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Abstract Some well known results of bivariate beta distribution have been reviwed. Corrected product moments and cumulative distribution function have been derived. These will be important for studying further characteristics of the distribution.

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1. Introduction

The bivariate beta distribution has found application in areas such as voting analysisi of political issues of two competing candidates and research on soil strength (see Hutchinson and Lai, 1990, 104). In this paper we derive some important centered moments that are important in studying further properties of the distribution. The product moment of order a and b for two random variables X and Y are defined by $\mu'_{a,b} = E(X^a Y^b)$ while the centered product moments (sometimes called central product moments, corrected moments or central mixed moments) are defined by

$$\mu_{a,b} = E[X - E(X^a)\, Y - E(Y^b)].$$

Interested readers may go through Johnson, Kotz and Kemp (1993, 46) or Johnson, Kotz and Balakrishnan (1997, 3). The former is often called product moments of order zero or raw product moments. Evidently $\mu'_{a,0} = E(X^a)$ is the a-th moment of X, and $\mu'_{0,b} = E(Y^b)$ is the b-th moment of Y. In case X and Y are independent $\mu'_{a,b} = E(X^a)E(Y^b) = \mu'_{a,0}\mu'_{0,b}$.

The correlation coefficient ρ $(-1 < \rho < 1)$ between $X$and Y is denoted by

$$\rho_{X,Y} = \frac{\mu_{1,1}}{\sqrt{\mu_{2,0}\mu_{0,2}}}. \qquad (1.1)$$

Note that $\mu_{2,0} = E(X - E(X))^2 = \sigma_{20}$ which is popularly denoted by σ_1^2 while the central product moment, $\mu_{1,1} = E[(X - E(X))(Y - E(Y))]$ denoted popularly by σ_{12}, is in fact the covariance between X and Y.

The importance of evaluating central moments of a bivariate distribution cannot be overlooked. In a series of papers, Mardia (1970, 1974, 1975) defined and discussed the properties of measures for kurtosis and skewness based on Mahalanobis distance. As it is difficult to derive distribution of Mahalanobis distance for many distributions and calculate moments thereof, Joarder (2006) derived Mahalanobis moments (or simply, standardized moments) in terms of central product moments. He showed that the central moments can be used as an alternative way to describe further important characteristics of a bivariate distribution such as Mahalanobis moments which includes bivariate skewness and kurtosis coefficients that are very difficult to derive. It should be mentioned that the central moments derived in this paper required meticulous calculation and cross-checking and, in and of itself, a formidable task to complete.

2. The Bivariate Beta Distribution

The bivariate Dirichlet is an extension of a univariate Beta distribution. The probability density function of the bivariate Dirichlet distribution is given by

$$f(x,y) = \frac{\Gamma(m+n+p)}{\Gamma(m)\Gamma(n)\Gamma(p)} x^{m-1} y^{n-1} (1-x-y)^{p-1}, \tag{2.1}$$

where $m,n,p > 0,\ x \geq 0,\ y \geq 0,$ and $x+y \leq 1.$

Proof. Let $A = \{(x,y) \in R^2 : x > 0, y > 0 \text{ and } x+y < 1\}$ and for $m,n,p > 0$

$$I(m,n,p) = \iint_A x^{m-1} y^{n-1} (1-x-y)^{p-1}\, dxdy.$$

With the transformation $u = \frac{x}{1-y},\ v = y$ and Jacobian $J(x,y \to u,v) = 1-v$ in the above integral, we have

$$\begin{aligned}
&\int_0^1 \int_0^{1-y} x^{m-1} (1-x-y)^{p-1}\, dx\, y^{n-1}\, dy \\
&= \int_0^1 \int_0^1 (u(1-v))^{m-1} v^{n-1} ((1-u)(1-v))^{p-1} (1-v)\, dv\, du, \\
&= B(m,p) \int_0^1 v^{n-1} (1-v)^{m+p-1}\, dv \\
&= B(m,p) B(n,m+p)
\end{aligned}$$

where $B(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m+n).$ This proves that the function in (2.1) is a joint probability density function.

Theorem 2.1 Let X and Y have the joint pdf given by (2.1). Then the marginal probability density functions of the bivariate beta distribution with pdf in (2.1) are given by:

(i) $X \sim Beta(m, n+p),$
(ii) $Y \sim Beta(n, m+p).$ (2.2)

Proof. The marginal p.d.f. of Y is given by

$$
\begin{aligned}
h(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \\
&= \int_{-\infty}^{\infty} \frac{\Gamma(m+n+p)}{\Gamma(m)\Gamma(n)\Gamma(p)} x^{m-1} y^{n-1} (1-x-y)^{p-1}\,dx \\
&= \frac{1}{B(n,m+p)} y^{n-1}(1-y)^{m+p-1}.
\end{aligned}
$$

Thus Y follows $Beta(n, m+p)$. Similarly, X follows $Beta(m, n+p)$.

We note that the mean and variance of Y are given by

$$E(Y) = \frac{n}{m+n+p},\ V(Y) = \frac{n(m+p)}{(t+1)t^2}$$

respectively, where $t = m+n+p$. mean and variance of X are given by

$$E(X) = \frac{n}{m+n+p},\ V(X) = \frac{m(n+p)}{(t+1)t^2}.$$

Theorem 2.2 Let X and Y have the joint pdf given by (2.1). Then the conditional p.d.f. of Y given $X = x$ is given by

$$k(y) = \frac{(1-x)}{B(n,p)}\left(\frac{y}{1-x}\right)^{n-1}\left(1-\frac{y}{1-x}\right)^{p-1}, \quad 0 < y < 1-x, 0 < x < 1. \qquad (2.3)$$

Proof. The conditional p.d.f. k_2 of Y given $X = x$ is defined as

$$
\begin{aligned}
f(x,y)/h(x) &= \frac{\Gamma(m+n+p)}{\Gamma(m)\Gamma(n)\Gamma(p)} x^{m-1} y^{n-1}(1-x-y)^{p-1}\left(\frac{1}{B(m,n+p)} x^{m-1}(1-x)^{n+p-1}\right)^{-1} \\
&= \frac{\Gamma(m+n+p)}{\Gamma(m)\Gamma(n)\Gamma(p)B(m,n+p)} y^{n-1}(1-x-y)^{p-1}/(1-x)^{n+p-1}
\end{aligned}
$$

which can be written as (2.3).

Thus, from (2.3), it can be seen that the conditional distribution of $Y/(1-X)$ given $X = x$ is $Beta(n,p)$ which implies that $E(Y \mid X = x) = (1-x)n/(n+p)$ which can also be written as

$$E(Y \mid X = x) = -\frac{n}{n+p}x + \frac{n}{n+p} \qquad (2.4)$$

in the regular regression format. Thus the regression of Y on X is linear. Also

$$Var(Y \mid X = x) = (1-x)^2 np/((n+p)^2(n+p+1))$$

which is not free from x. This means that the conditional variance for the linear regression of Y on X is not homoscedastic. The linear regression suggests that Y is not independent of X.

Theorem 2.3 Let X and Y have the joint pdf given by (2.1). Then $Y/(1-X)$ and X are independent.

Proof. Let $u = y/(1-x)$ and $v = x$ with Jacobian $J(x, y \to u, v) = -(1-v)$. The region A is mapped into the region
$\{(u,v) : v > 0, u(1-v) > 0, v + u(1-v) < 1\} = \{(u,v) : 0 < u < 1, 0 < v < 1\}$. Then the joint p.d.f. of $(U, V) := (Y/(1-X), X)$ is given by

$$g(u,v) = \frac{1}{B(n,p)} u^{n-1}(1-u)^{p-1} \cdot \frac{1}{B(m,n+p)} v^{m-1}(1-v)^{n+p-1}$$

Then U and V are independent Beta variables.

Theorem 2.4 Let (X, Y) follow the bivariate Dirichlet distribution with pdf given by (2.1). Also let $U = X + Y$ and $V = X/(X+Y)$. Then $U \sim Beta(m+n, p)$ is independent of $V \sim Beta(m, n)$.

Proof. Let us make the transformation $u = x + y$ and $uv = x$. The region A is mapped onto the region $\{(u,v) : uv > 0, u(1-v) > 0, u < 1\} = \{(u,v) : 0 < u < 1, 0 < v < 1\}$ with Jacobian $J(x, y \to u, v) = -u$. The theorem then follows in a straightforward manner.

In what follows we will define

$$\mu_{a,b} = E[(X-\xi)^a (Y-\theta)^b] \tag{2.5}$$

where $\xi = E(X)$ and $\theta = E(Y)$.

3. Raw Product Moments

For any non-negative integer a, we have Pochhammer factorials defined as

$$c_{\{a\}} = c(c+1)(c+2)\cdots(c+a-1) \text{ and}$$

$$c^{\{a\}} = c(c-1)(c-2)\cdots(c-a+1).$$

with $c_{\{0\}} = 1$, $c^{\{0\}} = 1$.

Also, the $(a,b)^{th}$ raw product moment of X and Y of the bivariate Dirichlet distribution is given by

$$E(X^a Y^b) = \int_0^1 \int_0^1 x^a y^b f(x,y)\,dxdy. \tag{3.1}$$

Lemma 3.1 Let X and Y have the joint pdf given by (2.1). Then

(i) the marginal density function of $X \sim Beta(m, n+p)$, has an expected value of

$$E(X^a) = \frac{m_{\{a\}}}{t_{\{a\}}},$$

(ii) the marginal density function of $Y \sim Beta(n, m+p)$, has an expected value of

$$E(Y^b) = \frac{n_{\{b\}}}{t_{\{b\}}},$$

(iii) and the raw product moment of order (a, b) is

$$E(X^a Y^b) = \frac{m_{\{a\}} n_{\{b\}}}{t_{\{a+b\}}}, \qquad \text{where } t = m + n + p.$$

Lemma 3.1 gives rise to some useful raw moments that will be used further in this article. In particular, some specific raw moments that are needed for the calculation of centered moments, bivariate skewness and kurtosis are given below.

$$E(Y) = \frac{n}{t},$$
$$E(Y^2) = \frac{n(n+1)}{t(t+1)},$$
$$E(Y^3) = \frac{n(n+1)(n+2)}{t(t+1)(t+2)},$$
$$E(Y^4) = \frac{n(n+1)(n+2)(n+3)}{t(t+1)(t+2)(t+3)},$$
$$E(Y^5) = \frac{n(n+1)(n+2)(n+3)(n+4)}{t(t+1)(t+2)(t+3)(t+4)},$$
$$E(Y^6) = \frac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{t(t+1)(t+2)(t+3)(t+4)(t+5)},$$
$$E(XY) = \frac{mn}{t(t+1)},$$
$$E(XY^2) = \frac{mn(n+1)}{t(t+1)(t+2)},$$
$$E(XY^3) = \frac{mn(n+1)(n+2)}{t(t+1)(t+2)(t+3)},$$
$$E(XY^4) = \frac{mn(n+1)(n+2)(n+3)}{t(t+1)(t+2)(t+3)(t+4)},$$
$$E(XY^5) = \frac{mn(n+1)(n+2)(n+3)(n+4)}{t(t+1)(t+2)(t+3)(t+4)(t+5)},$$
$$E(X^2 Y^2) = \frac{m(m+1)n(n+1)}{t(t+1)(t+2)(t+3)},$$
$$E(X^2 Y^3) = \frac{m(m+1)n(n+1)(n+2)}{t(t+1)(t+2)(t+3)(t+4)},$$
$$E(X^2 Y^4) = \frac{m(m+1)n(n+1)(n+2)(n+3)}{t(t+1)(t+2)(t+3)(t+4)(t+5)},$$
$$E(X^2 Y) = \frac{m(m+1)n}{t(t+1)(t+2)},$$
$$E(X^3 Y) = \frac{m(m+1)(m+2)n}{t(t+1)(t+2)(t+3)},$$
$$E(X^4 Y) = \frac{m(m+1)(m+2)(m+3)n}{t(t+1)(t+2)(t+3)(t+4)},$$
$$E(X^5 Y) = \frac{m(m+1)(m+2)(m+3)(m+4)n}{t(t+1)(t+2)(t+3)(t+4)(t+5)},$$
$$E(X^3 Y^2) = \frac{m(m+1)(m+2)n(n+1)}{t(t+1)(t+2)(t+3)(t+4)},$$

$$E(X^3Y^3) = \frac{m(m+1)(m+2)(n+2)(n+1)n}{t(t+1)(t+2)(t+3)(t+4)(t+5)},$$
$$E(X^4Y^2) = \frac{m(m+1)(m+2)(m+3)(n+1)n}{t(t+1)(t+2)(t+3)(t+4)(t+5)},$$
$$E(X) = \frac{m}{t},$$
$$E(X^2) = \frac{m(m+1)}{t(t+1)},$$
$$E(X^3) = \frac{m(m+1)(m+2)}{t(t+1)(t+2)},$$
$$E(X^4) = \frac{m(m+1)(m+2)(m+3)}{t(t+1)(t+2)(t+3)},$$
$$E(X^5) = \frac{m(m+1)(m+2)(m+3)(m+4)}{t(t+1)(t+2)(t+3)(t+4)},$$
$$E(X^6) = \frac{m(m+1)(m+2)(m+3)(m+4)(m+5)}{t(t+1)(t+2)(t+3)(t+4)(t+5)}.$$

4. Centered Moments

The centered product moments of a bivariate Dirichlet distribution, $\mu_{a,b}$ can be obtained by directly evaluating the following integral.

$$E\left[(X_1 - E(X_1))^a (X_2 - E(X_2))^b\right] = \int_0^1\int_0^1 (X_1 - E(X_1))^a (X_2 - E(X_2))^b f(x,y)\,dxdy.$$

For illustration, we derive the central moment $\mu_{1,2}$ below.

$$\begin{aligned}
\mu_{1,2} &= E\left[(X - E(X))^1 (Y - E(Y))^2\right] \\
&= \int_0^1\int_0^1 \left(x - \frac{m}{t}\right)\left(y - \frac{n}{t}\right)^2 f(x,y)\,dxdy \\
&= \int_0^1\int_0^1 \left(xy^2 - 2xy\frac{n}{t} + x\left(\frac{n}{t}\right)^2 - y^2\frac{m}{t} + 2y\frac{n}{t}\frac{m}{t} - \left(\frac{n}{t}\right)^2\frac{m}{t}\right) f(x,y)\,dxdy \\
&= \int_0^1\int_0^1 xy^2 f(x,y)\,dxdy - 2\frac{n}{t}\int_0^1\int_0^1 xyf(x,y)\,dxdy + \left(\frac{n}{t}\right)^2\int_0^1\int_0^1 xf(x,y)\,dxdy \\
&\quad - \frac{m}{t}\int_0^1\int_0^1 y^2 f(x,y)\,dxdy + 2\frac{mn}{t^2}\int_0^1\int_0^1 yf(x,y)\,dxdy - \frac{n^2m}{t^3}\int_0^1\int_0^1 f(x,y)\,dxdy \\
&= \frac{mn(n+1)}{t(t+1)(t+2)} - 2\left(\frac{n}{t}\right)\frac{mn}{t(t+1)} + \left(\frac{n}{t}\right)^2\left(\frac{m}{t}\right) - \left(\frac{m}{t}\right)\frac{n(n+1)}{t(t+1)} \\
&\quad + 2\left(\frac{m}{t}\right)\left(\frac{n}{t}\right)^2 - \frac{n^2m}{t^3} \\
&= \frac{mn(n+1)}{t(t+1)(t+2)} - 2\frac{mn^2}{t^2(t+1)} + \frac{mn^2}{t^3} - \frac{mn(n+1)}{t^2(t+1)} + 2\frac{mn^2}{t^3} - \frac{mn^2}{t^3} \\
&= -\frac{2(t-2n)mn}{(t+1)(t+2)t^3}.
\end{aligned}$$

The higher order moments generally require more meticulous integration and cross-checking of calculations. Some centered product moments of order $a + b = 2, 3, 4, 5, 6$ are given below:

$$\mu_{0,2} = \frac{(t-n)n}{(t+1)t^2},$$

$$\mu_{0,3} = 2\frac{(t-2n)(t-n)n}{(t+1)(t+2)t^3},$$
$$\mu_{0,4} = 3(t-n)n\frac{t^2(n+2) - n(n+6)t + 6n^2}{(t+1)(t+2)(t+3)t^4},$$
$$\mu_{0,5} = 4(t-2n)(t-n)n\frac{t^2(5n+6) - n(5n+12)t + 12n^2}{(t+1)(t+2)(t+3)(t+4)t^5},$$
$$\begin{aligned}\mu_{0,6} = & \frac{5(t-n)n}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6} \\ & \times (t^4(3n^2+26n+24) - 2n(3n^2+56n+60)t^3 \\ & \quad + n^2(3n^2+172n+240)t^2 - 2n^3(43n+120)t + 120n^4),\end{aligned}$$

$$\mu_{1,1} = -\frac{nm}{(t+1)t^2},$$
$$\mu_{1,2} = -2\frac{(t-2n)mn}{(t+1)(t+2)t^3},$$
$$\mu_{1,3} = -3\frac{mn}{t^4(t+1)(t+2)(t+3)}((n+2)t^2 - n(n+6)t + 6n^2),$$
$$\mu_{1,4} = -4\frac{mn(t-2n)}{t^5(t+1)(t+2)(t+3)(t+4)}(12n^2 - (12+5n)nt + (6+5n)t^2),$$

$$\begin{aligned}\mu_{1,5} = & -5\frac{mn}{t^6(t+1)(t+2)(t+3)(t+4)(t+5)} \\ & \quad \times ((3n^2+26n+24)t^4 - 2n(3n^2+56n+60)t^3 \\ & \quad + n^2(3n^2+172n+240)t^2 - 2n^3(43n+120)t + 120n^4),\end{aligned}$$

$$\mu_{2,2} = mn\frac{t^3 - (m+n)t^2 + 3(2m+2n+mn)t - 18mn}{(t+1)(t+2)(t+3)t^4},$$
$$\mu_{2,3} = 2mn\frac{t^4 - t^3(m+6n) + (12m+15mn+5n^2)t^2 - 4n(9m+3n+5mn)t + 48mn^2}{(t+1)(t+2)(t+3)(t+4)t^5},$$
$$\begin{aligned}\mu_{2,4} = & \frac{mn}{t^6(t+1)(t+2)(t+3)(t+4)(t+5)}(t^5(3n+6) + t^4(-6m-40n-3mn-6n^2) \\ & \quad + t^3(120m+164mn+120n^2+3n^3+18mn^2) + t^2(-480mn-86n^3-516mn^2-15mn \\ & \quad + t(120n^3+720mn^2+430mn^3) - 600mn^3),\end{aligned}$$

$$\begin{aligned}\mu_{3,3} = & \frac{mn}{(t+1)(t+2)(t+3)(t+4)(t+5)t} \\ & \times (4 - 3\frac{10m+10n+3mn}{t} + \frac{(26n^2 + m^2(9n+26) + 9mn(n+20))}{t^2} \\ & \quad - 3\frac{2n^2(43m+20) + m^2(86n+5n^2+40)}{t^3} \\ & \quad + 10\frac{mn(36m+36n+43mn)}{t^4} - 600\frac{m^2n^2}{t^5})\end{aligned}$$

Similar expressions for moments $\mu(b,a) = E[(X-\xi)^b(Y-\theta)^a]$ are provided below.

$$\mu_{2,0} = \frac{(t-m)m}{(t+1)t^2},$$
$$\mu_{3,0} = 2\frac{(t-2m)(t-m)m}{(t+1)(t+2)t^3},$$

$$\mu_{4,0} = 3(t-m)m\frac{t^2(m+2)-m(m+6)t+6m^2}{(t+1)(t+2)(t+3)t^4},$$
$$\mu_{5,0} = 4(t-m)(t-2m)m\frac{t^2(5m+6)-m(5m+12)t+12m^2}{(t+1)(t+2)(t+3)(t+4)t^5},$$
$$\begin{aligned}\mu_{6,0} = {} & \frac{5(t-m)m}{(t+1)(t+2)(t+3)(t+4)(t+5)t^6} \\ & \times ((3m^2+26m+24)t^4 - 2m(3m^2+56m+60)t^3 \\ & \quad + m^2(3m^2+172m+240)t^2 - 2m^3(43m+120)t + 120m^4),\end{aligned}$$

$$\mu_{2,1} = -2\frac{mn(t-2m)}{t^3(t+1)(t+2)},$$
$$\mu_{3,1} = -3mn\frac{(m+2)t^2-m(m+6)t+6m^2}{(t+1)(t+2)(t+3)t^4},$$
$$\mu_{4,1} = -4(t-2m)mn\frac{(5m+6)t^2-(12m+5m^2)t+12m^2}{(t+1)(t+2)(t+3)(t+4)t^5},$$
$$\begin{aligned}\mu_{5,1} = {} & -5\frac{mn}{t^6(t+1)(t+2)(t+3)(t+4)(t+5)} \\ & \times ((3m^2+26m+24)t^4 - 2m(3m^2+56m+60)t^3 \\ & \quad + m^2(3m^2+172m+240)t^2 - 2m^3(43m+120)t + 120m^4),\end{aligned}$$

$$\mu_{3,2} = 2mn\frac{t^4 - t^3(6m+n) + (12n+5m(m+3n))t^2 - 4m(m(5n+3)+9n)t + 48m^2n}{(t+1)(t+2)(t+3)(t+4)t^5},$$
$$\begin{aligned}\mu_{4,2} = {} & \frac{mn}{t^6(t+1)(t+2)(t+3)(t+4)(t+5)} \\ & \times ((3m+6)t^5 - (6m^2+m(3n+40)+6n)t^4 \\ & \quad + (3m^3+6(3n+20)m^2+4n(41m+30))t^3 \\ & \quad - m(m^2(15n+86)+12n(43m+40))t^2 \\ & \quad + 10m^2(12m+72n+43mn)t - 600m^3n).\end{aligned}$$

5. Correlation for the Bivariate Beta Distribution

Theorem 5.1 Let X and Y have the joint pdf given by (2.1). Then the product moment correlation coefficient between X and Y is given by

$$\rho = -\sqrt{\frac{mn}{(n+p)(m+p)}}.$$

Proof. Substituting the moments from Section 4 in $\rho\sqrt{\mu_{2,0}\mu_{0,2}} = \mu_{1,1}$ we have the theorem.

Note that Theorem 5.1 is a special case of results in Kotz, Balakrishnan & Johnson (2000, 488).

For a special case of a bivariate Dirichlet distribution defined in (2.1) where $m = n$, that is $(t = 2m+p)$ then the product moment correlation coefficient is $\rho = -m/(m+p)$. Note that when $m = n$, we have the case when the two bivariate marginal probability density functions are identical.

Another special case of a bivariate Dirichlet distribution defined in (2.1) occurs when $m = n = p$, that is $t = 3m$. In this case then the product moment correlation coefficient

$\rho = -\frac{1}{2}$. Note that when $m = n = p$, we have special case of Theorem 5.1 when the two bivariate marginal pdfs are identical with $p = m$.

It is also easy to check that

$$Cov(X, Y) = E(XY) - E(X)E(Y) = \frac{-mn}{(t+1)t^2}$$

since $E(XY) = E[XE(Y|X)] = E\left(\frac{1-X}{n(n+p)}\right) = \frac{mn}{t(t+1)}$.

Consequently, the product moment correlation ρ between X and Y is given by what we have in the theorem.

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