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Orthogonal Exponential Zero-Interpolants

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by

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Abstract

A class of *exponential orthogonal polynomials* is introduced which serves as an approximating subspace for certain type of constrained approximation problems over semi-infinite interval. We shall discuss theoretical as well as computational aspects of this class in this report.

1. Preliminaries

We shall introduce exponential polynomials and some relevant concepts in this section. This material will lead us to define *Orthogonal Exponential Zero-Interpolants* (*OEZI*).

1.1. Exponential polynomial

An expression of the form

$$E_n(x) := \sum_{j=0}^{n} a_j e^{-jx} \tag{1}$$

where a_j's are real numbers, will be called n^{th} degree *exponential polynomials.*

Notation: We shall denote the *class of all n^{th} degree exponential polynomials* by χ_n.

1.2. Lagrange exponential interpolants

Let $x_0, x_1, ..., x_k$ be $k+1$ distinct nonnegative real numbers. We set

$$\tilde{W}_k(x) = \prod_{j=0}^{k} (e^{-x} - e^{-x_j}). \tag{2}$$

The exponential polynomial given by

$$\lambda_k(x, f) := \sum_{j=0}^{k} f(x_j) e^{-x_j} \frac{\tilde{W}_k(x)}{\tilde{W}_k'(x_j)(e^{-x_j} - e^{-x})} \tag{3}$$

interpolates a function $f : [0, \infty) \to \Re$ at the points $x_0, x_1, ..., x_k$. We shall call $\lambda_k(., f)$ the k^{th} degree *Lagrange exponential interpolant* to f at the points $x_0, x_1, ..., x_k$. Note that $\lambda_k(., f) \in \chi_k$.

For computational purpose, the Lagrange exponential interpolant can be described by Newton's interpolation formula [28]. If we denote the divided difference[1] [5] of order j by $f[x_0, x_1, ..., x_j]$ then, we have

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[1] Note that for the points $x_0, x_1, ..., x_k$, the 0-order divided difference $f[x_i]$ is defined as $f[x_i] = f(x_i)$, i = 0,1,…,k; and the m^{th}-order divided difference is by the recursive formula $f[x_0, x_1, ..., x_m] = \dfrac{f[x_0, x_1, ..., x_{m-1}] - f[x_1, x_2 ..., x_m]}{x_0 - x_m}$, m = 1, 2, …,j

$$\lambda_k(x,f) = f[x_0] + \sum_{j=1}^{k} f[x_0, x_1, \dots, x_j] \prod_{i=0}^{j-1} (e^{-x} - e^{-x_i}) . \tag{4}$$

1.3. Exponential zero-interpolants (*EZI*)

Let

$$\tilde{\Lambda}_k := \{x_0, x_1, \dots, x_k\} \subset [0, \infty) . \tag{5}$$

We set

$$\chi_n(\tilde{\Lambda}_k) := \left\{ \tilde{W}_k E : E \in \chi_n \right\} \tag{6}$$

where $\tilde{W}_k$ is given by (2).

Remark 1. $\chi_n(\tilde{\Lambda}_k)$ is the *class of exponential polynomials which interpolates the zero-function at the set of nodes* x_i, *i* = 0,1,…,*k*. Note that $\chi_n(\tilde{\Lambda}_k)$ is an (*n* + 1) dimensional subspace of χ_{n+k+1}.

Definition 1. An exponential polynomial in $\chi_n(\tilde{\Lambda}_k)$ will be called an *exponential zero-interpolant* (*EZI*) of degree *n* + *k* at the set $\tilde{\Lambda}_k$.

2. Orthogonal Exponential Zero-Interpolants

In this section we shall introduce the concept orthogonal exponential zero-interpolant. It may be noted that the notion of orthogonality over an interval is always tied up with an *appropriate* weight function. In order to define an orthogonal exponential polynomial, say $E_n(x)$ over the semi-infinite interval $[0,\infty)$, we shall require a weight function *w* for which

$$\int_0^\infty E_n^2(x) w(x) dx < \infty . \tag{7}$$

Here, one may consider $w(x)$ to be the generalized Laguerre weight function [2] $\tilde{w}_{\alpha,\beta}(x) := x^\alpha e^{-\beta x}$ where $\alpha > -1$ and $\beta > 0$ and note that the improper integral

$\int_0^\infty E_n^2(x) x^\alpha e^{-\beta x} dx$ converges for this choice. Thus, $\tilde{w}_{\alpha,\beta}(x) := x^\alpha e^{-\beta x}$ is a suitable class of weight functions in order to define orthogonal exponential polynomials. Following definition of inner product will be used frequently:

$$\langle f, g \rangle_w := \int_a^b f(x) g(x) w(x) dx \tag{8}$$

Definition 2. An exponential polynomial $\tilde{\psi}_n \in \chi_n(\tilde{\Lambda}_k)$ will be called an *orthogonal exponential zero-interpolant* (OEZI) at $\tilde{\Lambda}_k$ with respect to weight function $\tilde{w}_{\alpha,\beta}$ over the interval $[0,\infty)$ if

(a) $\tilde{\psi}_n$ is a zero-interpolant at $\tilde{\Lambda}_k$ in the sense of Section 1.3.

(b) $\langle \psi_n, E \rangle_{\tilde{w}_{\alpha,\beta}} = 0$ for all $E \in \chi_m(\Lambda_k)$, $m = 0, 1, ... n-1$.

2.1 Orthogonal basis of $\chi_n(\tilde{\Lambda}_k)$

Since $\chi_n(\tilde{\Lambda}_k)$ is an n + 1 dimensional subspace of the linear space χ_{n+k+1}, it is natural to look for orthogonal basis, say "$\tilde{\psi}_0, \tilde{\psi}_1, ..., \tilde{\psi}_n$". We can achieve this by using the standard 3-term recurrence relation [2] as we do in case of classical orthogonal polynomials. More specifically, we set

$$\tilde{\psi}_{i+1}(x) = (e^{-x} - \alpha_i)\tilde{\psi}_i(x) - \beta_i \tilde{\psi}_{i-1}(x), \quad i = 1, 2, \ldots . \tag{9}$$

with

$$\tilde{\psi}_0(x) = \tilde{W}_k(x) \ , \qquad \tilde{\psi}_1(x) = (e^{-x} - \alpha_0)\tilde{W}_k(x); \quad \alpha_0 = \langle \tilde{W}_k, \tilde{W}_k \rangle_{\tilde{w}_{\alpha,\beta}}$$

The *recursion coefficients* in (9) are given by

$$\left.\begin{aligned} \alpha_i &= \frac{\langle x\tilde{\psi}_i, \tilde{\psi}_i \rangle_{\tilde{w}_{\alpha,\beta}}}{\langle \tilde{\psi}_i, \tilde{\psi}_i \rangle_{\tilde{w}_{\alpha,\beta}}}, \qquad i = 1, 2, \ldots \\ \beta_i &= \frac{\langle \tilde{\psi}_i, \tilde{\psi}_i \rangle_{\tilde{w}_{\alpha,\beta}}}{\langle \tilde{\psi}_{i-1}, \tilde{\psi}_{i-1} \rangle_{\tilde{w}_{\alpha,\beta}}}, \quad i = 1, 2, \ldots \end{aligned}\right\} .$$

Definition 3. The exponential polynomials "$\tilde{\psi}_0, \tilde{\psi}_1, ..., \tilde{\psi}_n$" defined by the recurrence relation (9) will be referred to as an *orthogonal basis* of $\chi_n(\tilde{\Lambda}_k)$.

2.2. A useful transformation

The *transformation* $\eta:[0,1]\to[0,\infty)$ defined as $x=\eta(t):=-\ln t$, $t\in[0,1]$ plays a key role in our study. It converts the exponential polynomials $E_n(x):=\sum_{j=0}^{n} a_j e^{-jx}$ to algebraic polynomials $p_n(t)=\sum_{j=0}^{n} a_j t^j$. The following remark relates the zeros of these two types of polynomials:

Remark 2 Consider the transformation $x\mapsto -\ln t$ and let $x^*=-\ln t^*$. Then $x^*\in(0,\infty)$ is a zero of an exponential polynomial $E_n(x)=\sum_{i=0}^{n} a_i e^{-ix}$ if and only t^* is a zero of the algebraic polynomial $p_n(t)=\sum_{i=0}^{n} a_i t^i$. To justify this, note that

x^* is a zero of exponential polynomial E_n

$$\Leftrightarrow E_n(x^*)=\sum_{i=0}^{n} a_i e^{-ix^*}=0.$$

$$\Leftrightarrow \sum_{i=0}^{n} a_i e^{-i(-\ln t^*)}=0 \Leftrightarrow \sum_{i=0}^{n} a_i (t^*)^i=0 \Leftrightarrow p_n(t^*)=0.$$

In general, $x\mapsto -\ln t$ transforms an OEZI to an orthogonal algebraic 0-interpolant [1, section 5].

2.3. Some properties of *OEZI*

We provide some characteristics of OEZI's in

Lemma 1. The *OEZI* $\tilde{\psi}_m$, m = 0,1,2,…, as described in Definition 3 have the following properties:

(A) $\tilde{\psi}_m = E_m^* \tilde{W}_k$ for some $E_m^* \in \chi_m$. (10)

(B) The factor polynomials E_m^*, m = 0,1,2,…,n, are orthogonal with respect to $\tilde{w}_{\alpha,\beta}\tilde{W}_k^2$ on $[0,\infty)$.

(C) E_m^* has m distinct real zeros lying in the interval $(0,\infty)$.

Proof. **(A)** It follows from the structure of $\chi_n(\tilde{\Lambda}_k)$ that

$$\tilde{\psi}_m(x) = \tilde{W}_k(x) E_m^*(x) \tag{11}$$

where $\tilde{W}_k(x)$ is given by (2) and

$$E_m^*(x) = \sum_{i=0}^{m} a_{i,m} e^{-ix} \tag{12}$$

for some $a_{i,m} \in \Re,\ i = 0,1,...,m$.

(B) Since $\tilde{\psi}_m,\ m = 0,1,2,....$, are orthogonal with respect to $\tilde{w}_{\alpha,\beta}$ over $[0,\infty)$, we have

$$0 = \int_0^\infty \tilde{\psi}_m(x)\tilde{\psi}_l(x)\tilde{w}_{\alpha,\beta}(x)dx = \int_0^\infty E_m^*(x)E_l^*(x)\tilde{W}_k^2(x)\tilde{w}_{\alpha,\beta}(x)dx \tag{13}$$

when $m \neq l$. Therefore, the factor exponential polynomials $E_m^*,\ m = 0,1,2,....,$ are orthogonal with respect to $\tilde{W}_k^2\tilde{w}_{\alpha,\beta}$ over $[0,\infty)$.

(C) As noted above, the setting $x = -\ln(t)$ for $x \in [0,\infty)$ transforms the exponential polynomial over $[0,\infty)$ to algebraic polynomials over $[0,1]$. Thus,

$$E_m^*(x) := \sum_{i=0}^{m} a_{i,m} e^{-ix} \mapsto p_m^*(t) := \sum_{i=0}^{m} a_{i,m} t^i , \tag{14}$$

$$\tilde{W}_k(x) := \prod_{i=0}^{k} (e^{-x} - e^{-x_i}) \mapsto W_k(t) := \prod_{i=0}^{k} (t - t_i) \text{ with } x_i = -\ln t_i , \tag{15}$$

and

$$\tilde{w}_{\alpha,\beta}(x)dx := x^\alpha e^{-\beta x} dx \mapsto w_{\alpha,\beta}(t)dt := \left|\ln t\right|^\alpha t^{\beta-1} dt . \tag{16}$$

Hence,

$$\int_0^1 p_m^*(t)p_l^*(t)W_k^2 w_{\alpha,\beta}(t)dt = \int_0^\infty \tilde{\psi}_m(x)\tilde{\psi}_l(t)\tilde{w}_{\alpha,\beta}(t)dt = 0 \tag{17}$$

where $m \neq l$. This implies that the set of algebraic polynomials $\left\{p_m^* : m = 0,1,...,n\right\}$ is orthogonal with respect to the weight function $W_k^2 w_{\alpha,\beta}$ over the interval $[0,1]$. Thus, each

p_m^*, except for m = 0, has m real and distinct zeros, say $t_{i,m}, i = 1, 2, \ldots, m$, in the interval $(0,1)$. Now using the inverse transformation $t \mapsto e^{-x}$, we note that $t_{i,m} = e^{-x_{i,m}}$ where $x_{i,m} \in (0, \infty)$. Thus, by Remark 2, $x_{i,m}, i = 1, 2, \ldots, m$, are the m positive and distinct zeros of the factor exponential polynomial E_m^*.

3. Computational Aspects

As observed in (9), the computation of *OEZI* is entirely based on the recursion coefficients α_j and β_j which in fact are quotients of following type of integrals:

$$\left.\begin{aligned} \left\langle e^{-x}\tilde{\psi}_j, \tilde{\psi}_j \right\rangle_{\tilde{w}_{\alpha,\beta}} &= \int_0^\infty e^{-x}\tilde{\psi}_j(x)\tilde{\psi}_j(x)\tilde{w}_{\alpha,\beta}(x)dx, \\ \left\langle \tilde{\psi}_j, \tilde{\psi}_j \right\rangle_{\tilde{w}_{\alpha,\beta}} &= \int_0^\infty \tilde{\psi}_j(x)\tilde{\psi}_j(x)\tilde{w}_{\alpha,\beta}(x)dx. \end{aligned}\right\} \qquad (18)$$

Note that the degree of *OEZI*'s grow higher and higher with successive applications of the recurrence relation (9). Therefore, the *propagation of round-off error* in the computation of the integrals (18) will cause severe *ill-conditioning* effect on the 3-term recurrence relation. This is a similar situation which we encounter in computing the classical orthogonal polynomials [3]. To overcome this problem, approximation of inner products with a suitable quadrature rule (discretization) is highly recommended [3]. The procedure explained below leads to the process of computing the OEZI's which avoids the impact of ill-conditioning.

A. Steiltjes procedure to compute recursion coefficients: The orthogonal exponential 0-interpolant computed at each stage by the recurrence relation (9) in return is used to compute the recursion coefficients for the next stage. This procedure is described in subsection 2.1 and is due to Steiltjes [4].

B. Transformation from infinite to finite interval: The most popular processes of discretiztion of integrals involve Chebyshev zeros. Since these zeros lie in the interval

$(-1,1)$, it is appropriate to covert the computations over the interval $[0,\infty)$ to the interval $[-1,1)$. For this, we use the substitution $x = h(u) := -\ln\frac{1-u}{2}$ to transform all integrals of the form $\int_0^\infty (.)dx$ to $\int_{-1}^1 (.)du$. The suggested substitution basically transforms

1. the weight function $\tilde{w}_{\alpha,\beta}(x) = x^\alpha e^{-\beta x}$ to $w_{\alpha,\beta}(u) = \left|\ln\left(\frac{1-u}{2}\right)\right|^\alpha \left(\frac{1-u}{2}\right)^\beta$.
2. the differential dx to $\frac{du}{1-u}$
3. the exponential polynomial $E_n(x) = \sum_{j=0}^n a_i e^{-jx}$ to an algebraic polynomial $p_n(u) = \sum_{j=0}^n \frac{a_j}{2^j}(1-u)^j$.
4. the product $\prod_{i=0}^k (e^{-x} - e^{-x_i})$ to $\prod_{i=0}^k \frac{u_j - u}{2}$ where $u_j = 1 - 2e^{-x_j}$

C. Discretization by Fejer quadrature rule: Once the integrals $\int_0^\infty (.)dx$ are transformed to $\int_{-1}^1 (.)du$, the inner products in (18) are computed by discretization of $\int_{-1}^1 (.)du$ using the following quadrature rule [3], [5]:

$$\int_{-1}^1 F(u)\omega(u)du \approx \sum_{j=1}^M w_j^M F(u_j^M)\omega(u_j^M) \tag{19}$$

with the following choice of nodes $u_j^M \in (-1,1)$ and weights $w_j^M > 0$:

$$u_i^M := \cos\theta_i^M; \quad w_i^M := \frac{2}{M}\left\{1 - 2\sum_{j=1}^{\lfloor M/2 \rfloor} \frac{\cos\left(2j\theta_i^M\right)}{4j^2 - 1}\right\} \tag{20}$$

where

$$\theta_i^M = \frac{2i-1}{2n}. \tag{21}$$

The method of approximating an integral by (19) is known as Fejer quadrature rule if the nodes u_i^M and weights w_i^M are based on (20) and (21).

D. Computational procedure: In order to implement the suggested computational procedure, the following steps may be implemented:

Step 1: Input required:

i. (α, β) = Parameters involved in the weight function $\tilde{w}_{\alpha,\beta}(x) = x_\alpha e^{-\beta x}$

ii. NC = Number of Chebyshev points used for Discretization

iii. NF = Number of points fixed on $[0,\infty)$ for the OEZI's

iv. (x_i) = NF-vector of fixed points required for interpolation

v. NOP = Number of Orthogonal polynomials used for approximation

Step 2: Computation of NC Nodes (Chebyshev points) and NC Weights by using (20) and (21).

Step 3: Transformation of the following functions by means of $x = -\ln\frac{1-t}{2}$ from $[0,\infty)$ to $[-1,1)$:

(i) weight function $\tilde{w}_{\alpha,\beta}(x) = x^\alpha e^{-\beta x}$

(ii) $\prod_{j=1}^{NF}(e^{-x} - e^{-x_j})$

Step 4: Discretization of the transformed inner-products by means of (19).

Step 5: Computation of NOP orthogonal 0-intepolants (algebraic) with respect to the transformed weight function $w_{\alpha,\beta}(u) = |h(u)|^\alpha \left(\frac{1-u}{2}\right)^\beta$.

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