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Abstract Moments of discrete distributions are usually calculated by averaging the probability mass function or calculating characteristic function and then differentiating it. In this paper we try to popularize a mass identity technique for calculating moments. Since the probability mass function adds to one, an identity called mass identity or density identity is obtained. Differentiating repeatedly by a continuous parameter in the density identity, we obtain higher order mass identities that may be useful beyond the calculation of moments.

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1. Introduction

Moments of discrete distributions are calculated variously by averaging the probability mass function, differentiating the characteristic function etc. In this paper we try to popularize a mass identity technique for calculating moments. Since the probability mass function adds to one, an identity called mass identity or density identity is obvious. In case, the probability mass function has at least one continuous parameter, one can differentiate the density identity repeatedly to obtain higher order mass identities. In case there is no continuous parameter in the probability mass function, one can be incorporated. The general mass identity of general order may be useful beyond the calculation of moments. The technique applies well to continuous probability density functions.

Consider a probability mass function $f\ (x;\theta) > 0,\ x \in D$ where θ is continuous. Then $\sum_{x \in D}^{\infty} f\ (x;\theta) = 1$ will be called the density identity. By differentiating the density identity, some new identities can be generated which is helpful in calculating factorial moments. Note that

$$\mu'_{\{r\}} = E(X^{\{r\}}), \\ = E[X(X-1)\cdots(X-r+1)], \quad r = 1,2,\cdots \tag{1}$$

is the r-th factorial moment. These can then be used to find raw moments by the relation

$$\mu'_r = E(X^r) \\ = \sum_{i=1}^{r} S(r,i)\mu'_{\{i\}}, \quad (r = 1,2,\cdots) \tag{2}$$

where $S(r,i)$ is the Stirling number of the second kind given by

$$S(r,i) = \frac{1}{i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j} j^r, \quad 0 \le i \le r.$$

The raw moments can be further used to find central moments by

$$\mu_r = \sum_{i=0}^{r}(-1)^i \binom{r}{i} \mu'_{r-i}(\mu'_1)^i \tag{3}$$

(Johnson, Kotz and Kemp, 1993, 42-44). For a combinatorial proof of Stirling numbers of the second kind, the reader is referred to Roberts (1984, pp. 182-183), and for an inductive proof , the reader is referred to Joarder and Mahmood (1997).

2. Identities Based on Mass Identities

For nonnegative integer a, we will use the notation:

$$x^{\{a\}} = x(x-1)\cdots(x-a+1), x^{\{0\}} = 1, \\ x_{\{a\}} = x(x+1)\cdots(x+a-1), \; x_{\{0\}} = 1.$$

In this section, we derive higher order mass identities for a number of univariate discrete distributions.

(i) Consider the geometric distribution with probability mass function

$$f(x) = pq^{x-1}, \; 0 < p < 1, \; p+q = 1, \; x = 1,2,\cdots$$

Since $f(x)$ is a probability mass function, we have $\sum_{x=1}^{\infty} pq^{x-1} = 1$ or equivalently

$$\sum_{x=1}^{\infty} q^x = q(1-q)^{-1} \tag{2.1}$$

which is an identity in q.

Theorem 2.1 For $0 < p < 1$ and $p + q = 1$, $\sum_{x=1}^{\infty} x^{\{a\}} q^x = a!\, p^{-(a+1)} q^a$ **.**

Proof. The first identity in (a) below is equivalent to (2.1). Differentiating (2.1) with respect to q repeatedly, we have the following identities from (b) to (e).

$(a) \sum_{x=1}^{\infty} q^x = p^{-1} q,$

$(b) \sum_{x=1}^{\infty} x q^x = p^{-2} q,$

$(c) \sum_{x=1}^{\infty} x^{\{2\}} q^x = 2 p^{-3} q^2,$

$(d) \sum_{x=1}^{\infty} x^{\{3\}} q^x = 6 p^{-4} q^3,$

$(e) \sum_{x=1}^{\infty} x^{\{4\}} q^x = 24 p^{-5} q^4.$

The generalization is obvious.

Corollary 2.1 For $0 < p < 1$ and $p + q = 1$, the following identities are true:

$(a) \sum_{x=1}^{\infty} x^2 q^x = p^{-3}(q^2 + q) = p^{-3} q(q+1),$

$(b) \sum_{x=1}^{\infty} x^3 q^x = p^{-4}(q^3 + 4q^2 + q),$

$(c) \sum_{x=1}^{\infty} x^4 q^x = p^{-5}(q^4 + 11q^3 + 11q^2 + q).$

Proof.

$$(a) \sum_{x=1}^{\infty} x^2 q^x = \sum_{x=1}^{\infty} (x^{\{2\}} + x) q^x$$
$$= 2p^{-3} q^2 + p^{-2} q,$$

$$(b) \sum_{x=1}^{\infty} x^3 q^x = \sum_{x=1}^{\infty} (x^{\{3\}} + 3x^{\{2\}} + x) q^x$$
$$= 6p^{-4} q^3 + 3(2p^{-3} q^2) + p^{-2} q,$$

$$(c)\sum_{x=1}^{\infty} x^4 q^x = \sum_{x=1}^{\infty}(x^{\{4\}} + 6x^{\{3\}} + 7x^{\{2\}} + x)q^x$$
$$= 24p^{-5}q^4 + 36p^{-4}q^3 + 7(2p^{-3}q^2) + p^{-2}q.$$

Raw moments are then calculated in the following way:

$$E(X) = \sum_{x=1}^{\infty} xf(x) = pq^{-1}\sum_{x=1}^{\infty} xq^x = pq^{-1}(p^{-2}q) = p^{-1},$$
$$E(X^2) = \sum_{x=1}^{\infty} x^2 f(x) = pq^{-1}\sum_{x=1}^{\infty} x^2 q^x = pq^{-1}[p^{-3}(q^2 + q)] = p^{-2}(q+1),$$
$$E(X^3) = \sum_{x=1}^{\infty} x^3 f(x) = pq^{-1}\sum_{x=1}^{\infty} x^3 q^x = pq^{-1}[p^{-4}(q^3 + 4q^2 + q)] = p^{-3}(q^2 + 4q + 1),$$

etc. The centered moments can then be calculated by the above raw moments by (3).

(ii) Consider the Binomial Distribution with the probability mass function

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \ 0 < p < 1, \ p + q = 1, \ x = 0,1,\cdots,n.$$

so that the mass identity is given by

$$\sum_{x=1}^{n} x^{\{a\}} \binom{n}{x}\left(\frac{p}{1-p}\right)^x = (1-p)^{-n}.$$

Theorem 2.2 For $0 < p < 1$ and $p + q = 1$, $\sum_{x=1}^{n} x^{\{a\}} \binom{n}{x}\left(\frac{p}{1-p}\right)^x = n^{\{a\}} p^a q^{-n}$.

(iii) Consider the Poisson Distribution with the probability mass function

$$f(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \ \lambda > 0, \ x = 0,1,\cdots,.$$

so that the mass identity is

$$\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}.$$

Theorem 2.3 For $\lambda > 0$, $\sum_{x=0}^{\infty} x^{\{a\}} \frac{\lambda^x}{x!} = \lambda^a e^{\lambda}$.

(iv) Consider the Negative Binomial Distribution with probability mass function

$$f(x)=\binom{k+x-1}{x}p^k q^x, k=1,2,3,...;0<p<1, p+q=1; x=0,1,2,...,$$

so that the mass identity

$$\sum_{x=0}\binom{k+x-1}{x}q^x=q^{-k}$$

Theorem 2.4 For $k=1,2,3,...;\ 0<p<1$ and $p+q=1$, $\sum_{x=0}x^{\{a\}}\binom{k+x-1}{x}q^x=k_{\{a\}}p^{-(k+a)}q^a$

(iv) Consider the Truncated Poisson Distribution with the probability mass function

$$f(x)=(e^{\lambda}-1)^{-1}\frac{\lambda^x}{x!},\ \lambda>0,\ x=1,2,\cdots,.$$

so that the mass identity is

$$\sum_{x=1}^{\infty}\frac{\lambda^x}{x!}=e^{\lambda}-1$$

Theorem 2.5 For $\lambda>0$, $\sum_{x=1}^{\infty}x^{\{a\}}\frac{\lambda^x}{x!}=\lambda^a e^{\lambda}$.

(vi) Consider the Truncated Binomial Distribution with the probability mass function

$$f(x)=(1-q^n)^{-1}\binom{n}{x}p^x q^{n-x}, 0<p<1, p+q=1; x=1,2,...,n$$

so that the mass identity is

$$\sum_{x=1}^{\infty}\binom{n}{x}\left(\frac{p}{q}\right)^x=q^{-n}-1.$$

Theorem 2.6 For $0<p<1$ and $p+q=1$, $\sum_{x=1}^{\infty}x^{\{a\}}\binom{n}{x}\left(\frac{p}{q}\right)^x=n^{\{a\}}p^a q^{-n}$.

(vii) Consider the Logarithmic Distribution with probability mass function

$$f(x)=-\left(\ln(1-\theta)\right)^{-1}\frac{\theta^x}{x},\quad 0<\theta<1,\ x=1,2,3,\cdots$$

so that the mass identity is

$$\sum_{x=1}\frac{\theta^x}{x}=-\ln(1-\theta).$$

Theorem 2.7: For $0<\theta<1$, $\sum_{x=1}^{\infty} x^{\{k\}} \frac{\theta^x}{x} = (k-1)!\left(\frac{\theta}{1-\theta}\right)^k$.

3. Some General Identities

The identities developed in this paper are applicable in a more general situations than in the context of deriving moments. The following identities are obtained from Section 2 by putting particular values of the continuous parameter involved.

$$(a)\sum_{x=1}^{\infty} x^{\{a\}}\left(\frac{1}{2}\right)^x = a!2,$$

$$(b)\sum_{x=1}^{n} x^{\{a\}}\binom{n}{x} = n^{\{a\}}2^{n-a},$$

$$(c)\sum_{x=0}^{\infty} \frac{x^{\{a\}}}{x!} = e,$$

$$(d)\sum_{x=0}^{\infty} x^{\{a\}}\binom{k+x-1}{x}\left(\frac{1}{2}\right)^x = k_{\{a\}}2^k,$$

$$(e)\sum_{x=1}^{\infty} x^{\{a\}}\frac{e^x}{x!} = e^{a+e},$$

$$(f)\sum_{x=1}^{\infty} x^{\{a\}}\frac{2^{-x}}{x!} = 2^{-a}e^{0.5},$$

$$(g)\sum_{x=1}^{\infty} x^{\{a\}}\binom{n}{x} = n^{\{a\}}2^{n-a},$$

$$(h)\sum_{x=1}^{\infty} x^{\{k\}}\frac{3^{-x}}{x} = (k-1)!2^{-k}.$$

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