \int_\Omega |\nabla u|^2 dx + \frac{1}{4\alpha}\int_\Omega u_t^2 dx, \qquad \alpha > 0,$$

we arrive at

$$\begin{aligned} \Psi'(t) &\leq \left[1 + \frac{1}{4\alpha}\left|\frac{\xi'(t)}{\xi(t)}\right|\right]\xi(t)\int_\Omega u_t^2 dx + \frac{1}{2}(1+\frac{1}{\eta})(1-l)\xi(t)(g\circ\nabla u)(t) \\ &\quad -\frac{1}{2}\left[1-(1+\eta)(1-l)^2 - 2\left|\frac{\xi'(t)}{\xi(t)}\right|\alpha C_p^2\right]\xi(t)\int_\Omega |\nabla u(t)|^2 dx \qquad (3.11) \\ &\leq \left[1+\frac{1}{4\alpha}k\right]\xi(t)\int_\Omega u_t^2 dx + \frac{1}{2}(1+\frac{1}{\eta})(1-l)\xi(t)(g\circ\nabla u)(t) \\ &\quad -\frac{1}{2}\left[1-(1+\eta)(1-l)^2 - 2k\alpha C_p^2\right]\xi(t)\int_\Omega |\nabla u(t)|^2 dx. \end{aligned}$$

By choosing $\eta = l/(1-l)$ and $\alpha = l/4kC_p^2$, (3.7) is established.
Lemma 3.5 *Under the assumptions (G1) and (G2), the functional*

$$\chi(t) := -\xi(t)\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx$$

satisfies, along the solution of (1.1),

$$\begin{aligned} \chi'(t) &\leq \delta\xi(t)\left[1+2(1-l)^2\right]\int_\Omega |\nabla u(t)|^2 dx \\ &\quad + \left[\{2\delta + \frac{1}{2\delta}\}(1-l) + \frac{C_p^2}{4\delta}k\right]\xi(t)(g\circ\nabla u)(t) \qquad (3.12) \\ &\quad + \frac{g(0)}{4\delta}C_p^2\xi(t)(-(g'\circ\nabla u)(t) + \left[\delta(k+1) - \int_0^t g(s)ds\right]\xi(t)\int_\Omega u_t^2 dx, \quad \delta > 0. \end{aligned}$$

Proof. Direct computations, using (1.1), yield

$$\begin{aligned}
\chi'(t) &= -\xi(t)\int_\Omega u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \\
&\quad -\xi(t)\int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dx - \xi(t)\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2 dx \\
&\quad -\xi'(t)\int_\Omega u_t\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \\
&= \xi(t)\int_\Omega \nabla u(t).\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)dx \qquad (3.13)\\
&\quad -\xi(t)\int_\Omega\left(\int_0^t g(t-\tau)\nabla u(\tau)d\tau\right).\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)dx \\
&\quad -\xi(t)\int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dx - \xi(t)\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2 dx \\
&\quad -\xi'(t)\int_\Omega u_t\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx.
\end{aligned}$$

Similarly to (3.8), we estimates the RHS terms of (3.13). So, by using Young's inequality, the first term gives

$$\begin{aligned}
&-\int_\Omega \nabla u(t).\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)dx \qquad (3.14)\\
&\le \delta\int_\Omega |\nabla u|^2 dx + \frac{1-l}{4\delta}(g\circ\nabla u)(t), \quad \forall\delta>0.
\end{aligned}$$

Similarly, the second term can be estimated as follows

$$\begin{aligned}
&\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right).\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)dx \\
&\le \delta\int_\Omega\left|\int_0^t g(t-s)\nabla u(s)ds\right|^2 dx + \frac{1}{4\delta}\int_\Omega\left|\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right|^2 dx \\
&\le \delta\int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(t)-\nabla u(s)|+|\nabla u(t)|\right)ds\right)^2 dx \qquad (3.15)\\
&\quad +\frac{1}{4\delta}\int_\Omega\left|\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right|^2 dx \\
&\le \left(2\delta+\frac{1}{4\delta}\right)\int_\Omega\left(\int_0^t g(t-s)\left|\nabla u(t)-\nabla u(s)\right|ds\right)^2 dx + 2\delta\left(1-l\right)^2\int_\Omega|\nabla u|^2 dx \\
&\le \left(2\delta+\frac{1}{4\delta}\right)(1-l)(g\circ\nabla u)(t) + 2\delta(1-l)^2\int_\Omega|\nabla u|^2 dx.
\end{aligned}$$

As for the third and the fourth terms we have

$$-\int_{\Omega} u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dx \le \delta \int_{\Omega} |u_t|^2 dx - \frac{g(0)}{4\delta} C_p^2 (g' \circ \nabla u)(t). \quad (3.16)$$

and

$$\int_{\Omega} u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \le \delta \int_{\Omega} |u_t|^2 dx + \frac{C_p^2}{4\delta}(g \circ \nabla u)(t). \quad (3.17)$$

By combining (3.13)-(3.17), the assertion of the lemma is established.

Theorem 3.6 *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that* g *and* ξ *satisfy* $(G1)$ *and* $(G2)$. *Then, for each* $t_0 > 0$, *there exist strictly positive constants* K *and* λ *such that the solution of* (1.1) *satisfies*

$$\mathcal{E}(t) \le K e^{-\lambda \int_{t_0}^t \xi(s)ds}, \qquad t \ge t_0. \quad (3.18)$$

Proof

Since g is positive and $g(0) > 0$ then for any $t_0 > 0$ we have

$$\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds = g_0 > 0, \qquad \forall t \ge t_0. \quad (3.19)$$

By using (3.1), (3.3), (3.7), (3.12) and (3.19), we obtain for $t \ge t_0$,

$$\begin{aligned} F'(t) \le & -\left[\varepsilon_2\{g_0 - \delta(1+k)\} - \varepsilon_1\left(1 + \frac{k^2 C_p^2}{l}\right)\right] \xi(t) \int_{\Omega} u_t^2 dx \\ & + \{\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_p^2 M\}(g' \circ \nabla u)(t) \\ & - \left[\frac{\varepsilon_1 l}{4} - \varepsilon_2 \delta\{1 + 2(1-l)^2\}\right] \xi(t) \|\nabla u\|_2^2 \\ & + \left(\frac{\varepsilon_1(1-l)}{2l} + \varepsilon_2\left(2\delta + \frac{1}{2\delta}\right)(1-l) + \varepsilon_2 \frac{C_p^2}{4\delta} k\right) \xi(t)(g \circ \nabla u)(t). \end{aligned} \quad (3.20)$$

At this point we choose δ so small that

$$\begin{aligned} g_0 - \delta(1+k) &> \frac{1}{2} g_0 \\ \frac{4}{l}\delta\left[1 + 2(1-l)^2\right] &< \frac{1}{4\left(1 + \frac{k^2 C_p^2}{l}\right)} g_0. \end{aligned}$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{g_0}{4\left(1 + \frac{k^2 C_p}{l}\right)} \varepsilon_2 < \varepsilon_1 < \frac{g_0}{2\left(1 + \frac{k^2 C_p^2}{l}\right)} \varepsilon_2 \quad (3.21)$$

will make

$$\begin{aligned} k_1 &:= \varepsilon_2\{g_0 - \delta(1+k)\} - \varepsilon_1\left(1 + \frac{k^2 C_p^2}{l}\right) > 0 \\ k_2 &:= \frac{\varepsilon_1 l}{4} - \varepsilon_2 \delta\left[1 + 2(1-l)^2\right] > 0. \end{aligned}$$

We then pick ε_1 and ε_2 so small that (3.4) and (3.21) remain valid and, further,

$$k_3 := \left(\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_p^2 M\right) - \left(\frac{\varepsilon_1}{l} + \varepsilon_2\{2\delta + \frac{1}{2\delta}\}\right)(1-l) - \varepsilon_2 \frac{C_p^2}{4\delta} k > 0.$$

Hence

$$\begin{aligned} &\{\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_p^2 M\}(g' \circ \nabla u)(t) \\ &+ \left(\frac{\varepsilon_1(1-l)}{2l} + \varepsilon_2\left(2\delta + \frac{1}{2\delta}\right)(1-l) + \varepsilon_2 \frac{C_p^2}{4\delta} k\right)\xi(t)(g \circ \nabla u)(t) \\ \le\ & -\{\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_p^2 M\} \int_\Omega \int_0^t \xi(t-\tau) g(t-\tau) |\nabla u(\tau) - \nabla u(t)|^2 d\tau dx \\ &+ \left(\frac{\varepsilon_1(1-l)}{2l} + \varepsilon_2\left(2\delta + \frac{1}{2\delta}\right)(1-l) + \varepsilon_2 \frac{C_p^2}{4\delta} k\right)\xi(t)(g \circ \nabla u)(t) \\ \le\ & -k_3 \xi(t)(g \circ \nabla u)(t), \end{aligned} \tag{3.22}$$

since ξ is nonincreasing. Therefore, by using (3.4), (3.20), and (3.22), we arrive at

$$F'(t) \le -\beta_1 \xi(t) \mathcal{E}(t) \le -\beta_1 \alpha_1 \xi(t) F(t) \qquad \forall t \ge t_0. \tag{3.23}$$

A simple integration of (3.23) leads to

$$F(t) \le F(t_0) e^{-\beta_1 \alpha_1 \int_{t_0}^t \xi(s) ds}, \qquad \forall t \ge t_0. \tag{3.24}$$

Thus (3.4), (3.24) yield

$$\mathcal{E}(t) \le \alpha_2 F(t_0) e^{-\beta_1 \alpha_1 \int_{t_0}^t \xi(s) ds} = K e^{-\lambda \int_{t_0}^t \xi(s) ds}, \qquad \forall t \ge t_0. \tag{3.25}$$

This completes the proof.

Remark 3.1 This result generalizes and improves the results of [1], [2], [5] and [6]. In particular, it allows some relaxation functions which satisfy $g' \le -a g^\rho$, $1 \le \rho < 2$. This improves early works [2] and [6], where it is assumed that $1 \le \rho < 3/2$.

Remark 3.2 Note that the exponential and the polynomial decay estimates, given in early works [1], [2], [5] and [6], are only particular cases of (3.25). More precisely, we obtain exponential decay for $\xi(t) \equiv a$ and polynomial decay for $\xi(t) = a(1+t)^{-1}$, where $a > 0$ is a constant.

Remark 3.3 Observe that our result is proved without any condition on g'' and g''' unlike what was assumed in (2.4) of [6]. We only need g to be differentiable satisfying

(G1) and (G2).
Remark 3.4 Estimates (3.18) are also true for $t \in [0, t_0]$ by virtue of continuity and boundedness of $\mathcal{E}(t)$ and $\xi(t)$.
Remark 3.5 A similar result can be established for the semilinear problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-\tau)\Delta u(x,\tau)d\tau + b|u|^{p-2}u(x,t) = 0, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \ x \in \partial\Omega \ , t \geq 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{3.26}$$

where $b > 0$ and $2 \leq p \leq 2(n-1)/(n-2)$, if $n \geq 3$.
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