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Tilting & Cotilting Modules over Commutative Rings
(Survey & Open Problems)

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(Survey & Open Problems)

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Contents

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Notation

- **General Notation:**

$\aleph_\alpha$	:=	the αth cardinal ($\aleph_0$ stands for countable)
ω_α	:=	the first ordinal of cardinality $\aleph_\alpha$
ω	:=	the first infinite ordinal
$\mathcal{PS}(A)$	:=	the *power set* of the set A
$\mathcal{V} = \mathcal{L}$	:=	Gödel's Axiom of Constructability
$\mathbb{Z}$	:=	the ring of integers
$\mathbb{Q}$	:=	the field of rational numbers
$\mathbb{P}$	:=	the set of prime positive integers
$\mathbb{Z}_n$	:=	$\mathbb{Z}/n\mathbb{Z}$ ($\simeq \{0, 1, ..., n-1\}$, the cyclic group of order n)
$\mathbb{Z}_{p^\infty}$	:=	$\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ (The *Prüfer p-Group*, $p \in \mathbb{P}$)
$\mathbb{J}_p$	:=	$\mathrm{End}(\mathbb{Z}_{p^\infty})$ (The ring of p-adic integers, $p \in \mathbb{P}$)

- **Rings:**

R	:=	an associative ring with $1_R \neq 0_R$
$Z(R)$	:=	$\{r \in R \mid rr' = r'r$ for all $r' \in R\}$ (the *center* of R)
$I \lhd R$	:=	I is a *two-sided* ideal of R
$\mathrm{ann}^l_R(a)$ $(\mathrm{ann}^r_R(a))$	:=	$\{r \in R \mid ra = 0\ (ar = 0)\}$
R^{reg}	:=	$\{a \in R \mid \mathrm{ann}^l_R(a) = 0 = \mathrm{ann}^r_R(a)\}$
R^{op}	:=	the opposite ring of R ($r \cdot_{R^{op}} \widetilde{r} = \widetilde{r} \cdot_R r\ \forall r, \widetilde{r} \in R$)
$R^{(m,n)}$	:=	The (R, R)-bimodule of $m \times n$-matrices over R
$R^{(n,n)}$	:=	The ring of all $n \times n$-matrices with entries in R
$\mathrm{Spec}(R)$	:=	the spectrum of all prime ideals of R
$\mathrm{Max}(R)$	:=	the spectrum of all maximal ideals of R
$\mathrm{Jac}(R)$	:= =	$\bigcap\{P \mid P \lhd R$ is a maximal left ideal of $R\}$ $\bigcap\{P \mid P \lhd R$ is a maximal right ideal of $R\}$

- **Modules:**

R-module	:=	left R-module (unless otherwise specified)
${}_RU$ (U_R)	:=	a left (right) R-module
${}_RU_S$	:=	an (R,S)-bimodule U (where R,S are rings)
${}_R\mathbb{M}$ ($\mathbb{M}_R$)	:=	the category of left (right) R-modules
${}_R^{f.g.}\mathbb{M}$ ($\mathbb{M}_R^{f.g.}$)	:=	the category of *finitely generated* left (right) R-modules
${}_R^{f.p.}\mathbb{M}$ ($\mathbb{M}_R^{f.p.}$)	:=	the category of *finitely presented* left (right) R-modules
${}_R^{s}\mathbb{M}$ ($\mathbb{M}_R^{s}$)	:=	the category of *small* left (right) R-modules
${}_R^{s.s.}\mathbb{M}$ ($\mathbb{M}_R^{s.s.}$)	:=	the category of *self-small* left (right) R-modules
${}_R\mathbb{M}_S$	:=	the category of (R,S)-bimodules (where R,S rings)
$L \leq_R M$	:=	L is an R-submodule of M
$L \lneqq_R M$	:=	L is a *proper* R-submodule of M
$L <_R^{\oplus} M$	:=	${}_RL$ is a *direct summand* of ${}_RM$
$L \leq_R^{ess} M$	:=	${}_RL$ is an *essential* R-submodule of ${}_RM$
$\mathrm{Soc}({}_RU)$	:= =	$\sum\{{}_RX \mid X \leq_R U$ is a *simple* R-submodule$\}$ (the *socle* of ${}_RM$) $\bigcap\{{}_RL \mid L \leq_R^{ess} U$ (an *essential* R-submodule)$\}$
$\mathrm{Rad}({}_RU)$	:= =	$\bigcap\{{}_RX \mid X \lneqq_R U$ is a *maximal* R-submodule$\}$ $\sum\{{}_RL \mid L \lneqq_R U$ is a *superfluous* R-submodule$\}$
$\mathrm{ann}_M(r)$	:=	$\{m \in M \mid rm = 0\}$ ($r \in R$, ${}_RM$ is a left R-module)
$\mathrm{ann}_S(m)$	:=	$\{s \in S \mid sm = 0\}$ ($M \in {}_R\mathbb{M}$ and $\varnothing \neq S \subseteq R$)
$\tau({}_RM)$	:=	$\bigcup\limits_{r \in R\backslash\{0\}} \mathrm{ann}_M(r)$ (the *torsion submodule* of ${}_RM$)
$\mathrm{ann}_R(M)$	:=	$\{r \in R \mid rM = 0\} = \bigcap\limits_{m \in M} \mathrm{ann}_R(m)$
$M^{\#}$	:=	$\{r \in R \mid rM \subsetneqq M\}$
$M_{\#}$	:=	$\bigcup\limits_{m \in M\backslash\{0\}} \mathrm{ann}(m)$
$D_R(M)$	:=	$\{r \in R \mid rM = M\}$ (the *divisibility set* of ${}_RM$)
$\mathrm{E}(M)$	:=	the *injective envelope* of ${}_RM$
$\widehat{M}$	:=	the *pure-injective envelope* of the R-module M
${}_R\mathbf{I}$ ($\mathbf{I}_R$)	:=	an injective cogenerator in ${}_R\mathbb{M}$ ($\mathbb{M}_R$), e.g. $\mathbf{I} = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$

- **Dual Modules**[2]:

M^c	:=	$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ for an R-module M (the *character module* of M)
${}^dM$ (M^d)	:=	$\mathrm{Hom}_R(M, \mathbf{I})$, where ${}_RM$ (M_R) is a left (right) R-module
${}^*M$ (M^*)	:=	$\mathrm{Hom}_R(M, R)$, where M is a left (right) R-module

Commutative Algebra:

- **Ab** : the category of Abelian groups (i.e. $\mathbb{Z}$-modules)
- For a commutative ring R set

R^{reg}	:=	$\{r \in R \mid 0 \neq r$ is NOT a zero-divisor (called also *regular element*)$\}$
$R^{\times}$	:=	$R\backslash\{0\}$ $(= R^{\mathrm{reg}}$ if and only if R is a *domain*)
$U(R)$	:=	$\{r \in R \mid r$ is a unit (invertible) in $R\}$
$\mathrm{Ass}(M)$	:=	$\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} = \mathrm{ann}_R(m)$ for some $m \in M\}$
$R_{\mathbf{S}}$	:=	$\{\frac{r}{s} \mid r \in R,\ s \in \mathbf{S}\}$ (the *localization* of R at $\mathbf{S}$)
$\mathbf{Q}(R)$	:=	the (*classical*) *total quotient ring* of R $(= R_{R^{\mathrm{reg}}})$
$\mathbf{Q}$	:=	the *quotient field* of an *integral domain* R $(:= R_{R^{\times}})$
I^{-1}	:=	$\{q \in \mathbf{Q}(R) \mid qI \subseteq R\}$ (for an ideal $I \lhd R$)

[2]The notion of a *dual module* of a given module differs from an author to another

Homological Algebra:

- For an R-module M we set

$\mathbf{P}_{\widetilde{M}}$	:=	$\ldots \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \overset{\partial_n}{\to} P_{n-1} \to \ldots \to P_1 \overset{\partial_1}{\to} P_0 \longrightarrow 0$
		(obtained from a *projective resolution* $\mathbf{P}_{\widetilde{M}} \to M \to 0$);
$\mathbf{F}_{\widetilde{M}}$	:=	$\ldots \to F_{n+1} \overset{\partial_{n+1}}{\to} F_n \overset{\partial_n}{\to} F_{n-1} \to \ldots \to F_1 \overset{\partial_1}{\to} F_0 \longrightarrow 0$;
		(obtained from a *flat resolution* $\mathbf{F}_{\widetilde{M}} \to M \to 0$);
$\mathbf{E}_{\widetilde{M}}$	:=	$0 \to E^0 \overset{\delta^1}{\to} E^1 \to \ldots \to E^{n-1} \overset{\delta^n}{\to} E^n \overset{\delta^{n+1}}{\to} E^{n+1} \to \ldots$
		(obtained from an *injective coresolution* $0 \to M \to \mathbf{E}_{\widetilde{M}}$);
$\mathfrak{P}_M$	:=	$0 \to X \to P \overset{\pi}{\to} M \to 0$ (a projective presentation of ${}_RM$);
$\mathfrak{F}_M$	:=	$0 \to X \to F \overset{\pi}{\to} M \to 0$ (a flat presentation of ${}_RM$);
$\mathfrak{E}_M$	:=	$0 \to M \overset{\iota}{\to} E \to Z \to 0$ (an injective copresentation of ${}_RM$);

- The following table provides a summary of the definitions presented in the sequel:

$\mathrm{Ext}_R^n(\bullet, B)$	:=	$(\mathcal{L}^n\mathrm{Hom}_R(-,B))(\bullet)$,	$\mathrm{Ext}_R^n(A,B)$	=	$H^n(\mathrm{Hom}_R(\mathbf{P}_{\widetilde{A}}, B))$;
$\overline{\mathrm{Ext}}_R^n(A, \bullet)$	:=	$(\mathcal{R}_n\mathrm{Hom}_R(A,-))(\bullet)$,	$\overline{\mathrm{Ext}}_R^n(A,B)$	=	$H^n(\mathrm{Hom}_R(A, \mathbf{E}_{\widetilde{B}}))$;
$\mathrm{Tor}_n^R(\bullet, B)$	:=	$(\mathcal{L}^n(- \otimes_R B))(\bullet)$,	$\mathrm{Tor}_n^R(A,B)$	=	$H_n(\mathbf{P}_{\widetilde{A}} \otimes_R B))$;
$\overline{\mathrm{Tor}}_n^R(A, \bullet)$	:=	$(\mathcal{L}^n(A \otimes_R -))(\bullet)$,	$\overline{\mathrm{Tor}}_n^R(A,B)$	=	$H_n(A \otimes_R \mathbf{P}_{\widetilde{B}}))$;
$\mathrm{tor}_n^R(A, \bullet)$	:=	$H_n(\mathbf{F}_{\widetilde{A}} \otimes_R -))$,	$\mathrm{tor}_n^R(A,B)$	=	$H_n(\mathbf{F}_{\widetilde{A}} \otimes_R B))$;
$\overline{\mathrm{tor}}_n^R(\bullet, B)$	:=	$H_n(A \otimes_R \mathbf{F}_{\widetilde{B}}))$,	$\overline{\mathrm{tor}}_n^R(A,B)$	=	$H_n(A \otimes_R \mathbf{F}_{\widetilde{B}}))$;

- For the ring R we set

LG.dim.(R)	:=	the *left global dimension* of R
RG.dim.(R)	:=	the *right global dimension* of R
W.dim.(R)	:=	the *weak dimension* of R

- For $\mathcal{U} \subseteq {}_RM$ and $i \geq 1$, set[3]

$\mathcal{U}^{\perp_i}$	$:=$	$\bigcap_{U \in \mathcal{U}} \{Y \in {}_R\mathbb{M} \mid \mathrm{Ext}^i_R(U,Y) = 0\};$		$\mathcal{U}^{\perp_\infty}$	$:=$	$\bigcap_{i=1}^{\infty} \mathcal{U}^{\perp_i}$
$\mathcal{U}^{\top_i}$	$:=$	$\bigcap_{U \in \mathcal{U}} \{Y \in \mathbb{M}_R \mid \mathrm{Tor}^R_i(U,Y) = 0\};$		$\mathcal{U}^{\top_\infty}$	$:=$	$\bigcap_{i=1}^{\infty} \mathcal{U}^{\top_i}$
${}^{\perp_i}\mathcal{U}$	$:=$	$\bigcap_{U \in \mathcal{U}} \{X \in {}_R\mathbb{M} \mid \mathrm{Ext}^i_R(X,U) = 0\};$		${}^{\perp_\infty}\mathcal{U}$	$:=$	$\bigcap_{i=1}^{\infty} {}^{\perp_i}\mathcal{U}$
${}^{\top_i}\mathcal{U}$	$:=$	$\bigcap_{U \in \mathcal{U}} \{X \in \mathbb{M}_R \mid \mathrm{Uor}^R_i(X,U) = 0\};$		${}^{\top_\infty}\mathcal{U}$	$:=$	$\bigcap_{i=1}^{\infty} {}^{\top_i}\mathcal{U}$

- For an R-module ${}_RU$, we set:

proj.dim.$({}_RU)$	$:=$	the *projective dimension* of ${}_RU$
inj.dim.$({}_RU)$	$:=$	the *injective dimension* of ${}_RU$
flat.dim.$({}_RU)$	$:=$	the *flat dimension* of ${}_RU$
U^+	$:=$	$\mathrm{Ke}(\mathrm{Hom}_R(U,-)) \cap \mathrm{Ke}(\mathrm{Ext}^1_R(U,-))$
${}^+U$	$:=$	$\mathrm{Ke}(\mathrm{Hom}_R(U,-)) \cap \mathrm{Ke}(\mathrm{Ext}^1_R(-,U))$

- For each $n \geq 0$ set

${}_R\mathcal{P}_n$	$:=$	$\{{}_RM \mid \text{proj.dim.}({}_RM) \leq n\};$		${}_R\mathcal{P}$	$:=$	$\bigcup_{n=0}^{\infty} {}_R\mathcal{P}_n$
${}_R\mathcal{I}_n$	$:=$	$\{{}_RM \mid \text{inj.dim.}({}_RM) \leq n\};$		${}_R\mathcal{I}$	$:=$	$\bigcup_{n=0}^{\infty} {}_R\mathcal{I}_n$
${}_R\mathcal{F}_n$	$:=$	$\{{}_RM \mid \text{flat.dim.}({}_RM) \leq n\};$		${}_R\mathcal{F}$	$:=$	$\bigcup_{n=0}^{\infty} {}_R\mathcal{F}_n$

[3] If i is dropped, then $i = 1$ is meant.

- We set

$$ {}_R\mathcal{PROJ} := {}_R\mathcal{P}_0,\ {}_R\mathcal{INJ} := {}_R\mathcal{I}_0 \text{ and } {}_R\mathcal{FLAT} := {}_R\mathcal{F}_0. $$

- For a non-empty class of left R-module $\varnothing \neq \mathcal{U} \subseteq {}_R\mathbb{M}$, we set

$\overline{\mathcal{U}}$:= the smallest *closed*[4] subcategory of ${}_R\mathbb{M}$ that contains $\mathcal{U}$;
$\overline{\mathcal{U}}_f$:= the smallest *finitely closed*[5] subcategory of ${}_R\mathbb{M}$ that contains $\mathcal{U}$.

- For an R-module ${}_RU$ and $n \geq 1$, let [6]

$\mathrm{Gen}_n(U)$	:=	$\{{}_RM \mid \exists$ exact sequence $U^{(\Lambda_n)} \to ... \to U^{(\Lambda_1)} \to M \to 0\}$
$\mathrm{Gen}_\infty(U)$	:=	$\{{}_RM \mid \exists$ e.s. $... \to U^{(\Lambda_n)} \to ... \to U^{(\Lambda_1)} \to M \to 0\}$
$\mathrm{gen}_n(U)$	:=	$\{{}_RM \mid \exists$ e.s. $U^{k_n} \to ... \to U^{k_1} \to M \to 0,\ k_i \in \mathbb{N}\}$
$\mathrm{gen}_\infty(U)$	:=	$\{{}_RM \mid \exists$ e.s. $... \to U^{k_n} \to ... \to U^{k_1} \to M \to 0,\ k_i \in \mathbb{N}\}$
$\mathrm{Add}(U)$	:=	$\{{}_RM \mid M \leq_R^{\oplus} U^{(\Lambda)}$ for some index set $\Lambda\}$
$\mathrm{add}(U)$	:=	$\{{}_RM \mid {}_RM \leq_R^{\oplus} U^k$ for some $k \in \mathbb{N}\}$
$\sigma_n[U]$	:=	$\{L \mid L \leq_R N$ for some $N \in \mathrm{Gen}_n({}_RU)\}$
$\sigma_n^f[U]$	:=	$\{L \mid L \leq_R N$ for some $N \in \mathrm{gen}_n({}_RU)\}$

- By convention, for any R-module ${}_RU$ we have:

$$ \mathrm{Gen}_0(U) = \sigma_0[U] = {}_R\mathbb{M}. $$

[4] i.e. closed under submodules, factor modules and arbitrary direct sums
[5] i.e. closed under submodules, factor modules and finite direct sums
[6] If n is dropped, then $n = 1$ is meant. We also set $\sigma_0[{}_RU] = \mathrm{Gen}_0({}_RU) := {}_R\mathbb{M}$.

- For an R-module ${}_RU$ and $n \geq 1$, let[7]

$\mathrm{Cogen}_n(U)$	$:=$	$\{{}_RM \mid \exists$ exact sequence $0 \to M \to U^{\Lambda_1} \to ... \to U^{\Lambda_n}\}$
$\mathrm{Cogen}_\infty(U)$	$:=$	$\{{}_RM \mid \exists$ e.s. $0 \to M \to U^{\Lambda_1} \to ... \to U^{\Lambda_n} \to ..., k_i \in \mathbb{N}\}$
$\mathrm{cogen}_n(U)$	$:=$	$\{{}_RM \mid \exists$ e.s. $0 \to M \to U^{k_1} \to ... \to U^{k_n}, k_i \in \mathbb{N}\}$
$\mathrm{cogen}_\infty(U)$	$:=$	$\{{}_RM \mid \exists$ e.s. $0 \to M \to U^{k_1} \to ... \to U^{k_n} \to ..., k_i \in \mathbb{N}\}$
$\mathrm{Prod}(U)$	$:=$	$\{{}_RM \mid M <_R^\oplus U^\Lambda$ for some index set $\Lambda\}$
$\mathrm{prod}(U)$	$:=$	$\{{}_RM \mid M <_R^\oplus U^k$ for some $k \in \mathbb{N}\}$
$\pi_n[U]$	$:=$	$\{N/L \mid {}_RL \leq {}_RN$ and $N \in \mathrm{Cogen}({}_RN)\}$
$\pi_n^f[U]$	$:=$	$\{N/L \mid {}_RL \leq {}_RN$ and $N \in \mathrm{cogen}({}_RN)\}$

- By convention, for any R-module ${}_RU$ we have:

$$\mathrm{Cogen}_0(U) = \pi_0[U] = {}_R\mathbb{M}.$$

- For ${}_RU$ and $n \geq 0$, set[8]

$\mathrm{Pres}_n(U)$	$:=$	$\mathrm{Gen}_{n+1}(U)$;		$\mathrm{pres}_n(U)$	$:=$	$\mathrm{gen}_{n+1}(U)$;
$\mathrm{Copres}_n(U)$	$:=$	$\mathrm{Cogen}_{n+1}(U)$;		$\mathrm{copres}_n(U)$	$:=$	$\mathrm{cogen}_{n+1}(U)$.

- For any subclass $\mathcal{U} \subseteq {}_R\mathbb{M}$ (respectively $\mathcal{U} \subseteq \mathbb{M}_R$) we set

${}^{f.g.}\mathcal{U}$	$:=$	$\mathcal{U} \cap {}^{f.g.}_R\mathbb{M}$;		$\mathcal{U}^{f.g.}$	$:=$	$\mathcal{U} \cap \mathbb{M}_R^{f.g.}$	
${}^{f.p.}\mathcal{U}$	$:=$	$\mathcal{U} \cap {}^{f.p.}_R\mathbb{M}$;		$\mathcal{U}^{f.p.}$	$:=$	$\mathcal{U} \cap \mathbb{M}_R^{f.p.}$	
${}^{s.s.}\mathcal{U}$	$:=$	$\mathcal{U} \cap {}^{s.s.}_R\mathbb{M}$;		$\mathcal{U}^{s.s.}$	$:=$	$\mathcal{U} \cap \mathbb{M}_R^{s.s.}$	

[7]If n is dropped, then $n = 1$ is meant.
[8]If n is dropped, then $n = 1$ is meant.

- We set

  $$R\text{mod} := \{_R U \mid U \text{ has a } \textit{projective resolution} \text{ consisting of f.g. } R\text{-modules}\}.$$

  The subcategory $\text{mod}R \subseteq \mathbb{M}_R^{f.g.}$ is analogously defined.

  Remark: Objects in Rmod (mod$R$) are called *strongly finitely presented*. If R is left (right) coherent, then

  $$R\text{mod} = {}_R^{f.p.}\mathbb{M} \ (\text{mod}R = \mathbb{M}_R^{f.p.}).$$

- For $\mathcal{U} \subseteq {}_R\mathbb{M}$ ($\mathcal{U} \subseteq \mathbb{M}_R$) set

  $$\mathcal{U}^{<\omega} := \mathcal{U} \cap R\text{mod} \ (\mathcal{U}^{<\omega} := \mathcal{U} \cap \text{mod}R).$$

- Set

  $$\begin{aligned} {}_R\mathcal{PROG} &:= \{_R P \mid P \text{ is a } \textit{progenerator} \text{ in } {}_R\mathbb{M}\}; \\ &= \{_R P \mid P \text{ is f.g., projective and a generator in } {}_R\mathbb{M}\}; \end{aligned}$$

- For $\mathcal{U} \subseteq {}_R\mathbb{M}$ we set

$\mathcal{PROJ}(\mathcal{U})$	:=	$\{_R U \mid \text{Hom}_R(U, -)$ respects short exact sequences in $\mathcal{U}\}$; (We say U is *projective on* $\mathcal{U}$ in this case);
$\mathcal{INJ}(\mathcal{U})$	:=	$\{_R U \mid \text{Hom}_R(-, U)$ respects short exact sequences in $\mathcal{U}\}$. (We say U is *injective on* $\mathcal{U}$ in this case);

- For $_R U$ and $n \geq 1$ we set[9]

${}_R\mathcal{Q}\text{-}\mathcal{PROG}_n$	:=	$\{_R U \mid U \text{ is a progenerator in } \sigma_n[U]\}$;
${}_R\mathcal{GEN}_n$	:=	$\{_R U \mid \text{Gen}_n(U) = {}_R\mathbb{M}\}$;
${}_R\mathcal{SUBGEN}_n$	:=	$\{_R U \mid \sigma_n[U] = {}_R\mathbb{M}\}$;
${}_R\mathcal{S}\text{-}\mathcal{G}_n$	:=	$\{_R U \mid \sigma_n[U] = \text{Gen}_n(U)\}$;
${}_R\mathcal{INJ}\text{-}\mathcal{GEN}_n$	:=	$\{_R U \mid {}_R\mathcal{INJ} \subseteq \text{Gen}_n(U)\}$.

[9] If n is dropped, then $n = 1$ is meant.

- **Remarks**:

1. $\mathcal{Q}\text{-}\mathcal{PROG}_0 = {}_R\mathcal{PROG}$ and

${}_R\mathcal{Q}\text{-}\mathcal{PROG}_1$	$:=$	$\{{}_RU \mid U$ is a progenerator in $\sigma[{}_RU]\}$;
	$=$	$\{{}_RU \mid U$ is a *quasi-progenerator*[10] in ${}_R\mathbb{M}\}$;

2. ${}_R\mathfrak{I}\text{-}\mathcal{GEN}_1 = {}_R\mathcal{SUBGEN}_1$.

3. $\sigma[{}_RU] = \overline{\mathrm{Gen}({}_RU)}$ and $\sigma_f[{}_RU] = \overline{\mathrm{gen}({}_RU)}_f$.

4. We have $\pi[{}_RU] = {}_{R/\mathrm{ann}_R(U)}\mathbb{M}$, whence a closed subcategory of ${}_R\mathbb{M}$ (i.e. $\sigma[{}_RU] \subseteq \pi[{}_RU]$).

5. We have $\mathrm{prod}({}_RU) = \mathrm{add}({}_RU) \subseteq \mathrm{Add}({}_RU) \cap \mathrm{Prod}({}_RU)$ and

$\mathrm{Add}({}_RU)$	$\subseteq$	$\mathrm{Pres}({}_RU)$	$\subseteq$	$\mathrm{Gen}({}_RU)$	$\subseteq$	$\sigma[{}_RU]$;
$\mathrm{add}({}_RU)$	$\subseteq$	$\mathrm{pres}({}_RU)$	$\subseteq$	$\mathrm{gen}({}_RU)$	$\subseteq$	$\sigma_f[{}_RU]$;
$\mathrm{Prod}({}_RU)$	$\subseteq$	$\mathrm{Copres}({}_RU)$	$\subseteq$	$\mathrm{Cogen}({}_RU)$	$\subseteq$	$\pi[{}_RU]$;
$\mathrm{prod}({}_RU)$	$\subseteq$	$\mathrm{copres}({}_RU)$	$\subseteq$	$\mathrm{cogen}({}_RU)$	$\subseteq$	$\pi_f[{}_RU]$;

[10] in the sense of Fuller [Ful:1974]

Tilting Theory:

- For $n \geq 0$[11], we set

${}_R\mathcal{STAR}_n$	$:= \{{}_RU \mid U$ is an *n-star R-module*$\}$
${}_R\mathcal{STAR}$	$:= \{{}_RU \mid U$ is an n-star R-module for some $n \in \mathbb{N}\}$;
${}_R\mathcal{STAR}_n^{s.s.}$	$:= \{{}_RU \mid U$ is a *self-small n-star R-module*$\}$
${}_R\mathcal{STAR}^{s.s.}$	$:= \{{}_RU \mid U$ is a *self-small n-star R-module for some $n \in \mathbb{N}$*$\}$
${}_R\mathcal{Q}\text{-}\mathcal{TILT}_n$	$:= \{{}_RT \mid T$ is *n-quasi-tilting*$\}$
${}_R\mathcal{Q}\text{-}\mathcal{TILT}$	$:= \{{}_RT \mid T$ is n-quasi-tilting for some $n \in \mathbb{N}\}$;
${}_R\mathcal{SELF}\text{-}\mathcal{TILT}_n$	$:= \{{}_RT \mid T$ is *n-self-tilting*$\}$
${}_R\mathcal{SELF}\text{-}\mathcal{TILT}$	$:= \{{}_RT \mid T$ is n-self-tilting $\}$;
	for some $n \in \mathbb{N}$
${}_R\mathcal{TILT}_n$	$:= \{{}_RT \mid T$ is *n-tilting*$\}$
${}_R\mathcal{TILT}$	$:= \{{}_RT \mid T$ is n-tilting for some $n \in \mathbb{N}\}$;
${}_R\mathcal{C}\text{-}\mathcal{TILT}_n$	$:= \{{}_RT \mid T$ is *classical n-tilting*$\}$
${}_R\mathcal{C}\text{-}\mathcal{TILT}$	$:= \{{}_RT \mid T$ is classical n-tilting for some $n \in \mathbb{N}\}$

[11] If n is dropped, then n is an arbitrary non-negative integer.

Abstract

Tilting (cotilting) modules were introduced by S. Brenner and M. Butler [BB:1980] as a natural generalization of progenerators (injective cogenerators). Since then, "Tilting (Cotilting) Theory" is attracting the attention of many researchers in different aspects of mathematics, including mainly "*Representation Theory*" of (finite dimensional, Artin) algebras, "*Categories of Modules*" and "*Commutative Algebra*".

This technical report includes a **survey** of the main results known on the structure of tilting (cotilting) modules, and some of their generalizations, over commutative rings. We studied intensively these results and presented some of them in several interactive talks at the "*Commutative Algebra Weekly Seminar*" (KFUPM) during the period December 2004 - June 2006.

At the end of the report we include a number of *open problems* concerning the structure of tilting (cotilting) modules over commutative rings. Some of these problems were formulated by us after careful investigation and intensive literature review, while the others were suggested by experts in "Tilting (Cotilting) Theory" or were highlighted in the literature.

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13D07: Homological functors on modules
13F05: Dedekind, Prüfer and Krull rings and their generalizations
13F30: Valuation rings

16D50: Injective modules, self-injective rings
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Introduction

In [Mor:1958], K. Morita presented the first major results on equivalences and dualities between categories of modules for a pair of rings R and S.

A left R-module $_RP$ is said to be a **progenerator**, iff $_RP$ is finitely generated, projective and a generator. A bimodule $_RP_S$ induces a *Morita equivalence* between the categories of modules $_R\mathbb{M}$ and $_S\mathbb{M}$ if and only if $_RP$ is a progenerator and $S \simeq \mathrm{End}(_RP)^{op}$ (equivalently, M_S is a progenerator and $R \simeq \mathrm{End}(M_S)$); and such an equivalence is induced by the *covariant* functors $\mathrm{Hom}_R(P,-) : {}_R\mathbb{M} \to {}_S\mathbb{M}$ and $P \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}$.

On the other hand, a bimodule $_RU_S$ induces a *Morita duality* between suitable full subcategories of $_R\mathbb{M}$ and $\mathbb{M}_S$ if and only if $_RU_S$ is *faithfully balanced* and $_RU$, U_S are **injective cogenerators;** and such a duality is obtained by restrictions of the *contravariant* functors $\mathrm{Hom}_R(-,U) : {}_R\mathbb{M} \to \mathbb{M}_S$ and $\mathrm{Hom}_S(-,U) : \mathbb{M}_S \to {}_R\mathbb{M}$. For details, the reader is referred to [AF:1974] and [CF:2004].

The definitions of *tilting* (*cotilting*) *modules* of arbitrary finite projective (injective) dimension, that we adopt, are due to L. Angeleri-Hügel and F.U. Coelho [AC:2001]. As clarified by S. Bazzoni [Baz:2004(b)], an n-**tilting** (n-**cotilting**) **module** (for $1 \leq n < \infty$) can be defined *briefly* as an R-module U for which $\mathrm{Gen}_n(U) = U^{\perp_\infty}$ ($\mathrm{Cogen}_n(U) = {}^{\perp_\infty}U$), where $\mathrm{Gen}_n(U)$ is the class of U-*n-generated*[12] R-modules ($\mathrm{Cogen}_n(U)$ is the class of U-*n-cogenerated*[13] R-modules) and

$$U^{\perp_\infty} := \bigcap_{i=1}^{\infty} \mathrm{Ker}(\mathrm{Ext}^i_R(U,-)) \;\; ({}^{\perp_\infty}U := \bigcap_{i=1}^{\infty} \mathrm{Ker}(\mathrm{Tor}^R_i(-,U))).$$

[12] i.e. of U-*codominant dimension* $\geq n$

[13] i.e. of U-*dominant dimension* $\geq n$

As mentioned above, *Tilting theory* is a generalization of *Morita theory* on equivalences between module categories. In fact, every progenerator ${}_RP$ is tilting and every tilting module ${}_RT$ satisfying *suitable finiteness conditions* induces an equivalence between suitable full subcategories of ${}_R\mathbb{M}$ and ${}_{\mathrm{End}({}_RT)^{op}}\mathbb{M}$ as shown by the **Tilting Theorem** [BB:1980] and its generalizations (e.g. [Miy:1986]).

Moreover, tilting modules play an important role in the representation theory of (finite dimensional, Artin) algebras (e.g. [Hap:1988], [ASS:2006]). According to I. Assem et. al. [ASS:2006, Page 184]: *one of the main ideas is that when the representation theory of an algebra A is difficult to study directly, it may be convenient to replace it with another algebra B and to reduce a problem on A to a problem on B. To do this we construct a module ${}_AT$, called a tilting module, which can be thought of as being close to the (Morita) progenerators such that, if $S := End({}_AT)^{op}$, then the categories of finitely generated left A-modules and finitely generated left S-modules are reasonably close to each other (although generally not equivalent).*

The first examples of tilting modules were *finitely generated* over finite dimensional and Artin algebras (e.g. [BB:1980], [HR:1982]), and were generalized later to finitely generated tilting modules of projective dimension at most 1 over arbitrary ground rings (which we call after [GT:2006] **classical 1-tilting modules).** However, many interesting examples of *infinitely generated* modules (over commutative rings) having the *tilting property* were discovered, among which are the Fuchs divisible module ∂ over an arbitrary integral domain R studied by A. Facchini (e.g. [Fac:1987], [Fac:1988]), the Bass tilting module $B := \bigoplus\limits_{\mathrm{ht}(\mathfrak{q})=0} \mathrm{E}(R/\mathfrak{q}) \oplus \bigoplus\limits_{\mathrm{ht}(\mathfrak{p})=1} \mathrm{E}(R/\mathfrak{p})$ over a commutative 1-Gorenstein ring R (e.g. [AHT:2006]), and the Abelian group $\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}$ (e.g. [GT:2000]).

The definition of (possibly *infinitely generated*) tilting modules of projective dimension at most 1 is due to R. Colpi and J. Trlifaj [CT:1995]; on the other hand, Y. Miyashita introduced in [Miy:1986] *strongly finitely presented* tilting modules with *arbitrary* finite projective dimension. Generalizing both definitions, *arbitrary* tilting modules of *arbitrary* finite projective dimension were introduced by L. Angeleri-Hügel and F.U. Coelho in [AC:2001]. These were studied further and characterized using special classes of modules by S. Bazzoni [Baz:2004(b)].

One of the first notions that generalized progenerators is that of *quasi-progenerators* (introduced by K. Fuller in [Ful:1974]). A left R-module ${}_TV$ with $S := \mathrm{End}({}_RV)^{op}$ is a said to be a **quasi-progenerator**, iff ${}_RV$ is finitely generated, self-projective and self-generator (roughly speaking, iff V is a progenerator in $\sigma[{}_RV]$). In fact, ${}_RV$ is a quasi-progenerator if and only if $\sigma[{}_RV] \underset{T\otimes_S -}{\overset{\mathrm{Hom}_R(T,-)}{\approx}} {}_S\mathbb{M}$.

A notion that generalizes progenerators, quasi-progenerators as well as classical 1-tilting modules is that of $*^1$-*modules* (introduced by C. Menini and A. Orsatti in [MO:1989] and named[14] in [Col:1990]): a left R-module ${}_RV$ with $S := \mathrm{End}({}_RV)^{op}$ is said to be a $*^1$**-module**, iff $\mathrm{Gen}({}_RV) \underset{T\otimes_S -}{\overset{\mathrm{Hom}_R(T,-)}{\approx}} \mathrm{Cogen}({}_S^dV)$ (where ${}^dV := \mathrm{Hom}_R(V, \mathbf{I})$ and ${}_R\mathbf{I}$ is an injective cogenerator). Such modules are necessarily finitely generated as shown by J. Trlifaj in [Trl1994].

A subclass of $*^1$-modules that generalizes tilting modules, similarly as quasi-progenerators generalize progenerators, is the notion of 1-*quasi-tilting modules* introduced by R. Colpi et. al. in [CDT:1997][15]. In particular, a self-small R-module T is said to be 1-**quasi-tilting**, iff $\mathrm{Pres}({}_RT) = \mathrm{Gen}({}_RT) \subseteq T^{\perp_1}$. Dropping the finiteness condition, R. Wisbauer introduced the (possibly infinitely generated) 1-*self-tilting modules*[16]: An R-module ${}_TT$ is 1-**self-tilting**, iff

$$\mathrm{Gen}({}_RT) = T^{\perp_1}_{\sigma[{}_RT]} := \{M \in \sigma[{}_RT] \mid \mathrm{Ext}^1_T(T, M) = 0\}.$$

(i.e. roughly speaking, iff T is a tilting object in the category $\sigma[{}_RT]$ of T-subgenerated R-modules). Moreover, he showed that $*^1$-modules coincide with the self-small 1-self-tilting modules.

Another generalization of tilting modules is due to T. Wakamatsu in [Wak:1988]: an R-module ${}_RW$ (possibly with infinite projective dimension) is said to be a **Wakamatsu tilting module**, iff $R \simeq \mathrm{End}(W_S)$, where $S := \mathrm{End}({}_RW)^{op}$, and ${}_RW_S$ is *self-orthogonal* in the sense that $\mathrm{Ext}^i_R(W, W) = 0 = \mathrm{Ext}^i_S(W, W)$ for all $i \geq 1$ (see [MR:2004] for more details).

[14] called originally $*$-*modules*
[15] called originally *quasi-tilting modules*
[16] called originally *self-tilting modules*

In a series of recent papers (e.g. [HHTW:2003], [Wei:2005a], [Wei:2005b]) J. Wei et. al. introduced further generalizations of the notion of tilting modules by presenting the notions of (not necessarily finitely generated) n-**star modules**, for $n \geq 2$. For $n = 1$, the class of (self-small) 1-star modules coincides with the class of ($*^1$-modules) 1-self-tilting modules. The relation between (strongly finitely presented) n-star modules and (classical) n-tilting modules was also clarified.

Cotilting theory is a generalization of *Morita duality* in the same sense that Tilting Theory is a generalization of Morita equivalences between categories of modules. A central point in cotilting theory was always to obtain a *Cotilting Theorem* dual to the Brenner-Butler *Tilting Theorem.* Several *cotilting theorems* were obtained by different authors (e.g. [Col:1989], [CF:1990], [Ang:2000], [Wis:2002]), using different notions of cotilting modules (several of which are different from the cotilting modules defined above).

Cotilting modules appeared first as vector space duals of tilting modules over finite dimensional algebras (e.g. [Hap:1988]). A first attempt to generalize cotilting modules is due to R. Colby [Col:1989]. In [CDT:1997], R. Colpi et. al. defined 1-cotilting modules by introducing conditions that are *formally* dual to those of 1-tilting modules in [CT:1995]. Generalizing several previous definitions, L. Angeleri-Hügel and F.U. Coelho introduced in [AC:2001] cotilting modules of *arbitrary* finite injective dimension were defined by dualizing the conditions that define tilting modules of arbitrary finite projective dimension. These were studied further by S. Bazzoni in several papers (e.g. [Baz:2004(a)], [Baz:2004(b)]).

It should be noticed that, although the conditions defining cotilting modules are *formal dual* of those defining tilting modules, a major difference between them is that "all tilting modules are of *finite type*" (as proved recently by S. Bazzoni and J. Šťovíček in [BS]), while not all cotilting modules are of *cofinite type* as indicated by S. Bazzoni in [Baz]. However, an interesting property of cotilting modules is that they are *pure-injective* as shown recently by J. Šťovíček [Sto:2006].

Although there is a wide literature on tilting (cotilting) modules since their appearance more than 25 years ago, their structure is still not well understood in most cases. This surprising fact becomes more clear, if one considers infinitely generated tilting (cotilting) modules.

For instance, restricting to modules over commutative integral domains a *complete* description for the structure of infinitely generated tilting (cotilting) modules is known only for Prüfer (Dedekind) domains, as characterized by L. Salce [Sal:2005] (S. Bazzoni [Baz]). Indeed, several *partial results* are known over other domains.[17]

The characterization of tilting (cotilting) modules over integral domains found its real start almost at the beginning of the current century, when R. Göbel and J. Trlifaj completed in [GT:2000] the characterization of tilting (cotilting) Abelian groups. This work was continued by J. Trlifaj et. al. (e.g. [TW:2002], [TW:2003], [BET:2005]), who characterized tilting (cotilting) modules over arbitrary Dedekind domains. Tilting modules over valuation domains were studied by L. Salce in [Sal:2004] and a complete characterization (removing previous set theoretic assumptions in [Sal:2005]) are included in [GT:2006]. A characterization of cotilting modules *of cofinite type* over Prüfer domains, e.g. strongly discrete valuation domains, is obtained by S. Bazzoni [Baz]. Elegant versions of the -so far- obtained characterizations of tilting (cotilting) modules over the above mentioned special classes of integral domains are included in the recent monograph [GT:2006].

So, a main *open problem* in "Tilting (Cotilting) Theory" is to

determine the structure of tilting (cotilting) modules over special classes of commutative non Prüfer integral domains.

The main goal of this *technical report* is to present a survey on the results that are known -so far- on the structure of tiling (cotilting) modules over commutative rings. Moreover, we formulate some problems related to the main open problem highlighted above.

We assume familiarity with the foundations of "Categories of Modules" (e.g. [AF:1974], [Fai:1981], [Wis:1991]), "Modules over Commutative Rings" (e.g. [FS:2000], [Mat:2004]), "Homological Algebra" (e.g. [Osb:2000] and [Rot:1979]) and "Relative Homological Algebra" (e.g. [EJ:2000]). However, we include several definitions and results that are used in the report in Part I (Preliminaries). The appendix contains also a "brief introduction" to the $\mathrm{Ext}^n_R(\bullet,\bullet)$ and $\mathrm{Tor}^n_R(\bullet,\bullet)$ bifunctors for $n \in \mathbb{N}$.

[17] see also [GT:2006] for more on this issue.

We don't provide proofs of the results included in this brief survey. However, we provide reference(s) to each result (with exception to those that can be considered nowadays as *folklore*).

This *technical report* consists of four main parts:

Part I contains mainly *preliminaries* on modules over associative and commutative rings, in addition to a brief introduction to Morita theory on equivalences (dualities).

Part II is the *core* of the report and consists of three chapters. In Chapter 4, we include a summary of the basic definitions and results on tilting (cotilting) modules over arbitrary associative rings that are needed in the sequel. We also include some of the generalizations of the titling (cotilting) modules, e.g. *star modules*, *self-tilting modules* (*finitely cotilting modules*). Some of the "*Tilting* (*Cotilting*) *Theorems*" that are known so far are also included. In Chapter 5, we restrict our attention to the structure of tilting (cotilting) modules over commutative rings and integral domains. In particular, we present results on the structure of tilting (cotilting) modules over Prüfer domains, elementary divisor domains, commutative 1-Gorenstein rings, Dedekind domains and valuation domains. Chapter 6 contains a brief literature review of tilting (cotilting) modules along with some historical notes.

Part III is the *main goal* of the report. It includes a number of *open problems* on tilting (cotilting) modules over commutative rings that were formulated after careful investigation and intensive literature review.

Part IV is an appendix on "(*Co*)*Homology*" and contains mainly the different (equivalent ways) to define $\mathrm{Ext}_R^n(\bullet,\bullet)$- and $\mathrm{Tor}_R^n(\bullet,\bullet)$-bifunctors.

An *updated* list of references is included at the end of the report. Indeed, there are many other important and interesting articles on tilting (cotilting) modules that were not included in this list because the results in such articles were beyond the scope of this brief survey.

Part I

Preliminaries

Chapter 1

Modules over Associative Rings

In what follows, we include some properties of modules over rings defined using the Hom and the Tensor functors.

Throughout, R denotes an associative ring with $1_R \neq 0_R$. Ideals of R are assumed to be *two-sided* (unless otherwise explicitly specified). All modules are assumed to be unitary.

1.1 Generators (Cogenerators) and Projectives (Injectives)

Definition 1. Let $\mathcal{U} \neq \varnothing$ be a non-empty class of R-modules and M be an R-module.

1. The **trace** of $\mathcal{U}$ in M is

$$\mathrm{Tr}(\mathcal{U}, M) := \sum \{\mathrm{Im}(f) \mid f \in \mathrm{Hom}_R(U, M) \text{ for some } U \in \mathcal{U}\}.$$

2. The **reject** of $\mathcal{U}$ in M is

$$\mathrm{Rej}(M, \mathcal{U}) := \bigcap \{\mathrm{Ker}(h) \mid h \in \mathrm{Hom}_R(M, U) \text{ for some } U \in \mathcal{U}\}.$$

Definition 2. Let $\mathcal{U} \neq \varnothing$ be a non-empty class of R-modules. We say an R-module M is **$\mathcal{U}$-generated** (**$\mathcal{U}$-cogenerated**), iff for any ${}_RN$ (${}_RL$) and any distinct R-linear morphisms $f, g : M \to N$ ($f, g : L \to M$), there exists some $U \in \mathcal{U}$ and $h : U \to M$ ($h : M \to U$) such that $f \circ h \neq g \circ h$ ($h \circ f \neq h \circ g$).

Remark 3. Let $\mathcal{U} \neq \varnothing$ be a non-empty class of R-modules and M be an R-module. The R-submodule $\mathrm{Tr}(\mathcal{U}, M) \subseteq M$ ($\mathrm{Rej}(M,\mathcal{U}) \subseteq M$) is the largest R-submodule of M that is $\mathcal{U}$-generated (the smallest R-submodule of M such that $M/\mathrm{Rej}(M,\mathcal{U})$ is $\mathcal{U}$-cogenerated). In particular, M is $\mathcal{U}$-generated ($\mathcal{U}$-cogenerated) if and only if $\mathrm{Tr}(\mathcal{U}, M) = M$ ($\mathrm{Rej}(M,\mathcal{U}) = 0$).

Definition 4. We call an R-module U :

1. **self-generator** (**self-cogenerator**), iff U generates all of its submodules (cogenerates all of its factor modules).

2. **generator** (**cogenerator**), iff every R-module M is U-generated (U-cogenerated).

Definition 5. An R-module M is said to be **cyclically presented**, iff $M \simeq R/(Rr)$ for some $r \in R$.

Definition 6. An R-module M is **κ-generated** for some cardinal κ, iff M admits a *spanning set* of cardinality κ. In particular, ${}_RM$ is said to be **finitely** (**countably**) **generated**, iff M admits a *finite* (*countable*) spanning set.

Definition 7. Let Y be an R-module. We say an R-module U is

Y-projective , iff for every *epimorphism* $Y \xrightarrow{\pi} Z \to 0$ and any R-linear morphism $f : U \to Z$, there exists an R-linear morphism $\widetilde{f} : U \to Y$, such that $\pi \circ \widetilde{f} = f$;

Y-injective, iff for very *monomorphism* $0 \to X \xrightarrow{\iota} Y$ and any R-linear morphism $g : X \to U$, there exists an R-linear morphism $\widetilde{g} : Y \to U$, such that $\widetilde{g} \circ \iota = g$;

weakly Y-injective , iff for very *monomorphism* $0 \to X \xrightarrow{\iota} Y^{(\mathbb{N})}$ with ${}_RX$ finitely generated and any R-linear morphism $g : X \to U$, there exists an R-linear morphism $\widetilde{g} : Y^{(\mathbb{N})} \to U$, such that $\widetilde{g} \circ \iota = g$;

Definition 8. We call an R-module U :

1. **quasi-projective** (**quasi-injective**), iff U is U-projective (U-injective);

2. **FP-injective**, iff U is weakly R-injective;

3. **projective** (**injective**), iff U is M-projective (M-injective) for every left R-module M.

Proposition 9. ([Wis:1991, Sections 13, 18]) *Let P be a left R-module and $S := \mathrm{End}({}_RP)^{op}$. The following are equivalent:*

1. *${}_RP$ is a generator;*

2. $\mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_\mathbb{Z}\mathbb{M}$ *is* faithful; *i.e. for any left R-modules A, B the following canonical set mapping is injective:*

$$\mathrm{Hom}_R(P, -) : \mathrm{Hom}_R(A, B) \to \mathrm{Map}(\mathrm{Hom}_R(P, A), \mathrm{Hom}_R(P, B));$$

3. $\mathrm{Hom}_R(P, -)$ *reflects zero morphisms in* ${}_R\mathbb{M}$;

4. $\mathrm{Hom}_R(P, -)$ *reflects epimorphisms in* ${}_R\mathbb{M}$;

5. $\mathrm{Hom}_R(P, -)$ *reflects exact sequences in* ${}_R\mathbb{M}$;

6. *for every left R-module M, there exists an index set Λ and an epimorphism $P^{(\Lambda)} \to M \to 0$;*

7. *for every left R-module M we have $\mathrm{Tr}(P, M) = M$.*

8. *${}_RP$ generates all finitely generated (cyclic) R-modules;*

9. *${}_RP$ generates R;*

10. $\mathrm{Tr}(P, R) = R$ *(i.e. ${}^*PP = R$);*

11. *there exists $\{p_1, ..., p_n\} \subseteq P$ and $\{\varphi_1, ..., \varphi_n\} \subseteq {}^*P$ such that*

$$\varphi_1(p_1) + ... + \varphi_n(p_n) = 1_R.$$

12. *${}_RR$ is a direct summand of ${}_RP^k$ for some $k \in \mathbb{N}$ (i.e. $\exists$ an R-module N such that $R \oplus N = P^k$);*

13. *P_S is finitely generated, projective and $R \simeq \mathrm{End}(P_S)$.*

Corollary 10. *Let P be a left R-module and $S := \mathrm{End}({}_RP)^{op}$. If ${}_RP$ is a generator, then P_S is a direct summand of S^k (i.e. there exists a right S-module L_S, such that $P \oplus L = S^k$).*

Lemma 11. ([Wis:1991, 18.13. (8)]) *Let P be a left R-module, $S := \mathrm{End}({}_RP)^{op}$ and $B := \mathrm{End}(P_S)$. If ${}_RP$ is finitely generated and projective, then*

1. *${}_BP$ is a finitely generated and projective.*
2. *P_S is a generator.*

Definition 12. An R-module U is said to be **faithful**, iff $\mathrm{ann}_R(U) = 0$.

Proposition 13. ([Wis:1991, 14.10 (1, ii)]) *The following are equivalent for an R-module U :*

1. *${}_RU$ is faithful;*
2. *U cogenerates R;*
3. *$R \hookrightarrow U^\Lambda$ for some index set Λ;*
4. *U cogenerates a generator in ${}_R\mathbb{M}$.*

Proposition 14. *For an R-module U the following are equivalent:*

1. *${}_RU$ is a cogenerator;*
2. $\mathrm{Hom}_R(-,U) : {}_R\mathbb{M} \to {}_\mathbb{Z}\mathbb{M}$ *is* faithful; *i.e. for any left R-modules A and B, the following canonical set mapping is injective:*

   $$\mathrm{Hom}_R(-,U) : \mathrm{Hom}_R(A,B) \to \mathrm{Map}(\mathrm{Hom}_R(B,U), \mathrm{Hom}_R(A,U));$$

3. *for every left R-module M, there exists an index set Λ and a monomorphism $0 \to M \to U^\Lambda$;*
4. *for every left R-module M we have $\mathrm{Rej}(M,U) = 0$.*

Definition 15. Let $\mathcal{U} \neq \varnothing$ be a non-empty class of R-modules. We say an R-module U is

$\mathcal{U}$-**projective** , iff U is Y-projective for every $Y \in \mathcal{U}$;
projective on $\mathcal{U}$, iff $\mathrm{Hom}_R(U, -)$ respects short exact sequences in $\mathcal{U}$;
Ext-**projective in** $\mathcal{U}$, iff $\mathrm{Ext}^1_R(U,\mathcal{U}) = 0$ (i.e.$\mathcal{U} \subseteq U^{\perp_1}$);
$\mathcal{U}$-**injective** , iff U is Y-injective for every $Y \in \mathcal{U}$;
injective on $\mathcal{U}$, iff $\mathrm{Hom}_R(-, U)$ respects short exact sequences in $\mathcal{U}$;
Ext-**injective in** $\mathcal{U}$, iff $\mathrm{Ext}^1_R(\mathcal{U}, U) = 0$ (i.e. $\mathcal{U} \subseteq {}^{\perp_1}U$);

Proposition 16. *The following are equivalent for a left R-module P :*

1. ${}_RP$ *is projective;*
2. $\mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_\mathbb{Z}\mathbb{M}$ *is* (right) exact*; i.e. for every short exact sequence of left R-modules*
$$0 \to X \to Y \to Z \to 0,$$
the following sequence of Abelian groups is exact
$$0 \to \mathrm{Hom}_R(P, X) \to \mathrm{Hom}_R(P, Y) \to \mathrm{Hom}_R(P, Z) \to 0;$$
3. *Every short exact sequence of left R-modules*
$$0 \to X \to Y \to P \to 0$$
splits, i.e. $Y \simeq X \oplus P$;
4. *P is a direct summand of a free left R-submodule, i.e. there exists some index set Λ and an R-module K such that $R^{(\Lambda)} \simeq K \oplus P$;*
5. *there exists a class $\{f_\lambda, p_\lambda\} \in {}^*P \times P$, such that every $p \in P$ can be written (not necessarily uniquely) as*
$$p = \sum_{i=1}^{n} f_{\lambda_i}(p)p_{\lambda_i} \;\; (\{\lambda_{1,\ldots,}\lambda_n\} \subset \Lambda).$$

Proposition 17. *For a left R-module U the following are equivalent:*

1. ${}_RU$ *is injective;*

2. $\mathrm{Hom}_R(-, U) : {}_R\mathbb{M} \to {}_{\mathbb{Z}}\mathbb{M}$ *is* (right) exact*; i.e. for every short exact sequence of left R-modules*

$$0 \to X \to Y \to Z \to 0,$$

the following sequence of Abelian groups is exact

$$0 \to \mathrm{Hom}_R(Z, U) \to \mathrm{Hom}_R(Y, U) \to \mathrm{Hom}_R(X, U) \to 0;$$

3. Every short exact sequence of left R-modules

$$0 \to U \to Y \to Z \to 0$$

splits, i.e. $Y \simeq U \oplus Z$.

Definition 18. A short *exact* sequence of left R-modules

$$0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0 \tag{1.1}$$

is said to be **pure-exact** (or that $X \subseteq Y$ is a **pure submodule**), iff for every right R-module M the following sequence of Abelian groups is exact

$$0 \to M \otimes_R X \to M \otimes_R Y \to M \otimes_R Z \to 0.$$

Definition 19. An R-module U is said to be

pure-projective, iff for all pure-exact sequences of the form (1.1), for every R-linear morphism $f : U \to Z$ there exists an R-linear morphism $\widetilde{f} : U \to Y$ such that $f = \pi \circ \widetilde{f}$;

pure-injective, iff for all pure-exact sequences of the form (1.1), every R-linear morphism $g : X \to U$ can be extended to an R-linear morphism $\widetilde{g} : Y \to U$ such that $g = \widetilde{g} \circ \iota$.

Definition 20. Let $k \geq 1$. We say an R-module U is

k-$\sum$-quasi-projective, iff for any short exact sequence of R-modules

$$0 \to K \to U' \to N \to 0 \tag{1.2}$$

with $U' \in \mathrm{Add}(U)$ and $K \in \mathrm{Gen}_{k-1}(U)$, the following induced sequence is exact

$$0 \to \mathrm{Hom}_R(U, K) \to \mathrm{Hom}_R(U, L) \to \mathrm{Hom}_R(U, N) \to 0; \tag{1.3}$$

strictly k-$\sum$-quasi-projective , iff for any short exact sequence of R-modules of the form (1.2) with $U' \in \mathrm{Add}({}_RU)$ and $N \in \mathrm{Gen}_{k-1}(U)$, the induced sequence (1.3) is exact if and only if $K \in \mathrm{Gen}_{k-1}(U)$.

After R. Colpi in [Col:1990], we present

Definition 21. We call and R-module U :
(**strictly**) **w-$\sum$-quasi-projective**, iff ${}_RU$ is (strictly) 2-$\sum$-quasi-projective;
(**strictly**) **$\sum$-quasi-projective** , iff ${}_RU$ is (strictly) 1-$\sum$-quasi-projective.

Definition 22. An R-module U is called **w-$\prod$-quasi-injective**, iff the contravariant functor $\mathrm{Hom}_R(-,U)$ respects exactness of short exact sequences of the form

$$0 \to X \to U^{\Lambda} \to Z \to 0 \text{ (where } Z \in \mathrm{Cogen}(U),\ \Lambda \text{ any index set).}$$

23. We call an R-module U **locally projective** (in the sense of B. Zimmermann-Huisgen [Z-H:1976]), iff for every diagram of R-modules

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\iota"] \arrow[dr, dotted, "g'\circ\iota"'] & U \arrow[d, dotted, "g'"] \arrow[dr, "g"] & & \\
& & L \arrow[r, "\pi"'] & N \arrow[r] & 0
\end{tikzcd}$$

with exact rows and finitely generated R-submodule $F \subseteq U$: for every R-linear morphism $g : U \to N$, there exists an R-linear morphism $g' : U \to L$, such that the entstanding parallelogram is commutative (i.e. $g \circ \iota = \pi \circ g' \circ \iota$).

Definition 24. Let M, N be R-modules. An R-linear morphism $\varphi : M \to N$ is said to be

n**-splitting**, iff for each subset $F \subseteq M$ with $|F| = n$, there exists some $\psi \in \mathrm{Hom}_R(N, M)$ such that $F(1 - \varphi\psi)\varphi = 0$;

splitting, iff φ is n-splitting for every $n \in \mathbb{N}$.

Notation. Let U be a left R-module, $S := \mathrm{End}({}_RU)^{op}$ and consider for every $f \in {}^*U$ the R-linear morphism

$$\phi_f : U \to S,\ u \mapsto f(-)u.$$

We consider also the two ideals

$$\nabla_U := \sum_{f \in {}^*U} \mathrm{Im}(f) \lhd R \text{ and } \Delta_U := \sum_{f \in {}^*U} \mathrm{Im}(\phi_f). \tag{1.4}$$

Proofs of the following characterizations of *locally projective* R-modules can be found in [Abu:2005], [Z-H:1976] and [GT:2006]:

Proposition 25. *The following are equivalent for a left R-module ${}_RU$:*

1. *${}_RU$ is locally projective;*
2. *${}_RU$ satisfies the α-*condition*, i.e. for every right R-module M the following canonical map is injective*

$$\alpha_M^U : M \otimes_R U \to \mathrm{Hom}_R({}^*U, M),\ m \otimes_R u \mapsto [f \mapsto mf(u)];$$

3. *α_M^U is injective for every cyclic right R-module M;*
4. *Any R-linear morphism $\varphi : M \to U$ is finitely splitting;*
5. *${}_RU$ has a local dual basis (in the sense that For any finite subset $E = \{u_1, ..., u_n\} \subseteq U$, there exists $\{\varphi_1, ..., \varphi_n\} \subseteq {}^*U$ such that $e = \sum_{k=1}^{n} \varphi_k(e)u_k$ for every $e \in E$);*
6. *For any $u \in U$, there exists $\{u_1, ..., u_n\} \subseteq U$ and $\{\varphi_1, ..., \varphi_n\} \subseteq {}^*U$, such that $u = \sum_{k=1}^{n} \varphi_k(u)u_k$;*
7. *$U = \nabla_U U$, and S/Δ is flat as a left S-module;*
8. *$U = \nabla_U U$, and U_S is a generator in $\sigma[U_S]$.*

Remark 26. A modules U satisfying the equivalent conditions of Proposition 25 were studied by several authors under several names (e.g. *universally torsionless (UTL) in* [Gar1967], *trace modules* in [BO1972, BO1972] and *modules plats et strictement de Mittag-Leffler* in [GR1971].

Definition 27. An R-module U is said to be

torsion-less, iff U is R-cogenerated (i.e. $U \hookrightarrow R^{\Lambda}$ for some index set Λ);

freely separable, iff every finitely generated R-submodule $K \leq_R U$ can be embedded in a free direct summand of U;

projectively separable, iff every finitely generated R-submodule $K \leq_R U$ can be embedded in a projective direct summand of U;

Remark 28. ([Abu:2005], [GT:2006, Proposition 1.3.12.]) Every locally projective R-module is flat, R-cogenerated (i.e. torsion-less) and moreover isomorphic to a pure R-submodule of R^{Λ} for some index set Λ.

Proposition 29. *Let R be a PID and U and R-module. The following are equivalent:*

1. *${}_RU$ is locally projective;*
2. *${}_RU$ is freely separable;*
3. *every pure R-submodule $L <_R U$ or finite rank is a free direct summand (i.e. $L <_R^{\oplus} U$);*
4. *${}_RU$ is a pure R-submodule of R^{Λ} for some index set Λ.*

Finiteness Conditions

In what follows, we consider various finiteness conditions for modules over arbitrary rings.

Lemma 30. *The following are equivalent for a left R-module ${}_RU$:*

1. *${}_RU$ is finitely generated;*
2. *for every class of left R-modules $\{U_\lambda\}_\Lambda$ with an epimorphism of R-modules $\varphi : \bigoplus\limits_{\Lambda} U_\lambda \to U$, there exists a finite subset $\Lambda' := \{\lambda_1, ..., \lambda_n\} \subseteq \Lambda$, such that the following map is an epimorphism*
$$\varphi \circ \varepsilon_{\Lambda'} : \bigoplus_{i=1}^{n} U_{\lambda_i} \overset{\varepsilon_{\Lambda'}}{\hookrightarrow} \bigoplus_{\Lambda} U_\lambda \overset{\varphi}{\longrightarrow} U.$$
3. *For every class $\mathcal{L} = \{L_\lambda\}_\lambda$ of right R-modules, the following canonical map is surjective*
$$\varphi_{U,\mathcal{L}} : (\prod_{\Lambda} L_\lambda) \otimes_R U \to \prod_{\Lambda} (L_\lambda \otimes_R U).$$

4. *For any set Λ, the canonical map $\varphi_U : R^\Lambda \otimes_R U \to U^\Lambda$ is surjective.*

Lemma 30 motivates the following

Definition 31. An R-module ${}_RU$ is said to be **finitely cogenerated**, iff for every *monomorphism* $f : U \to \prod_{\lambda\in\Lambda} U_\lambda$ in ${}_R\mathbb{M}$, there exists a *finite subset* $\Lambda' = \{\lambda_1, ..., \lambda_n\} \subseteq \Lambda$ such that the following map (with π canonical) is injective:

$$\widetilde{f} : U \overset{f}{\hookrightarrow} \prod_{\lambda\in\Lambda} U_\lambda \overset{\pi_{\Lambda'}}{\to} \prod_{i=1}^{n} U_{\lambda_i}$$

(equivalently, iff for any class $\{U_\lambda\}_{\lambda\in\Lambda}$ of R-submodules of U with $\bigcap\limits_{\lambda\in\Lambda} U_\lambda = 0$, there exists a finite subset $\{U_{\lambda_1}, ..., U_{\lambda_n}\} \subseteq \{U_\lambda\}_{\lambda\in\Lambda}$ such that $\bigcap\limits_{i=1}^{n} U_{\lambda_i} = 0$).

Definition 32. An R-module ${}_RU$ is **simple**, iff $U \neq 0$ and U has no non-trivial R-submodules.

Definition 33. We define the **socle** of an R-module V as

$$\mathrm{Soc}({}_RV) := \sum\{U \mid U \subseteq V \text{ is a simple } R\text{-submodule}\}.$$

We call ${}_RV$ **semisimple**, iff $V = \mathrm{Soc}({}_RV)$ (equivalently, iff every R-submodule $U \leq_R V$ is a direct summand).

Definition 34. Let V be an R-module.

An R-submodule $U \leq_R V$ is said to be **essential**[1] (written $U \leq_R^{ess} V$), iff for every non-zero R-submodule $0 \neq L \leq_R V$ we have $L \cap U \neq 0$;

An R-submodule $U \leq_R V$ is said to be **superfluous**[2] (written $U \ll V$), iff whenever $L \leq_R V$ satisfies $L + U = V$ we have also $U = V$.

Proposition 35. ([Wis:1991, 21.3.])

1. *An R-module U is finitely cogenerated if and only if* $\mathrm{Soc}(U)$ *is finitely generated and essential in U.*

---

[1] called also *large submodule* (e.g. [Wis:1991])

[2] called also *small submodule* (e.g. [Wis:1991])

2. *Every finitely cogenerated R-module is a (finite) direct sum of indecomposable R-modules.*

Definition 36. A left R-module ${}_RU$ is called **finitely presented**, iff ${}_RU$ is finitely generated and for every short exact sequence of R-modules

$$0 \to X \to Y \to U \to 0$$

with ${}_RY$ finitely generated, the left R-module ${}_RX$ is also finitely generated.

Proposition 37. ([Wis:1991, 25.4.]) *The following are equivalent for a left R-module ${}_RU$:*

1. *${}_RU$ is finitely presented;*
2. *There exists a short exact sequence*
$$0 \to K \to R^n \to U \to 0$$
with ${}_RK$ finitely generated;
3. *There exists an exact sequence*
$$R^m \to R^n \to U \to 0.$$
4. $\mathrm{Hom}_R(U, -)$ *preserves direct limits;*
5. *$U \otimes_R -$ preserves direct products, i.e. for every class right R-modules $\mathcal{L} = \{L_\lambda\}_\Lambda$, we have a canonical isomorphism*
$$(\prod_\Lambda L_\lambda) \otimes_R U \overset{\varphi_{U,\mathcal{L}}}{\simeq} \prod_\Lambda (L_\lambda \otimes_R U).$$
6. *We have a canonical isomorphism $R^\Lambda \otimes_R U \overset{\varphi_U}{\simeq} U^\Lambda$.*

Definition 38. A left R-module ${}_RU$ is **coherent**, iff ${}_RU$ is finitely generated and every finitely generated R-submodule $L \leq_R U$ is finitely presented.

Remark 39. For every class $\{M_\lambda\}_{\lambda\in\Lambda}$ of R-modules, there is a canonical monomorphism

$$\psi : \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_R(U, M_\lambda) \hookrightarrow \mathrm{Hom}_R(U, \bigoplus_{\lambda\in\Lambda} M_\lambda),\ (f_\lambda)_\Lambda \mapsto [u \mapsto (f_\lambda(u))_\Lambda].$$

Definition 40. An R-module U is called

small, iff $\mathrm{Hom}_R(U,-)$ respects direct sums, i.e. for every class $\{M_\lambda\}_{\lambda\in\Lambda}$ of R-modules, the following canonical map is an isomorphism

$$\bigoplus_{\lambda\in\Lambda}\mathrm{Hom}_R(U,M_\lambda)\overset{\psi}{\simeq}\mathrm{Hom}_R(U,\bigoplus_{\lambda\in\Lambda}M_\lambda);$$

self-small, iff for every index set Λ we have

$$\mathrm{Hom}_R(U,U^{(\Lambda)})\simeq\mathrm{End}_R(U)^{(\Lambda)}.$$

Lemma 41. ([Zem:2005, Lemma 1.1.]) *The following are equivalent for an R-module U :*

1. *U is a small R-module;*
2. *If $U=\bigcup\limits_{k=1}^{\infty}U_k$ for an increasing chain of R-submodules*
$$U_1\leq U_2\leq\ ...\ \leq U_k\leq U_{k+1}\leq\ ...,$$
then there exists some $n\in\mathbb{N}$, with $U=U_n$.
3. *If $U=\sum\limits_{k=1}^{\infty}U_k$ for a system of R-submodules $U_k\leq U$, then there exists some $n\in\mathbb{N}$, with $U=\sum\limits_{k=1}^{n}U_k$.*

Notation. With ${}_R^s\mathbb{M}\subseteq{}_R\mathbb{M}$ ($\mathbb{M}_R^s\subseteq\mathbb{M}_R$) we denote the full subcategory of small left (right) R-submodules.

Proposition 42. ([CT:1994, Corollary 1.2.]) *The class ${}_R^s\mathbb{M}$ is closed under taking quotients and extensions.*

Remarks 43. The following should be remarked concerning small R-modules.

1. *Small modules* were introduced first by H. Bass in [Bas:1968], and are known in the literature under several names (e.g. *dually slender modules*, ($\sum$-)*compact modules* $\sum$*-modules*, *modules of type* $\sum$: see [Zem:2005] for more details).

2. The notion of small modules we use is different from that of *small submodules,* which we call here *superfluous* (e.g. [Wis:1991, 19.1.]).

3. Every finitely generated R-modules is small (e.g. [Wis:1991, 13.9., 2 (ii)]), while the converse is not true (R. Rentschler gave in [Ren:1996] an example of an infinitely generated small module).

4. A projective R-module is small if and only if it is finitely generated.

5. In general, ${}^{s}_{R}\mathbb{M}$ is not closed under taking submodules.

Chain Conditions

Definition 44. An R-module U is called

Noetherian, iff every *ascending* chain of R-submodules of U is stationary, i.e. for any non-decreasing chain of R-submodules

$$U_1 \leq_R U_2 \leq_R \ \dots \ \leq_R U_k \leq_R U_{k+1} \leq_R \ \dots,$$

there exists some $n \in \mathbb{N}$ such that $U_n = U_{n+i}$ for all $i \geq 1$.

Artinian, iff every *descending* chain of R-submodules of U is stationary, i.e. for any non-decreasing chain of R-submodules

$$U_1 \geq_R U_2 \geq_R \ \dots \ \geq_R U_k \geq_R U_{k+1} \geq_R \ \dots,$$

there exists some $n \in \mathbb{N}$ such that $U_n = U_{n+i}$ for all $i \geq 1$.

Proposition 45. ([Wis:1991, 27.1., 31.1.]) *Let U be an R-module.*

1. *U is Noetherian if and only if every R-submodule of U is finitely generated.*

2. *U is Artinian if and only if every factor R-module of M is finitely cogenerated.*

Proposition 46. ([CT:1994, Proposition 1.3.]) *An R-module U is Noetherian if and only if each (essential) R-submodule of U is small.*

1.2 Torsion Theories

Definition 47. A non-empty class $\mathcal{U}$ of R-modules is said to be **closed under extensions**, iff for every short exact sequence of R-modules

$$0 \to X \to Y \to Z \to 0 \tag{1.5}$$

with $X, Z \in \mathcal{U}$ also $Y \in \mathcal{U}$.

Definition 48. A non-empty class $\mathcal{U} \neq \varnothing$ of R-modules is said to be
pretorsion, iff $\mathcal{U}$ is closed under epimorphic images and direct sums;
torsion, iff $\mathcal{U}$ is closed under epimorphic images, direct sums and extensions;
pretorsion-free, iff $\mathcal{U}$ is closed under submodules and direct products;
torsion-free, iff $\mathcal{U}$ is closed under submodules, direct products and extensions;
hereditary, iff $\mathcal{U}$ is closed under submodules.

Definition 49. A pair $(\mathfrak{T}, \mathfrak{F})$ of non-empty classes of R-modules is called a **torsion theory**, iff

1. We have

$$\begin{array}{lll} \mathfrak{T} & := & \{{}_RT \mid \mathrm{Hom}_R(T, F) = 0 \text{ for all } F \in \mathfrak{F}\}; \\ \mathfrak{F} & := & \{{}_RF \mid \mathrm{Hom}_R(T, F) = 0 \text{ for all } T \in \mathfrak{T}\}. \end{array}$$

2. For every ${}_RM$ there exists an R-submodule $L <_R M$ such that

$$L \in \mathfrak{T} \text{ and } M/L \in \mathfrak{F}. \tag{1.6}$$

In case L in (1.6) is unique, it is called the **torsion-submodule of** M **w.r.t.** $(\mathfrak{T}, \mathfrak{F})$.

Subgenerators

Definition 50. A *subcategory* $\mathfrak{G}$ of the category of R-modules is said to be

closed, iff $\mathcal{G}$ is closed under submodules, quotients and arbitrary direct sums (equivalently, iff $\mathcal{G}$ is a hereditary pretorsion class);

finitely closed, iff $\mathcal{G}$ is closed under submodules, quotients and finite direct sums.

Notation. For every non-empty class $\mathcal{U} \neq \varnothing$ of left R-modules, set

$$\begin{array}{lcl} \overline{\mathcal{U}} & := & \text{the smallest } \textit{closed} \text{ subcategory of } {}_R\mathbb{M} \text{ containing } \mathcal{U}; \\ \overline{\mathcal{U}}^f & := & \text{the smallest } \textit{finitely closed} \text{ subcategory of } {}_R\mathbb{M} \text{ containing } \mathcal{U}. \end{array}$$

51. Let U be a left R-module. A left R-module ${}_RN$ is said to be U-**subgenerated**, iff N is a submodule of a U-generated R-module. With $\sigma[{}_RU] \subseteq {}_R\mathbb{M}$ we denote the *full* subcategory of R-modules subgenerated by ${}_RU$. We call a left R-module ${}_RU$ a **subgenerator** (called also a **cofaithful module**), iff $\sigma[{}_RU] = {}_R\mathbb{M}$.

Proposition 52. *The category $\sigma[{}_RU]$ is the smallest closed subcategory of ${}_R\mathbb{M}$ containing ${}_RU$ (i.e. $\sigma[{}_RU] = \overline{\mathrm{Gen}({}_RU)}$).*

Weak Subgenerators

53. For an R-module U consider the category

$$\pi[{}_RU] := \{M/L \mid M \in \mathrm{Cogen}({}_RU) \text{ and } L \leq_R M\}.$$

We call ${}_RU$ a **weak subgenerator**, iff $\pi[{}_RU] = {}_R\mathbb{M}$.

Proposition 54. ([Wis:2002, 1.10.])

1. *For any left R-module U, we have $\pi[{}_RU] = {}_{R/\mathrm{ann}_R(C)}\mathbb{M}$.*

2. *${}_RU$ is a weak subgenerator if and only if ${}_RU$ is faithful.*

3. *If ${}_RR$ is finitely cogenerated, then $\pi[{}_RU] = \sigma[{}_RU]$ for every left R-module ${}_RU$.*

4. *R is left Artinian if and only if, for any left R-module ${}_RU$, $\sigma[{}_RU] = \pi[{}_RU]$.*

Lemma 55. ([Wis:1991, Exercise 16.12. (4)]) *For a self injective left R-module ${}_RU$ with $S := \mathrm{End}({}_RU)^{op}$ we have*

$$\sigma[{}_RU] = {}_{R/\mathrm{Ann}_R(U)}\mathbb{M} \Leftrightarrow M_S \text{ is finitely generated.}$$

Lemma 56. *The following are equivalent for a left R-module ${}_RU$ with $S := \mathrm{End}({}_RU)^{op}$:*

1. *U_S is flat;*

2. *for every R-linear morphism $f : U^n \to U^k$ (where $n, k \in \mathbb{N}$), we have $\mathrm{Ker}(f) \in \mathrm{Gen}({}_RU)$;*

3. *or every R-linear morphism $f : U^n \to U$ (where $n \in \mathbb{N}$), we have $\mathrm{Ker}(f) \in \mathrm{Gen}({}_RU)$.*

Proposition 57. ([Wis:1991, 15.4.]) *Let ${}_RU$ be a left R-module, $\overline{R} := R/\mathrm{ann}_R(U)$ and $S := \mathrm{End}({}_RU)^{op}$.*

1. *If U_S is finitely generated, then $\sigma[{}_RU] = {}_{\overline{R}}\mathbb{M}$;*

2. *If R is commutative and ${}_RU$ is finitely generated, then $\sigma[{}_RU] = {}_{\overline{R}}\mathbb{M}$.*

The following results characterizes the *subgenerators* in the category of R-module and plays an important role in the sequel:

Lemma 58. ([Col:1990, Proposition 4.5.], [Wis:1991, 15.3.], [Wis:2000, 2.3.]) *The following are equivalent for a left R-module ${}_RU$:*

1. *$\sigma[{}_RU] = {}_R\mathbb{M}$ (i.e. ${}_RU$ is a subgenerator, or cofaithful);*

2. *$R \hookrightarrow U^k$ for some $k \in \mathbb{N}$;*

3. *$\{L \mid L <_R U^{(\mathbb{N})}$ is a cyclic R-module$\}$ is a set of generators in ${}_R\mathbb{M}$;*

4. *${}_RU$ generates $\mathrm{E}({}_RR)$ (the injective envelope of ${}_RR$);*

5. *${}_RU$ generates all injective R-modules.*

Lemma 59. ([Col:1990, Proposition 4.5.], [Wis:1991, 15.3., 15.4.], [Wis:2000, 2.3.]) *Let ${}_RU$ be a left R-module and $S := \mathrm{End}({}_RU)^{op}$. Then $\sigma[{}_RU] = {}_R\mathbb{M}$, if (for example) ${}_RU$ is faithful and any of the following conditions holds:*

1. *U_S is finitely generated (equivalently, $\mathrm{Gen}({}_RU)$ is closed under direct products);*

2. *$\sigma[{}_RU]$ is closed under direct products;*

3. *R is commutative and ${}_RU$ is finitely generated;*

4. *${}_RR$ is finitely cogenerated (e.g. R is left Artinian).*

Properties of $\mathrm{Gen}({}_RU)$

Lemma 60. ([CF:2004, Chapter 1]) *Let ${}_RU$ be a left R-module and $S := \mathrm{End}({}_RU)^{op}$.*

1. *$\mathrm{Gen}({}_RU)$ is closed under epimorphic images and direct sums (i.e. a pretorsion class).*

2. *If $\mathrm{Gen}({}_RU) \subseteq U^{\perp_1}$, then $\mathrm{Gen}({}_RU)$ is closed under extensions (whence a torsion class);*

3. *If $\mathrm{Gen}({}_RU)$ is closed under submodules (i.e. $\mathrm{Gen}({}_RU) = \sigma[{}_RU]$), then U_S is flat.*

4. *$\mathrm{Gen}({}_RU)$ is closed under direct products if and only if U_S is finitely generated.*

Definition 61. An R-module U is said to be **product complete**, iff $\mathrm{Add}({}_RU)$ is closed under arbitrary direct sums.

Proposition 62. ([KS:1998]) *Let ${}_RU$ be a finitely generated left R-module. Then ${}_RU$ is product complete if and only if $S := \mathrm{End}({}_RU)^{op}$ is right coherent, left perfect and U_S is finitely presented.*

1.3 Special classes of Associative Rings

For the convention of the reader, we include in what follows the definitions and some of the main results on special classes of associative rings that are used in this report.

Finiteness Conditions

Definition 63. The ring R is called

left (right) coherent, iff ${}_RR$ (R_R) is coherent; and **coherent,** iff R is left and right coherent.

strongly left (right) coherent, iff the *right* (*left*) R-module R^R is locally projective; and strongly coherent, iff R is left and right strongly coherent.

Proposition 64. *The following are equivalent for the ring* R :

1. *R is left coherent;*
2. *every finitely presented R-module is coherent;*
3. *for every $r \in R$ the annihilator $\mathrm{ann}_R(r)$ is finitely generated, and the intersection of a (cyclic) finitely generated left ideal with a finitely generated left ideal is also finitely generated;*
4. *every product of flat right R-modules is flat;*
5. *for every set Λ, the right R-module R_R^Λ is flat.*

Proposition 65. ([GT:2006, Theorem 1.3.15.]) *The following are equivalent for the ring* R :

1. *R is strongly left coherent;*
2. *any product of locally projective right R-modules is locally projective;*
3. *all left R-ideals of the form mM^*, where $m \in M_R$, are finitely generated.*

Lemma 66. ([Wis:1991, Exercise 17.15. (12)])

1. *The following assertions are equivalent for the ring R :*

   (a) *$_RR$ is finitely cogenerated;*

   (b) *every cogenerator in $_R\mathbb{M}$ is a subgenerator;*

   (c) *every faithful left R-module is a subgenerator.*

2. *$_RR$ is injective if and only if every subgenerator in $_R\mathbb{M}$ is a generator.*

Chain Conditions

Definition 67. The ring R is called

left (right) Noetherian, iff $_RR$ (R_R) is **Noetherian**; and Noetherian, iff R is left and right Noetherian.

left (right) Artinian, iff $_RR$ (R_R) is Artinian; and **Artinian**, iff R is left and right Artinian.

Definition 68. The ring R is called **left** (**right**) **steady**, iff every small left (right) R-module is finitely generated, i.e. ${}_R^s\mathbb{M} = {}_R^{f.g.}\mathbb{M}$ ($\mathbb{M}_R^s = \mathbb{M}_R^{f.g.}$); and **steady**, iff R is left and right steady.

Proposition 69.

1. ([CT:1994, Corollaries 1.4., 1.6.]) *Every left* (*right*) *Noetherian ring is left* (*right*) *steady.*

2. *Every left* (*right*) *perfect ring is left* (*right*) *steady.*

Proposition 70. ([Zem:2005, Theorem 1.4.], [CT:1994, Proposition 1.3.]) *The following are equivalent for the ring R :*

1. *R is left Noetherian* (*i.e. every ascending chain of left R-ideals is stationary*)*;*

2. *every left ideal of R is finitely generated;*

3. *every* (*essential*) *left ideal of R is small;*

4. *every finitely generated left R-module is finitely presented;*

5. *every small left R-module is finitely presented.*

6. ${}^{f.g.}_{R}\mathbb{M} = {}^{f.p.}_{R}\mathbb{M} = {}^{s}_{R}\mathbb{M}$.

Proposition 71. ([Wis:1991, 31.4., Exercises 17.15 (12), 31.15. (4)], [Wis:2002, 1.10 (4)]) *The following are equivalent for the ring R :*

1. *R is left Artinian;*
2. *every finitely generated (cyclic) left R-module is finitely cogenerated;*
3. *R is left Noetherian,* $\mathrm{Jac}(R)$ *is nilpotent and* $R/\mathrm{Jac}(R)$ *is left semisimple;*
4. *every factor ring of R is left finitely cogenerated;*
5. *every self-injective left R-module U is finitely generated over* $\mathrm{End}({}_RU)^{op}$;
6. $\sigma[{}_RU] = \pi[{}_RU]$ $(= {}_{R/\mathrm{ann}_R(U)}\mathbb{M})$ *for every ${}_RU$.*

Corollary 72. *If R is left Artinian, then ${}_RR$ is finitely cogenerated.*

Definition 73. The ring R is said to be an **Artin algebra**, iff $Z(R)$ is Artinian and R is finitely generated a $Z(R)$-module.

Remark 74. Every Artin algebra is an *Artinian ring.*

Homological Rings

Definition 75. The ring R is said to be

left (**right**) **hereditary**, iff every left (right) ideal $I \lhd R$ is projective; and **hereditary**, iff R is left and right hereditary.

left (**right**) **semihereditray**, iff every finitely generated left (right) ideal $I \lhd R$ is projective; and **semi-hereditary**, iff R is left and right semi-hereditary.

Definition 76. The ring R is said to be (**strongly**) **von Neumann regular**, iff for every $r \in R$, there exists $s \in R$ such that $r = rsr$ $(r = r^2 s)$.

Proposition 77. ([Wis:1991, 3.10., 37.6.]) *The following are equivalent for the ring R :*

1. *R is von Neumann regular;*

2. *every left (right) principal ideal of R is generated by an idempotent;*

3. *every left (right) principal ideal is a direct summand of R;*

4. *every finitely generated left (right) ideal is a direct summand of R.*

5. *every left (right) R-module is flat;*

6. *every finitely presented left (right) R-module is projective;*

7. *every cyclic left (right) R-module is flat.*

Proposition 78. ([Wis:1991, 3.11.]) *The following are equivalent:*

1. *R is strongly von Neumann regular;*

2. *R is von Neumann regular and contains no non-zero nilpotent elements;*

3. *every left (right) principal ideal is generated by a central idempotent;*

4. *R is von Neumann regular and every left (right) ideal is a* two-sided *ideal.*

(Semi-)Perfect Rings

Definition 79. Two idempotents $e, e' \in R$ are said to be **orthogonal**, iff $ee' = 0$.

Definition 80. An idempotent $e \in R$ is said to be
local, iff eRe is a local ring;
primitive, iff for each pair e_1, e_2 of orthogonal idempotents

$$e = e_1 + e_2 \Rightarrow e_1 = 0 \text{ or } e_2 = 0.$$

Definition 81. ([Wis:1991, 42.6.]) The ring R is said to be **semiperfect**, iff there exists a set of local orthogonal idempotents $\{e_1, ..., e_k\}$ such that

$$R = Re_1 \oplus ... \oplus Re_k.$$

Definition 82. Let R be semiperfect. A left R-module M is said to be **primitive**, iff $M \simeq Re$ for some primitive idempotent $e \in R$.

Definition 83. A *pairwise orthogonal* set $\{e_1, ..., e_m\}$ of idempotents in R is said to be

complete, iff $1_R = e_1 + ... + e_m$;

basic, iff $\{Re_1, ..., Re_m\}$ is a complete irredundant set of representatives of the *primitive* left R-modules.

Definition 84. An idempotent $e \in R$ is said to be **basic**, iff there exists a *basic* set $\{e_1, ..., e_m\}$ of primitive idempotents of R such that

$$e = e_1 + ... + e_m.$$

Definition 85. A ring S is said to be a **basic ring for the ring** R, iff $S \simeq eRe$ for some basic idempotent $e \in R$.

Remark 86. Let R be semiperfect. Then R has a basic idempotent e and a basic ring $S \simeq eRe$ for R exists, which is unique up to isomorphism.

Proposition 87. ([AF:1974, Proposition 27.4.]) *A semiperfect ring R with basic idempotent e is Morita equivalent*[3] *to its basic ring eRe. Moreover, two semiperfect rings $R, \widetilde{R}$ with basic idempotents $e, \widetilde{e}$, respectively, are Morita equivalent if and only if their basic rings eRe and $\widetilde{e}\widetilde{R}\widetilde{e}$ are Morita equivalent.*

Definition 88. The ring R is said to be **left** (**right**) **perfect**, iff R satisfying the descending chain condition on its principal right (left) ideals; and perfect, iff R is left and right perfect.

Proposition 89. ([Wis:1991, 43.9.], [Fai1976, Theorem 22.29]) *The following are equivalent for the ring R :*

1. *R is a left (right) perfect ring;*
2. *the descending chain condition on the principal right (left) ideals of R holds;*
3. *the descending chain condition on the finitely generated right (left) ideals of R holds;*

[3]in the sense of 108

4. *the ascending chain condition for finitely generated left (right) R-modules holds;*

5. *the ascending chain condition for cyclic left (right) R-modules holds;*

6. *every flat left (right) R-module is projective.*

(Quasi-)Frobenius Rings

Definition 90. The ring R is called

left (right) pseudo-Frobenius, iff every faithful left (right) R-module is a generator in ${}_R\mathbb{M}$; and **pseudo Frobenius**, iff R is left and right Pseudo-Frobenius;

left (right) finitely pseudo-Frobenius ring, iff every faithful finitely generated left (right) R-module is a generator in ${}_R\mathbb{M}$; and **finitely pseudo-Frobenius**, iff R is left and right finitely pseudo-Frobenius.

Theorem 91. ([Wis:1991, 48.12.]) *The following are equivalent:*

1. *R is left Pseudo-Frobenius;*

2. *every cogenerator in ${}_R\mathbb{M}$ is a generator;*

3. *${}_RR$ is injective and finitely cogenerated;*

4. *${}_RR$ is injective, semiperfect and* $\mathrm{Soc}({}_RR) \leq^{ess}_R R$;

5. *${}_RR$ is a cogenerator and there are only finitely many non-isomorphic simple modules in ${}_R\mathbb{M}$;*

6. *${}_RR$ is a cogenerator in ${}_R\mathbb{M}$ and R_R cogenerates all simple right R-modules;*

7. *${}_RR$ is an injective cogenerator.*

Definition 92. A ring R is said to be **quasi-Frobenius,** or a **QF ring**, iff ${}_RR$ (R_R) is Noetherian and injective.

Theorem 93. ([Wis:1991, 48.12.]) *The following are equivalent:*

1. *R is a QF ring, i.e. ${}_RR$ (R_R) is Noetherian and injective;*
2. *${}_RR$ (R_R) is Noetherian and a cogenerator;*
3. *${}_RR$ is a cogenerator and R_R is Noetherian;*
4. *R_R is cogenerator and ${}_RR$ is Noetherian;*
5. *${}_RR$ is a cogenerator and R_R is Artinian;*
6. *R_R is cogenerator and ${}_RR$ is Artinian;*
7. *${}_RR$ (R_R) is Artinian and a cogenerator;*
8. *${}_RR$ (R_R) is Artinian, and injective envelopes of simple left (right) R-modules are projective;*
9. *every injective left (right) R-module is projective;*
10. *every projective left (right) R-module is injective;*
11. *$R^{(\mathbb{N})}$ is an injective cogenerator in ${}_R\mathbb{M}$ (in $\mathbb{M}_R$);*
12. *R is left (right) perfect and every FP-injective left (right) R-module is flat.*

Gorenstein Rings

Definition 94. The ring R is called an (**Iwangsawa-**)**Gorenstein ring**, iff R is left and right Noetherian and the left and the right injective dimensions of R are finite. In this case $\text{inj.dim.}({}_RR) = n = \text{inj.dim.}(R_R)$ for some $n < \infty$, and R is called n**-Gorenstein**.

Example 95. The class of 0-*Gorenstein rings* coincides with the class of QF-rings.

Lemma 96. ([EJ:2000, §9]) *Let R be an n-Gorenstein ring. Then*

$$ {}_R\mathcal{P} = {}_R\mathcal{P}_n = {}_R\mathcal{I} = {}_R\mathcal{I}_n = {}_R\mathcal{F} = {}_R\mathcal{F}_n. $$

Definition 97. An R-module ${}_RM$ over an *arbitrary* ring R is said to be

Gorenstein-injective, iff $M = \mathrm{Ker}(f)$ for some long exact sequence

$$... \to E_1 \to E_0 \xrightarrow{f} E^0 \to E^1 \to ...,$$

of injective left R-modules that stays exact under $\mathrm{Hom}_R(E, -)$ for any injective left R-module E;

Gorenstein-projective, iff $M = \mathrm{Ker}(g)$ for some long exact sequence

$$... \to P_1 \to P_0 \xrightarrow{g} P^0 \to P^1 \to ...,$$

of projective left R-modules that stays exact under $\mathrm{Hom}_R(-, P)$ for any projective left R-module P;

Gorenstein-flat, iff $M = \mathrm{Ker}(f)$ for some long exact sequence

$$... \to F_1 \to F_0 \xrightarrow{f} F^0 \to F^1 \to ...,$$

of flat left R-modules that stays exact under $F \otimes_R -$ for any injective right R-module F.

Notation. For the ring R we set

$$\begin{array}{lcl} \mathcal{GI}(R) & := & \{{}_RM \mid M \text{ is Gorenstein-injective}\}; \\ \mathcal{GP}(R) & := & \{{}_RM \mid M \text{ is Gorenstein-projective}\}; \\ \mathcal{GF}(R) & := & \{{}_RM \mid M \text{ is Gorenstein-flat}\}. \end{array}$$

Proposition 98. ([EJ:2000, 10.1.2., 10.2.3., 10.3.4.])

1. *If ${}_RM$ is Gorenstein-injective, then ${}_RM$ is injective or* $\mathrm{inj.dim.}({}_RM) = \infty$;
2. *If ${}_RM$ is Gorenstein-projective, then ${}_RM$ is projective or* $\mathrm{proj.dim.}({}_RM) = \infty$;
3. *If ${}_RM$ is Gorenstein-flat, then ${}_RM$ is flat or* $\mathrm{flat.dim.}({}_RM) = \infty$.

Chapter 2

An Introduction to Morita Theory

99. Two arbitrary categories $\mathcal{C}$ and $\mathcal{D}$ are said to be **equivalent** (and we write $\mathcal{C} \approx \mathcal{D}$), iff there exists a *covariant* functor $F : \mathcal{C} \to \mathcal{D}$ and a (necessarily covariant) functor $G : \mathcal{D} \to \mathcal{C}$ with *natural isomorphisms*

$$G \circ F \simeq \mathrm{id}_{\mathcal{C}} \text{ and } F \circ G \simeq \mathrm{id}_{\mathcal{D}}.$$

2.1 (Ad)Static Modules

100. For rings R, S and a bimodule ${}_RP_S$ we consider the covariant functors

$$\mathrm{Hom}_R(P, -) \;:\; {}_R\mathbb{M} \;\longrightarrow\; {}_S\mathbb{M}; \qquad P \otimes_S - \;:\; {}_S\mathbb{M} \;\longrightarrow\; {}_R\mathbb{M}.$$

For every ${}_RM$ and ${}_SN$ we have the canonical morphisms

$$\begin{array}{llllll} \nu_M & : & P \otimes_S \mathrm{Hom}_R(P, M) & \longrightarrow & M, & p \otimes_S f \;\mapsto\; f(p); \\ \eta_N & : & N & \longrightarrow & \mathrm{Hom}_R(P, P \otimes_S N), & n \;\mapsto\; [p \mapsto p \otimes_S n]. \end{array}$$

We say ${}_RM$ (${}_SN$) is P**-static** (P**-adstatic**), iff $P \otimes_S \mathrm{Hom}_R(P, M) \overset{\nu_M}{\simeq} M$ ($N \overset{\eta_N}{\simeq} \mathrm{Hom}_R(P, P \otimes_S N)$). Moreover, we set

$$\begin{array}{lll} \mathrm{Stat}^l({}_RP_S) & := & \{{}_RM \mid P \otimes_S \mathrm{Hom}_R(P, M) \overset{\nu_M}{\simeq} M\}; \\ \mathrm{Adstat}^l({}_RP_S) & := & \{{}_SN \mid N \overset{\eta_N}{\simeq} \mathrm{Hom}_R(P, P \otimes_S N)\}. \end{array}$$

In case $S := \mathrm{End}_R(P)^{op}$, we set:

$$\mathrm{Stat}({}_RP) := \mathrm{Stat}^l({}_RP_{\mathrm{End}_R(P)^{op}}) \text{ and } \mathrm{Adstat}({}_RP) := \mathrm{Adstat}^l({}_RP_{\mathrm{End}_R(P)^{op}}).$$

Lemma 101. (Compare [CF:2004, Lemmata 2.1.2., 2.1.3.]) *Let R, S be rings and P an (R, S)-bimodule.*

1. *${}_RK$ is P-generated if and only if $\nu_K : P \otimes_S \mathrm{Hom}_R(P, K) \to K$ is surjective;*
2. *$P \otimes_S N \subseteq \mathrm{Pres}({}_RP) \subseteq \mathrm{Gen}({}_RP)$ for every ${}_SN$.*
3. *${}_SL$ is ${}_S^dP$-cogenerated if and only if $\eta_L : L \to \mathrm{Hom}_R(P, P \otimes_S L)$ is injective.*
4. *$\mathrm{Hom}_R(P, M) \subseteq \mathrm{Copres}({}_S^dP) \subseteq \mathrm{Cogen}({}_S^dP)$ for every ${}_RM$.*

Definition 102. Let K be a left R-module.

1. Let M be a right R-module. An R-submodule $L \leq_R M$ is called K-**pure**, iff the following sequence of Abelian groups is exact
$$0 \to L \otimes_R K \to M \otimes_R K \to M/L \otimes_R K \to 0;$$
2. Let M be a left R-module. An R-submodule $L \leq_R M$ is called K-**copure** , iff the following sequence of Abelian groups is exact
$$0 \to \mathrm{Hom}_R(M/L, K) \to \mathrm{Hom}_R(M, K) \to \mathrm{Hom}_R(L, K) \to 0.$$

Theorem 103. ([Nau:1990]) *For every (R, S)-bimodule ${}_RP_S$, the adjoint pair of covariant functors $(-\otimes_S, \mathrm{Hom}_R(P, -))$ induces an equivalence of subcategories*
$$\mathrm{Stat}^l({}_RP_S) \approx \mathrm{Adstat}^l({}_RP_S).$$
In particular, we have
$$\mathrm{Stat}({}_RP) \approx \mathrm{Adstat}({}_RP).$$

Remark 104. Let ${}_RP_S$ be an (R, S)-bimodule. By definition, $\mathrm{Stat}^l({}_RP_S) \subseteq {}_R\mathbb{M}$ and $\mathrm{Adstat}^l({}_RP_S) \subseteq {}_S\mathbb{M}$ are the *largest subcategories*, between which the adjoint pair of covariant functors $(-\otimes_S, \mathrm{Hom}_R(P, -))$ induces an equivalence.

2.2 Morita Equivalences

Definition 105. ([Ful:1974]) An R-module U is called

progenerator, iff $_RU$ is finitely generated (small), projective and a generator;

quasi-progenerator, iff $_RP$ is finitely generated, quasi-projective and a self-generator.

Remarks 106.
1. By [Bas:1968, Cotollary 4.8.], an R-module U is **faithfully projective** (i.e. small, projective and a generator) if and only if $_RU$ is a progenerator.
2. Roughly speaking, an R-module $_RP$ is a quasi-progenerator if and only if P is a progenerator in $\sigma[_RP]$.

Theorem 107. (Morita) *The following are equivalent for rings R and S :*

1. $_R\mathbb{M} \underset{G}{\overset{F}{\approx}} {}_S\mathbb{M}$;
2. *There exists a progenerator $_RP$ and $S \simeq \mathrm{End}(_RP)^{op}$.*

In this case, $F \simeq \mathrm{Hom}_R(P, -)$ and $G \simeq P \otimes_S -$.

Definition 108. Two rings R and S are said to be **Morita equivalent**, or to be **similar** (and we write $R \sim S$), iff $_R\mathbb{M} \approx {}_S\mathbb{M}$.

Theorem 109. (e.g. [Fai:1981], [Wis:1991]) *For two rings R and S, we have*

1. $_R\mathbb{M} \approx {}_S\mathbb{M} \Leftrightarrow \mathbb{M}_R \approx \mathbb{M}_S$.
2. *If $R \sim S$, then $Z(R) \simeq Z(S)$.*
3. *If R and S are commutative, then $R \sim S$ if and only if $R \simeq S$.*
4. *For every $n \in \mathbb{N}$, we have $R \sim \mathrm{M}_n(R)$.*
5. *If $R \sim S$, then the left (right) global dimensions of R and S are equal. In particular, for every $n \in \mathbb{N}$, the rings R and $\mathrm{M}_n(R)$ have equal left (right) global dimensions.*

Morita Contexts

110. With a **Morita semi-context** we mean a datum

$$\mathbf{m} = (R, S, P, Q, <, >_R),$$

where R, S are rings, ${}_RP_S$ is an (R, S)-bimodule, Q an (S, R)-bimodule and $<, >_R$: $P \otimes_S Q \to R$ is a (R, R)-bilinear morphism. The ideal $I := \mathrm{Im}(<, >_R) \lhd R$ is called the **trace ideal** associated to the Morita semi-context $\mathbf{m}$.

111. With a **Morita context** we mean a datum

$$\mathcal{M} = (R, S, P, Q, <, >_R, <, >_S),$$

where $(R, S, P, Q, <, >_R)$, $(S, R, Q, P, <, >_S)$ are Morita semi-contexts and the bilinear morphisms $<, >_R$: $P \otimes_S Q \to R$, $< -, >_S$: $Q \otimes_R P \to S$ are *compatible,* in the sense that

$$< p, q >_R \widetilde{p} = p < q, \widetilde{p} >_S \;\; \& \;\; < q, p >_S \widetilde{q} = q < p, \widetilde{q} >_R, \forall\, p, \widetilde{p} \in P, q, \widetilde{q} \in Q. \tag{2.1}$$

112. Let $\mathbf{m} = (R, S, P, Q, <, >_R)$, $\widetilde{\mathbf{m}} = (\widetilde{R}, \widetilde{S}, \widetilde{P}, \widetilde{Q}, <, >_{\widetilde{R}})$ be Morita semi-contexts. With a **morphism of Morita semi-contexts** from $\mathbf{m}$ to $\mathbf{m}'$ we mean a four fold set of maps

$$(\beta, \gamma, \phi, \psi) : (R, S, P, Q) \to (\widetilde{R}, \widetilde{S}, \widetilde{P}, \widetilde{Q}),$$

where $\beta : R \to \widetilde{R}$ and $\gamma : S \to \widetilde{S}$ are morphisms of rings, $\phi : P \to \widetilde{P}$ is (R, S)-bilinear and $\psi : Q \to \widetilde{Q}$ is (S, R)-bilinear, such that

$$\beta(< p, q >_R) =< \phi(p), \psi(q) >_{\widetilde{R}} \text{ for all } p \in P, q \in Q\ .$$

Notice that we consider $\widetilde{P}$ as an (R, S)-bimodule and $\widetilde{Q}$ as a (S, R)-bimodule with actions induced by the morphism of rings β and γ.

113. Let $\mathcal{M} = (R, S, P, Q, <, >_R, <, >_S)$, $\widetilde{\mathcal{M}} = (\widetilde{R}, \widetilde{S}, \widetilde{P}, \widetilde{Q}, <, >_{\widetilde{R}}, <, >_{\widetilde{S}})$ be Morita contexts. Following [Ami:1971, Page 275], we mean by a **morphism of Morita contexts** from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$ a four fold set

$$(\beta, \gamma, \phi, \psi) : (R, S, P, Q) \to (\widetilde{R}, \widetilde{S}, \widetilde{P}, \widetilde{Q}),$$

where $\beta : R \to \widetilde{R}$, $\gamma : S \to \widetilde{S}$ are ring morphisms, $\phi : P \to \widetilde{P}$ is (R,S)-bilinear and $\psi : Q \to \widetilde{Q}$ is (S,R)-bilinear, such that for all $p \in P$ and $q \in Q$ we have

$$\beta(< p, q >_R) =< \phi(p), \psi(q) >_{\widetilde{R}} \text{ and } \gamma(< q, p >_S) =< \psi(q), \phi(p) >_{\widetilde{S}} .$$

114. Let $(R, S, P, Q, <, >_R)$ be a Morita semi-context and consider the isomorphisms of Abelian groups

$$\mathrm{Hom}_{(R,S)}(P, Q^*) \overset{\zeta}{\simeq} \mathrm{Hom}_{(R,R)}(P \otimes_S Q, R) \overset{\xi}{\simeq} \mathrm{Hom}_{(S,R)}(Q, {}^* P).$$

Then we have the *dual pairings* $P_l := (Q_R, {}_RP)$ and $P_r := ({}_RP, Q_R)$, induced by the canonical morphisms

$$\kappa_{\mathbf{P}_l} := \xi(<, >_R) : Q_R \to ({}^*P)_R \text{ and } \kappa_{\mathbf{P}_r} := \zeta^{-1}(<, >_R) : \ {}_RP \to \ {}_R(Q^*).$$

On the otherhand, let $(S, R, Q, P, <, >_S)$ be a Morita semi-context and consider the isomorphisms of Abelian groups

$$\mathrm{Hom}_{(R,S)}(P, {}^* Q) \overset{\zeta'}{\simeq} \mathrm{Hom}_{(S,S)}(Q \otimes_R P, S) \overset{\xi'}{\simeq} \mathrm{Hom}_{(S,R)}(Q, P^*).$$

Then we have the *dual pairings* $\mathbf{P}_r := ({}_SQ, P_S)$ and $P_l := (P_S, {}_SQ)$, induced by the canonical morphisms

$$\kappa_{\mathbf{P}_r} := \xi'(<, >_S) :_S Q \to_S (P^*) \text{ and } \kappa_{\mathbf{Q}_r} := (\zeta')^{-1}(<, >_S) : P_S \to ({}^*Q)_S.$$

In what follows we include some of the classical results about generators and progenerators (e.g. [Fai:1981], [Wis:1991] and [CF:2004]).

Proposition 115. *Let R, S be rings and $\mathcal{M} = (R, S, P, Q, <, >_R, <, >_S)$ be a Morita context.*

1. *If $<, >_R: P \otimes_S Q \to R$ is surjective, then:*
   - *(a) $<, >_R$ is injective (whence $P \otimes_S Q \overset{<,>_R}{\simeq} R$);*
   - *(b) ${}_RP$ and Q_R are generators;*
   - *(c) P_S and ${}_SQ$ are finitely generated and projective;*
   - *(d) $P \simeq {}^*Q$ and $Q \simeq P^*$;*
   - *(e) We have isomorphisms of rings* $\mathrm{End}(P_S) \simeq R \simeq \mathrm{End}({}_SQ)^{op}$.

2. *If* $<,>_S: Q \otimes_R P \to S$ *is surjective, then:*

   (a) $<,>_S$ *is injective, hence* $Q \otimes_R P \overset{<,>_S}{\simeq} S$;

   (b) P_S *and* ${}_SQ$ *are generators;*

   (c) ${}_RP$ *and* Q_R *are finitely generated and projective;*

   (d) $P \simeq Q^*$ *and* $Q \simeq {}^*P$;

   (e) *We have isomorphisms of rings* $\mathrm{End}({}_RP)^{op} \simeq S \simeq \mathrm{End}(Q_R)$.

Proposition 116. *Let* R, S *be rings. The following are equivalent for a Morita context* $\mathcal{M} = (R, S, P, Q, <,>_R, <,>_S)$:

1. *The* (R, R)*-bilinear morphisms* $<,>_R: P \otimes_S Q \to R$ *and* $Q \otimes_R P \to S$ *are surjective;*

2. $P \otimes_S Q \overset{<,>_R}{\simeq} R$ *and* $Q \otimes_R P \overset{<,>_S}{\simeq} S$ *as bimodules;*

3. ${}_RP$ *is a progenerator and* $S \simeq \mathrm{End}({}_RP)^{op}$;

4. ${}_RP$ *and* P_S *are generators and* $S \simeq \mathrm{End}({}_RP)^{op}$;

5. ${}_SQ$ *is a progenerator and* $R \simeq \mathrm{End}({}_SQ)^{op}$;

6. P_S *is a progenerator and* $R \simeq \mathrm{End}(P_S)$.

Proposition 117. *Let* P *a left* R*-module,* $S := \mathrm{End}({}_RP)^{op}$ *and consider the bilinear morphisms*

$$[,]_P : P \otimes_S {}^*P \to R \text{ and } (,)_P : {}^*P \otimes_R P \to S.$$

1. $(R, S, P, {}^*P, [,]_P, (,)_P)$ *is a Morita context.*

2. *The following are equivalent:*

   (a) *The canonical* (R, R)*-bilinear morphism* $[,]_P : P \otimes_S {}^*P \to R$ *is surjective;*

   (b) ${}_RP$ *is a generator;*

   (c) P_S *is finitely generated projective and* $R \simeq \mathrm{End}(P_S)$.

In this case, $P \otimes_S {}^*P \overset{[,]_P}{\simeq} R$ *as* (R,R)*-bimodules,* $R \simeq \mathrm{End}(P_S)$ *and* ${}^*P_R$ *is a generator.*

3. *The following are equivalent:*

   (a) *the canonical* (S,S)*-bilinear morphism* $(,)_P : {}^*P \otimes_R P \to S$ *is surjective;*

   (b) ${}_RP$ *is finitely generated projective.*

   In this case, ${}^*P \otimes_R P \overset{(,)_P}{\simeq} S$ *as* (S,S)*-bimodules,* $S \simeq \mathrm{End}({}^*P_R)$ *as rings, and* P_S *is a generator.*

4. *The following are equivalent:*

   (a) ${}_RP$ *is a progenerator;*

   (b) ${}_RP$ *and* P_S *are generators;*

   (c) ${}_RP$ *and* P_S *are finitely generated projective;*

   (d) P_S *is a progenerator and* $R \simeq \mathrm{End}(P_S)$.

Proposition 118. *Let* Q *be a right* R*-module,* $S := \mathrm{End}(Q_R)$ *and consider the canonical bilinear morphisms*

$$(,)_Q : Q^* \otimes_S Q \to R \text{ and } [,]_Q : Q \otimes_R Q^* \to S.$$

1. $(R, S, Q^*, Q, (,)_Q, [,]_Q)$ *is a Morita context.*

2. *The following are equivalent:*

   (a) *the canonical* (S,S)*-bilinear morphism* $(,)_Q : Q^* \otimes_S Q \to R$ *is surjective;*

   (b) Q_R *is a generator;*

   (c) ${}_SQ$ *is finitely generated projective and* $R \simeq \mathrm{End}({}_SQ)^{op}$.

   In this case, $Q^* \otimes_S Q \overset{(,)_Q}{\simeq} R$ *as bimodules,* $R \simeq \mathrm{End}({}_SQ)^{op}$ *as rings, and* ${}_RQ^*$ *is a generator.*

3. *The following are equivalent:*

(a) *the canonical (S,S)-bilinear morphism* $[,]_Q : Q \otimes_R Q^* \to S$ *is surjective;*

(b) Q_R *is finitely generated and projective; in this case* $Q \otimes_R Q^* \overset{[,]_Q}{\simeq} S$ *as (S,S)-bimodules,* $S \simeq \mathrm{End}({}_RQ^*)^{op}$ *as rings, and* ${}_SQ$ *is a generator.*

4. *The following are equivalent:*

   (a) Q_R *is a progenerator;*

   (b) Q_R *and* ${}_SQ$ *are generators;*

   (c) Q_R *and* ${}_SQ$ *are finitely generated projective;*

   (d) ${}_SQ$ *is a progenerator and* $R \simeq \mathrm{End}({}_SQ)^{op}$.

2.3 Morita Duality

119. Two arbitrary categories $\mathcal{C}$ and $\mathcal{D}$ are said to be **dual** (and we write $\mathcal{C} \circlearrowright \mathcal{D}$), iff there exists a *contravariant* functor $F : \mathcal{C} \to \mathcal{D}$ and a (necessarily contravariant) functor $G : \mathcal{D} \to \mathcal{C}$ with *natural isomorphisms*

$$G \circ F \simeq \mathrm{id}_{\mathcal{C}} \text{ and } F \circ G \simeq \mathrm{id}_{\mathcal{D}}.$$

Remark 120. It is obvious from the definitions, that for two categories $\mathcal{C}$ and $\mathcal{D}$ we have

$$\mathcal{C} \circlearrowright \mathcal{D} \Leftrightarrow \mathcal{C}^{op} \simeq \mathcal{D}^{op}.$$

For no pair of rings R, S is there a duality between the full module categories ${}_R\mathbb{M}$ and $\mathbb{M}_S$ (or ${}_S\mathbb{M}$), since $({}_R\mathbb{M})^{op}$ is not equivalent to ${}_S\mathbb{M}$ for any ring S. However, for a pair of rings R and S one is still interested in duality between *suitable subcategories* of ${}_R\mathbb{M}$ and $\mathbb{M}_S$.

121. For rings R, S and an (R, S)-bimodule ${}_RU_S$ we consider the functors

$$\begin{array}{rclccc} \mathbf{H}({}_RU) & := & \mathrm{Hom}_S(-, \mathrm{Hom}_R(-, U)), & {}_R\mathbb{M} & \to & {}_R\mathbb{M}; \\ \mathbf{H}(U_S) & := & \mathrm{Hom}_R(-, \mathrm{Hom}_S(-, U)), & \mathbb{M}_S & \to & \mathbb{M}_S. \end{array}$$

For every ${}_RM$ and N_S we have the canonical morphisms

$$\begin{array}{rcccccl} \beta_M & : & M & \to & \mathrm{Hom}_S(\mathrm{Hom}_R(M, U), U), & m \mapsto & [f \mapsto f(m)]; \\ \beta_N & : & N & \to & \mathrm{Hom}_R(\mathrm{Hom}_S(N, U), U), & n \mapsto & [g \mapsto g(n)]. \end{array}$$

A left R-module ${}_RM$ is called U-**torsionless** (respectively **semi** U**-reflexive,** U**-reflexive**), iff β_M is injective (respectively surjective, bijective). Similar definitions can be given for right S-modules. Set

$$\begin{array}{rcl} \mathbf{Ref}({}_RU) & := & \{{}_RM \mid M \overset{\beta_M}{\simeq} \mathrm{Hom}_S(\mathrm{Hom}_R(M, U), U)\}; \\ \mathbf{Ref}(U_S) & := & \{N_S \mid N \overset{\beta_N}{\simeq} \mathrm{Hom}_R(\mathrm{Hom}_S(N, U), U)\}. \end{array}$$

Proposition 122. *Let R, S be rings. For a bimodule ${}_RU_S$, the adjoint pair of contravariant functors $(\mathbf{H}({}_RU), \mathbf{H}(U_S))$ induces a duality*

$$\mathbf{Ref}({}_RU) \circlearrowright \mathbf{Ref}(U_S).$$

Theorem 123. (Morita) *Let R, S be rings, $\mathcal{C} \subseteq {}_R\mathbb{M}$ and $\mathcal{D} \subseteq \mathbb{M}_S$ be full subcategories that are closed under isomorphisms and such that ${}_RR \in \mathcal{C}$, $S_S \in \mathcal{D}$. If (F, G) is a pair of contravariant functors inducing a duality $\mathcal{C} \circlearrowright \mathcal{D}$, then there exists a bimodule ${}_RU_S$ such that*

1. ${}_RU \simeq G(S)$ *and* $U_S \simeq F(R)$;

2. *There are natural isomorphisms*

$$F \simeq \mathrm{Hom}_R(-, {}_RU) \text{ and } G \simeq \mathrm{Hom}_S(-, U_S);$$

3. $\mathcal{C} \subseteq \mathbf{Ref}({}_RU)$ *and* $\mathcal{D} \subseteq \mathbf{Ref}(U_S)$.

Definition 124. An (R, S)-bimodule ${}_RU_S$ is said to be **faithfully balanced**, iff $S \simeq \mathrm{End}({}_RU)^{op}$and $R \simeq \mathrm{End}(U_S)$.

Definition 125. A bimodule ${}_RU_S$ is said to be a **Morita bimodule**, iff

1. ${}_RM_S$ is faithfully balanced;

2. ${}_RR \in \mathbf{Ref}({}_RU)$ and $S_S \in \mathbf{Ref}(U_S)$;

3. $\mathbf{Ref}({}_RU)$ and $\mathbf{Ref}(U_S)$ are closed under submodules and factor modules.

Theorem 126. *For an (R, S)-bimodule ${}_RU_S$ the following are equivalent:*

1. ${}_RU_S$ *is a Morita (R, R)-bimodule;*

2. *Every factor module of ${}_RR, S_S$, ${}_RU$ and U_S is U-reflexive;*

3. ${}_RU_S$ *is faithfully balanced such that ${}_RU$ and U_S are injective cogenerators.*

Chapter 3

Modules over Commutative Rings

Throughout this chapter, R denotes a commutative ring with $1_R \neq 0_R$. Non-zero divisors of R are said to be **regular elements**, and we set

$$R^{\times} := R \backslash \{0\} \text{ and } R^{\mathrm{reg}} := \{r \in R \mid r \text{ is a } \textit{regular element}\}.$$

With an R-module U we mean a *left* R-module, considered as an (R, R)-*bimodule* with right action given by

$$ur := ru \text{ for every } r \in R \text{ and } u \in U.$$

3.1 Preliminaries

Definition 127. The **(classical) total quotient ring** of R is

$$\mathbf{Q}(R) := R_{R^{\mathrm{reg}}} = \{\frac{r}{s} \mid r \in R \text{ and } s \in R^{\mathrm{reg}}\}.$$

If R is an integral domain, then $\mathbf{Q} := R_{R^{\times}}$ is a field (called the **quotient field** of R).

Theorem 128. *For any integral domain R we have*

$$\mathbf{Q} \simeq \mathrm{E}({}_R R) \text{ (\textit{the} injective envelope \textit{of} } {}_R R).$$

In particular, ${}_R\mathbf{Q}$ is injective and $R \leq_R \mathbf{Q}$ is an essential *R-submodule.*

Definition 129. Let R be a commutative ring. Any intermediate ring $R \subseteq \widetilde{R} \subseteq \mathbf{Q}(R)$ is called an **overring** of R.

Definition 130. Let M be an R-module. The **divisibility set** of ${}_R M$ is

$$D_R(M) := \{r \in R \mid rM = M\}.$$

Remark 131. For any R-module M, then $D_R(M) \subseteq R$ is an admissible multiplicatively closed set.

Definition 132. Let M be an R-module and $\varnothing \neq \mathbf{S} \subseteq R$.

1. The **S-divisible submodule** of M is

$$d_{\mathbf{S}}(M) := \bigcup \{N \leq_R M \mid sN = N \text{ for every } s \in \mathbf{S}\}.$$

2. If $\mathbf{S} \subseteq R$ is multiplicatively closed, then the **S-torsion submodule** of M is

$$\tau_{\mathbf{S}}(M) := \{m \in M \mid \exists\, s \in \mathbf{S} \text{ with } sm = 0\} = \bigcup_{s \in \mathbf{S}} \mathrm{ann}_M(s).$$

Definition 133. Let $\varnothing \neq \mathbf{S} \subseteq R$ be a non-empty set. An R-module M is called **S-divisible** (**S-reduced**), iff $d_{\mathbf{S}}(M) = M$ $(d_{\mathbf{S}}(M) = 0)$;

Definition 134. Let $\varnothing \neq \mathbf{S} \subseteq R$ be a non-empty multiplicatively closed set. An R-module M is called **S-torsion** (**S-torsion-free**), iff $\tau_{\mathbf{S}}(M) = M$ $(\tau_{\mathbf{S}}(M) = 0)$.

Notation. Let $\varnothing \neq \mathbf{S} \subseteq R$ be a non-empty set. We define

$$\mathcal{DI}(\mathbf{S}) := \{{}_R M \mid M \text{ is } \mathbf{S}\text{-divisible}\} \text{ and } \mathcal{R}(\mathbf{S}) := \{{}_R M \mid M \text{ is } \mathbf{S}\text{-reduced}\}.$$

If $\mathbf{S}$ is multiplicatively closed, then we define

$$\mathcal{T}(\mathbf{S}) := \{{}_R M \mid M \text{ is } \mathbf{S}\text{-torsion}\} \text{ and } \mathcal{TF}(\mathbf{S}) := \{{}_R M \mid M \text{ is } \mathbf{S}\text{-torsion-free}\}.$$

Definition 135. An R-module M is called **divisible** (respectively **reduced, torsion**, **torsion-free**), iff M is R^{reg}-divisible (respectively R^{reg}-reduced, R^{reg}-torsion, R^{reg}-torsion-free).

Notation. We set

$$\begin{array}{llll} {}_R\mathcal{DI} & := \{{}_RM \mid M \text{ is divisible}\}; & {}_R\mathcal{R} & := \{{}_RM \mid M \text{ is reduced}\}; \\ {}_R\mathcal{T} & := \{{}_RM \mid M \text{ is torsion}\}; & {}_R\mathcal{TF} & := \{{}_RM \mid M \text{ is torsion-free}\}. \end{array}$$

Definition 136. An R-module M is said to be **mixed**, iff $0 \neq \tau(M) \subsetneqq M$ (i.e. M is neither torsion nor torsion-free).

Notation. For an R-module ${}_RF$ we denote with $\text{gen}({}_RF)$ the minimal cardinality of generating sets of F as an R-module.

Definition 137. Let R be an integral domain. The **rank** of ${}_RM$ is defined as

$$\text{rk}({}_RM) := \sup\{\text{gen}({}_RF) \mid F \leq_R M \text{ is a free } R\text{-submodule}\}.$$

An R-module M is said to be of **finite rank** (**countable rank**), iff $\text{rk}({}_RM) < \infty$ ($\text{rk}({}_RM) \leq \infty$).

Definition 138. Let R be an integral domain. A prime ideal $J \lhd R$ is said to be **branched**, iff it is *not* the union of two prime ideals properly contained in J; otherwise, it is called **unbranched**.

Definition 139. An R-module M is said to be
cotorsion, iff $\text{Ext}^1_R(F, M) = 0$ for every flat R-module ${}_RF$;
cotorsion-free, iff M has no non-zero cotorsion R-submodules.

Definition 140. Let M be an R-module. An R-submodule $L \leq_R M$ is called **tight**, iff

$$\text{proj.dim.}(M/L) \leq \text{proj.dim.}(M) \text{ (and necessarily then proj.dim.}(L) \leq \text{proj.dim.}(M)).$$

3.2 Localizing Systems (Gabriel Filters)

Definition 141. Let R be an integral domain with quotient field $\mathbf{Q}$. A non-empty *multiplicatively closed* collection of R-ideals $\mathcal{S} \neq \varnothing$ is called a **generalized multiplicative system** of R, and the associated overring

$$R_{\mathcal{S}} := \bigcup\{q \in \mathbf{Q} \mid qI \subseteq R \text{ for some } I \in \mathcal{S}\} = \bigcup_{I \in \mathcal{S}} I^{-1}.$$

is called the **generalized $\mathcal{S}$-transform of** R.

Definition 142. Let R be an integral domain. A non-empty set $\mathcal{F} \neq \varnothing$ of non-zero ideals of R is called a **localizing system** (**Gabriel filter**) of R, iff the following two conditions are satisfied:

1. if $I \in \mathfrak{F}$ and $I \subseteq J \lhd R$, then $J \in \mathfrak{F}$;
2. if $I \in \mathfrak{F}$ and $J \lhd R$ is such that $a^{-1}J \cap R \subseteq I$ for all $0 \neq a \in I$, then $J \in \mathfrak{F}$.

Remark 143. Let R be an integral domain. By [FHP:1997, Proposition 5.1.1.], every localizing system of R is multiplicatively closed (i.e. a generalized multiplicative system of R). In particular, every localizing system of R is closed under finite intersections.

Definition 144. A localizing system $\mathfrak{F}$ of an integral domain R is said to be **finitely generated** (**principal**), iff every ideal $I \in \mathfrak{F}$ contains a finitely generated (principal) ideal $J \in \mathfrak{F}$.

Definition 145. Let R be an integral domain and $\mathcal{S}$ a generalized multiplicative system of R-ideals. An R-module M is said to be

1. **$\mathcal{S}$-divisible** , iff $IM = M$ for every $I \in \mathcal{S}$;
2. **$h_{\mathcal{S}}$-divisible** , iff $M \in \mathrm{Gen}(R_{\mathcal{S}})$.

Notation. Let R be an integral domain and $\mathfrak{F}$ a localizing system of R-ideals. We set

$$\begin{array}{lll} \mathcal{DI}(\mathfrak{F}) & := & \{{}_R M \mid IM = M \text{ for all } I \in \mathfrak{F}\}; \\ h_{\mathfrak{F}}(R) & := & \{{}_R M \mid M \in \mathrm{Gen}(R_{\mathfrak{F}})\}; \end{array}$$

Proposition 146. *Let R be an integral domain and $\mathfrak{F}$ a localizing system of R.*

1. *$\mathcal{DI}(\mathfrak{F})$ is closed under epimorphic images, direct sums and extensions.*
2. *$h_{\mathfrak{F}}(\mathfrak{F})$ is closed under epimorphic images and direct sums.*
3. *If $R_{\mathfrak{F}}$ is flat as an R-module, then $h_{\mathfrak{F}}(R) \subseteq \mathcal{DI}(\mathfrak{F})$.*

Definition 147. Let R be an integral domain. An R-module M is said to be **I-divisible** for some ideal $I \lhd R$, iff $IM = M$.

Notation. For an R-module M we set

$$\begin{array}{lcl} \mathcal{D}(M) & := & \{I \lhd R \mid IM = M\}; \\ \mathcal{D}_0(M) & := & \{I \in \mathcal{D}(M) \mid I \text{ contains a finitely generated ideal } J \in \mathcal{D}(M)\}; \\ \mathcal{D}_p(M) & := & \{I \in \mathcal{D}(M) \mid I \text{ contains a principal ideal } (a) \in \mathcal{D}(M)\}. \end{array}$$

Lemma 148. ([Sal:2005, Lemma 1.1.]) *Let R be an integral domain and M an R-module. Then $\mathcal{D}(M)$, $\mathcal{D}_0(M)$ and $\mathcal{D}_p(M)$ are localizing systems.*

Proposition 149. ([FHP:1997, Proposition 5.1.10. & 5.1.11.]) *Let R be an integral domain and $R \subseteq \widetilde{R} \subseteq \mathbf{Q}$ be an overring of R.*

1. *If ${}_R\widetilde{R}$ is flat, then $\mathcal{D}(\widetilde{R})$ is a finitely generated localizing system of R and $\widetilde{R} = R_{\mathcal{D}(\widetilde{R})}$.*
2. *The following are equivalent:*
   (a) *${}_R\widetilde{R}$ is flat;*
   (b) *$\widetilde{R} = R_{\mathcal{S}}$ for some generalized multiplicative system $\mathcal{S} \subseteq \mathcal{D}(\widetilde{R})$;*
   (c) *$\widetilde{R} = R_{\mathfrak{F}}$ for some (finitely generated) localizing system $\mathfrak{F} \subseteq \mathcal{D}(\widetilde{R})$.*

3.3 The (Krull) Dimension of Commutative Rings

Definition 150. Let R be commutative ring. The **height** of a prime ideal $\mathfrak{p} \lhd R$ is defined as

$$\mathrm{ht}(\mathfrak{p}) := \max\{n \mid \exists \text{ a chain of prime ideals } \mathfrak{p} := \mathfrak{p}_0 \supsetneqq \ldots \supsetneqq \mathfrak{p}_n\}.$$

The (**Krull**) **dimension** of R is defined as

$$\dim(R) = \sup\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

Definition 151. Let R be a commutative ring and M an R-module. A prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ is said to be an **associated prime ideal of** M, iff $\mathfrak{p} = \mathrm{ann}_R(m)$ for some $m \in M$ (equivalently, iff M contains a cyclic R-submodule $N := Rm \simeq R/\mathfrak{p}$).

Notation. Let M be an R-module. We set

$$\mathrm{Ass}(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} = \mathrm{ann}_R(m) \text{ for some } m \in M\}.$$

3.4 (Sub)Generators over Commutative Rings

In what follows, we restate some of the main results on the structure of modules over commutative rings that will be needed in the sequel.

Proposition 152. ([Wis:1991, 15.4.], [Wis:2000, 2.3.]) *Let R be commutative and U a* finitely generated *R-module. Then*

$$\sigma[{}_RU] = {}_{R/\mathrm{ann}_R(U)}\mathbb{M}.$$

Moreover, the following are equivalent:

1. *${}_RU$ is faithful;*
2. *$\sigma[{}_RU] = {}_R\mathbb{M}$ (i.e. ${}_RU$ is a subgenerator);*
3. *${}_RU$ generates $\mathrm{E}({}_RR)$ (the injective envelope of ${}_RR$);*
4. *${}_RU$ generates all injective R-modules;*
5. *$R \hookrightarrow U^k$ for some $k \in \mathbb{N}$;*
6. *$\{L \mid L <_R U^{(\mathbb{N})}$ is a cyclic R-module$\}$ is a set of generators in ${}_R\mathbb{M}$.*

Theorem 153. ([Wis:1991, 48.12., 48.17.]) *The following are equivalent for a commutative ring R :*

1. *R is a pseudo-Frobenius ring (i.e. every faithful R-module is a generator);*
2. *${}_RR$ is a cogenerator;*
3. *${}_RR$ is an injective cogenerator;*
4. *${}_RR$ is injective and finitely cogenerated;*
5. *${}_RR$ is a cogenerator and there are only finitely many non-isomorphic simple modules in ${}_R\mathbb{M}$;*
6. *every cogenerator in ${}_R\mathbb{M}$ is a generator;*

7. ${}_R R$ *is injective and finitely cogenerated;*

8. ${}_R R$ *is injective, semiperfect and* $\mathrm{Soc}({}_R R) \leq_R^{ess} R$.

Proposition 154. ([FP:1984]) *Every self-injective commutative ring is a finitely pseudo-Frobenius ring.*

Proposition 155. ([FP:1984]) *For a commutative ring R, the following are equivalent:*

1. *R is a finitely pseudo-Frobenius ring* (*i.e. every finitely generated, faithful R-module is a generator*)*;*

2. *every finitely generated faithful ideal is projective and R has a classical ring of quotients $\mathbf{Q}_c(R)$, which is* (*self-*)*injective.*

Proposition 156. ([Wis:1991, 37.12. (5)]) *For a commutative von-Neumann regular ring R, the following are equivalent:*

1. *R is a finitely pseudo-Frobenius ring* (*i.e. every finitely generated, faithful R-module is a generator*)*;*

2. *R is self-injective;*

3. *for every faithful finitely generated R-module N, the trace ideal* $\mathrm{Tr}(N, R)$ *is finitely generated.*

Corollary 157. *An integral domain R is finitely pseudo-Frobenius if and only if R is a Prüfer domain.*

Proposition 158. *Let R be commutative.*

1. *If $I \lhd R$ is an ideal and U is an R-module with $IU = U$, then there exists $r \in I$ such that $(1 - r)M = 0$.*

2. *If ${}_R U$ is a faithful, then U generates all simple R-modules.*

Proposition 159. *Let R be commutative. A non-zero projective R-module $P \neq 0$ is a generator, if one of the following conditions hold:*

1. *${}_R P$ is finitely generated, and R contains no non-trivial idempotents;*

2. $_RP$ *is finitely generated and faithful;*

3. $_RP$ *is faithful and R is Noetherian.*

Proposition 160. ([Bas:1968, Cotollary 4.8.], [Fai:1981], [Wis:1991, 18.11.]) *Let R be a commutative ring. For an R-module P with $S := \mathrm{End}(_RP)^{op}$ the following are equivalent:*

1. $_RP$ *is a progenerator (i.e. $_RP$ is finitely generated, projective and a generator);*

2. $_RP$ *is a faithfully projective (i.e. $_RP$ is small, projective and a generator);*

3. $_RP$ *is finitely generated and projective, P_S is finitely generated projective, and $R \simeq \mathrm{End}(P_S)^{op}$;*

4. $_RP$ *and P_S are generators;*

5. $_RP$ *is finitely generated, projective and faithful.*

   If R has no non-trivial idempotents, then "2" (and "1") are equivalent to:

2'. $_RP$ *is finitely generated and projective.*

   If R is Noetherian and $_RP \neq 0$, then "2" (and "1") are equivalent to:

2''. $_RP$ *is projective and faithful.*

The Picard Group

161. Let R be a commutative ring and consider the category

$$\begin{array}{rcl} \underline{\underline{\mathrm{Pic}}}(R) & := & \{{}_RP \mid {}_RP_R \text{ is an } \textit{equivalence } (R,R)\textit{-bimodule}\} \\ & = & \{{}_RP \mid P \otimes_R Q \simeq R \text{ for some } R\text{-module } {}_RQ\}. \end{array}$$

We define the **Picard group** $\mathrm{Pic}(R)$ as the *isomorphism classes* of $\underline{\underline{\mathrm{Pic}}}(R)$, i.e.

$$\mathrm{Pic}(R) := \{[P] \mid \ {}_RP \text{ is an equivalence } (R,R)\text{-bimodule}\}.$$

The multiplication of *Abelian group* $\mathrm{Pic}(R)$ is given by

$$[P] \bullet [Q] := [P \otimes_R Q], \text{ for all } P, Q \in \underline{\underline{\mathrm{Pic}}}(R),$$

and the unity is $[R]$. For every $P \in \underline{\underline{\mathrm{Pic}}}(R)$, we have $[P]^{-1} = [P^*]$. The groups $\mathrm{Pic}(\mathbf{Q}(R)/R)$ is called the **relative Picard group** of R.

162. Let $\phi : R \to S$ be a morphism of commutative rings. Then ϕ induced a map

$$- \otimes_R S : \underline{\underline{\mathrm{Pic}}}(R) \to \underline{\underline{\mathrm{Pic}}}(S),$$

as well as a group morphism

$$\phi_* : \mathrm{Pic}(R) \to \mathrm{Pic}(S), \ [P] \mapsto [- \otimes_R S].$$

This shows that we have covariant functors

$$\underline{\underline{\mathrm{Pic}}}(\bullet) : \mathbf{CR} \to \mathbf{SET} \text{ and } \mathrm{Pic}(\bullet) : \mathbf{CR} \to \mathbf{Ab},$$

where **CR** is the category of *commutative rings*, **SET** is the category of *sets* and **Ab** is the category of *Abelian groups*.

Proposition 163. ([Bas:1968, Proposition III.7.5.]) *Let R be a commutative ring. The following are equivalent for an R-module P :*

1. $P \in \underline{\underline{\mathrm{Pic}}}(R)$;
2. ${}_RP$ *is finitely generated projective of rank* 1;
3. ${}_RP$ *is finitely generated projective and* $\mathrm{End}({}_RP) \simeq R$;
4. ${}_RP$ *is finitely generated and* $P_{\mathfrak{m}} \simeq R_m$ *for every* $\mathfrak{m} \in \mathrm{Max}(R)$.

Example 164. ([Lam:1999, Examples 2.22 (C), (F)]) Let R be a commutative ring. If R is local, or R is a UFD, then $\mathrm{Pic}(R) = \{1\}$.

Invertible Ideals

Definition 165. Let R be a commutative ring. We say $\varnothing \neq I \subseteq \mathbf{Q}(R)$ is

a **pre-fractional ideal**, iff $I \leq_R \mathbf{Q}(R)$ is an R-submodule;

a **fractional ideal**, iff $I \leq_R \mathbf{Q}(R)$ is an R-submodule and there exists some $0 \neq \widetilde{r} \in R$ with $\widetilde{r}I \subseteq R$;

an (**integral**) **ideal**, iff $I \lhd R$ is an ideal of R.

Definition 166. Let R be a commutative ring. We call a (pre-)fractional ideal of R an **invertible**, iff $IJ = R$ for some (pre-)fractional ideal J of R.

Notation. Let R be a commutative ring. We set

$$\mathcal{F}(R) := \{I \mid 0 \neq I \leq_R \mathbf{Q}(R) \text{ is a non-zero fractional ideal of } R\}.$$

We also set

$$\mathbb{I}(R) := \{I \in \mathcal{F}(R) \mid I \text{ is invertible}\} \text{ and } \mathbb{P}(R) := \{Rq \mid q \in \mathbf{Q}(R)\}.$$

For $\varnothing \neq I \subseteq R$, we set

$$I^{-1} := (R :_{\mathbf{Q}(R)} I) := \{q \in \mathbf{Q}(R) \mid qI \subseteq R\}.$$

Definition 167. Let R be a commutative ring. We call a (pre-)fractional ideal of R **regular** (or **non-degenerate**), iff $I \cap R^{\text{reg}} \neq \varnothing$. In particular, an (integral) ideal $I \lhd R$ is regular, iff I contains a regular element.

Lemma 168. (Compare with [Lam:1999, Lemma 2.16, Theorem 2.14]) *Let R be a commutative ring. If I is a regular (pre-)fractional ideal of R, then:*

1. $\mathrm{Hom}_R(I, \mathbf{Q}(R)) \simeq \mathbf{Q}(R)$ *as* $\mathbf{Q}(R)$*-modules.*
2. $I^* \simeq I^{-1}$ *as R-modules.*
3. *If I is invertible, then* $\mathrm{End}_R(I) \simeq R$.

Theorem 169. (Compare with [Lam:1999, Lemma 2.13, Theorem 2.17]) *Let R be a commutative ring. The following are equivalent for a (pre-)fractional ideal I of R :*

1. *I is invertible;*

2. $II^{-1} = R$;

3. ${}_RI$ *is (finitely generated) projective and* I *is regular.*

Remarks 170. Let R be a commutative ring.

1. Several authors define fractional ideals of R to be the (*finitely generated*) *pre-fractional* ones defined above, e.g. [Lam:1999] ([BK:2000]). Although non-standard, we add the prefix "*pre*" to distinguish between the two different classes of R-modules and to avoid confusion.

2. The group of invertible pre-fractional ideals and the group of invertible fractional ideals coincide. In particular, An (integral) ideal $I \lhd R$ is invertible as a pre-fractional ideal of R if and only if I is invertible as a fractional ideal of R.

Definition 171. Let R be a commutative ring. We call $\mathbb{I}(R)/\mathbb{P}(R)$ the **class group** of R

Proposition 172. ([Lam:1999, Theorem 2.14]) *Let* R *be a commutative ring. If* I *is an invertible (pre-)fractional ideal of* R*, then:*

1. ${}_RI$ *is finitely generated projective of rank* 1;

2. For any (pre-)fractional ideal J *of* R*, we have a canonical isomorphism of* R*-modules* $I \otimes_R J \simeq IJ$;

3. ${}_RI$ *is free if and only if* $I = qR$ *for some* $q \in \mathbf{Q}(R)$ *(necessarily a unit).*

Definition 173. Let R be a commutative ring. The **idealizer** of an ideal $I \lhd R$ is

$$\mathrm{E}(I) = \{q \in \mathbf{Q}(R) \mid qI \subseteq I\}.$$

Remark 174. Let R be a commutative ring. The *idealizer* $\mathrm{E}(I)$ is the largest overring of R, in which I is an ideal.

Definition 175. Let R be a commutative ring. An element $q \in \mathbf{Q}(R)$ is said to be **integral over** R, iff $\exists\ f(x) = a_0 + \ldots + a_{n-1}x^{n-1} + x^n \in R[x]$ with $f(q) = 0$. The **integral closure** of R is defined as

$$\mathrm{IC}(R) := \{q \in \mathbf{Q}(R) \mid q \text{ is integral over } R\}.$$

The commutative ring R is said to be **integrally closed**, iff $R = \mathrm{IC}(R)$.

Theorem 176. (Compare with [FS:2000, 3.7.]) *For an integral domain R, the following are equivalent:*

1. *R is integrally closed;*
2. $\mathrm{E}(I) = R$ *for every finitely generated (fractional) ideal* $0 \neq I \lhd R$;
3. $\mathrm{End}_R(I) \simeq R$ *for every finitely generated (fractional) ideal* $0 \neq I \lhd R$.

Proposition 177. ([Lam:1999, Page 35]) *Let R be a commutative ring.*

1. *Every finitely generated projective R-module P has the property that the function*
$$[P : R] : \mathrm{Spec}(R) \to \mathbb{Z},\ \mathfrak{p} \mapsto \mathrm{rk}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$$
is constant *if and only if R has no non-trivial idempotents.*
2. *If R is an integral domain, the for every finitely projective R-module P we have*
$$[P : R] = \dim_{\mathbf{Q}}(P \otimes_R \mathbf{Q}).$$

Definition 178. Let R be a commutative ring. We say a finitely generated projective R-module M has **rank** n, iff each of the $\{\mathrm{rk}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\} = \{n\}$, where $\mathrm{rk}(M_{\mathfrak{p}})$ is the rank of the free $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$.

Proposition 179. (Compare with [Bas:1968, Chapter III]) *Let R be commutative and $I \lhd R$ an ideal. Then $\mathcal{M}_I := (R, R, I, I^*, <,>_R, <,>^R)$ is a Morita context with canonical R-bilinear morphisms*
$$<,>_R : I \otimes_R I^* \to R \text{ and } <,>^R : I \otimes_R I \to R$$
Moreover, the following are equivalent:

1. $I \in \underline{\underline{\mathrm{Pic}}}(R)$;
2. $I \otimes_R I^* \overset{<,>_R}{\simeq} R$;
3. *${}_RI$ is finitely generated projective and faithful;*
4. *${}_RI$ is finitely generated projective of rank* 1;
5. *${}_RI$ is finitely generated and $I_{\mathfrak{m}} \simeq R_{\mathfrak{m}}$ for every* $\mathfrak{m} \in \mathrm{Max}(R)$.
6. *${}_RI$ is a generator and* $R \simeq \mathrm{End}({}_RI)$;
7. *${}_RI$ is a progenerator and* $R \simeq \mathrm{End}({}_RI)$.

In view of Proposition 160, the following result is easy to prove (in fact, it seems to be folklore, e.g. [Wis:1991, 40.1., 18.11], [BK:2000, Proposition 4.1.17.], [Fai:1981, Chapter 2, page 458], [Bas:1968, Chapter III]):

Theorem 180. *Let R be commutative and $0 \neq I \lhd R$ be a non-zero ideal. If I is regular (e.g. R is an integral domain), then the following are equivalent:*

1. *I is invertible;*
2. *$II^{-1} = R$;*
3. *${}_RI$ is projective;*
4. *${}_RI$ is finitely generated projective;*
5. *${}_RI$ is finitely generated projective and faithful;*
6. *${}_RI$ is a progenerator (i.e. ${}_RI$ is finitely generated, projective and a generator);*
7. *${}_RI$ is a faithfully projective (i.e. ${}_RI$ is small, projective and a generator);*
8. *$I \otimes_S I^* \simeq R$ as (R,R)-bimodules;*
9. *${}_RI$ is finitely generated projective and $\mathrm{End}({}_RI) \simeq R$;*
10. *${}_RI$ is finitely generated projective of rank 1;*
11. *${}_RI$ is a generator and $R \simeq \mathrm{End}({}_RI)$;*
12. *${}_RI$ is finitely generated and $I_{\mathfrak{m}} \simeq R_m$ for every $\mathfrak{m} \in \mathrm{Max}(R)$.*
13. *$I \in \underline{\underline{\mathrm{Pic}}}(R)$.*

181. Let R be a commutative ring. The embedding $\iota : R \hookrightarrow \mathbf{Q}(R)$ induces a map of set

$$- \otimes_R \mathbf{Q}(R) : \underline{\underline{\mathrm{Pic}}}(R) \to \underline{\underline{\mathrm{Pic}}}(\mathbf{Q}(R))$$

a morphism of Abelian groups

$$\iota_* : \mathrm{Pic}(R) \to \mathrm{Pic}(\mathbf{Q}(R)),\ [P] \mapsto [P \otimes_R \mathbf{Q}(R)].$$

Theorem 182. ([Lam:1999, Theorem 2.21.]) *Let R be a commutative ring. Then the embedding $R \overset{\iota}{\hookrightarrow} \mathbf{Q}(R)$ induced an exact sequence of Abelian groups*

$$1 \longrightarrow U(R) \longrightarrow U(\mathbf{Q}(R)) \xrightarrow{\beta} \mathbb{I}(R) \xrightarrow{\gamma} \mathrm{Pic}(R) \xrightarrow{\iota_*} \mathrm{Pic}(\mathbf{Q}(R)).$$

Where

$$\beta(q) := Rq \text{ for every } q \in U(\mathbf{Q}(R)) \text{ and } \gamma(I) := [I] \text{ for every } I \in \mathbb{I}(R).$$

Theorem 183. ([Lam:1999, Corollary 2.21.']) *Let R be a commutative ring and consider the embedding $R \overset{\iota}{\hookrightarrow} \mathbf{Q}(R)$. If the induced morphism $\iota_* : \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(\mathbf{Q}(R))$ is trivial, then we have an isomorphism of Abelian groups*

$$\mathrm{Pic}(R) \simeq \mathbb{I}(R)/\mathbb{P}(R).$$

Every invertible (pre-)fractional ideal of R is an equivalence (R, R)-bimodule; however the converse is not true in general, e.g. [Lam:1999, Example 2.22(A)]. The following results provides examples of rings for which every equivalence (R, R)-bimodule is isomorphic to an invertible (pre-)fractional ideal of R, whence the Picard and class groups of R are isomorphic.

Corollary 184. ([Lam:1999, Corollary 2.21.', Example 2.22(E)]) *If R is an integral domain, or if R is a Noetherian commutative ring, then* $\mathrm{Pic}(K) = \{1\}$ *whence* $\mathrm{Pic}(R) \simeq \mathbb{I}(R)/\mathbb{P}(R)$.

3.5 Special Classes of Commutative Rings

For the convention of the reader, we include in this section the definitions and the main results on some special classes of commutative rings and integral domain that will be used in the sequel.

Definition 185. A commutative ring R is said to be
local, iff R has exactly one maximal ideal;
semilocal, iff R has a finite number of maximal ideals;
principal ideal ring (**PIR**), iff every ideal of R is principal;
Bézout, iff every finitely generated ideal of R is principal.

Definition 186. An integral domain is said to be **maximal**, iff R is *linearly compact* in the discrete topology, i.e. every class of cosets $\{r_\lambda + I_\lambda\}_\Lambda$ with the finite intersection property has a non-empty intersection (where $r_\lambda \in R$, $I_\lambda \lhd R$ for every $\lambda \in \Lambda$ and Λ an arbitrary index set).

Definition 187. An integral domain R is said to be a **Matlis domain**, iff

$$\text{proj.dim.}({}_R\mathbf{Q}) = 1.$$

Prüfer Rings

Definition 188. A commutative ring R is called a **Prüfer ring**

Proposition 189. ([FHP:1997, Theorems 1.1.1, 5.1.15.]) *The following are equivalent for an integral domain R :*

1. *R is a Prüfer domain;*
2. *${}_RR$ is semi-hereditary (i.e. every finitely generated ideal of R is projective);*
3. *Each 2-generated ideal of R is invertible;*
4. *For each $P \in \mathrm{Spec}(R)$, R_P is a valuation domain;*

5. *For each* $\mathfrak{m} \in \mathrm{Max}(R)$, $R_{\mathfrak{m}}$ *is a valuation domain;*

6. *Each overring of* R *is integrally closed;*

7. *Each overring of* R *is flat as an* R*-module;*

8. *Each overring of* R *is the intersection of localizations of* R;

9. *If* $R \subseteq \widetilde{R} \subseteq \mathbf{Q}$ *is an overring of* R, *then*

$$\mathrm{Spec}(\widetilde{R}) = \{\mathfrak{p}\widetilde{R} \mid \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

10. R *is integrally closed and for each overring* $R \subseteq \widetilde{R} \subseteq \mathbf{Q}$ *there exists a localizing system* $\mathfrak{F}$ *of* R *such that* $\widetilde{R} = R_{\mathfrak{F}}$;

11. *We have a 1-1 correspondence*

$$\{\mathfrak{F} \mid \mathfrak{F} \text{ is finitely generated localizing system}\} \underset{\mathcal{D}(-)}{\overset{R \mapsto R_{\mathfrak{F}}}{\rightleftarrows}} \{\widetilde{R} \mid \widetilde{R} \text{ overring of } R\}$$

12. *Every ideal* $I \lhd R$ *is flat;*

13. *every finitely generated torsion-free* R*-module is projective;*

14. *every finitely generated torsion-free* R*-module is flat;*

15. *every torsion-free* R*-module is flat;*

16. *the tensor product of any two torsion-free* R*-modules is torsion-free;*

17. *the tensor product of any two ideals of* R *is torsion-free;*

18. *every finitely presented cyclic* R*-module is a direct summand of a direct sum of cyclically presented modules.*

Corollary 190. (Compare [FS:2000, Ch. VI, Exercise 1.10.]) *Let* R *be a Prüfer domain. If* $P \neq 0$ *is a projective* R*-module, then* ${}_RP$ *is a progenerator.*

Lemma 191. ([FS:2000, VI.6.2. & VI.6.4.]) *Let* R *be a Prüfer domain.*

1. *A finitely generated* R*-module* M *is finitely presented if and only if* $\mathrm{proj.dim.}({}_RM) \leq 1$;

2. *If* proj.dim.$({}_RM) \leq 1$, *then* ${}_RM$ *is coherent and all finitely generated* R*-submodules of* M *are tight.*

192. Let R be a commutative ring. Two rectangular matrices $A, B \in \mathbf{M}_{m\times n}(R)$ are said to be **equivalent**, iff there exist two *invertible* square matrices $G \in \mathbf{M}_m(R)$ and $H \in \mathbf{M}_n(R)$, such that $GAH = B$. A rectangular matrix $A \in \mathbf{M}_{m\times n}(R)$ admits a **diagonal reduction**, iff A is equivalent to some *diagonal* matrix $D \in \mathbf{M}_{m\times n}(R)$, with diagonal entries d_i satisfying the divisibility relations $d_{i+1} \mid d_i$ for all $i < \min\{m, n\}$.

Definition 193. A commutative ring R is said to be an **elementary divisor ring** (**EDR**), iff every rectangular matrix with entries in R admits a **diagonal reduction**.

Lemma 194. ([FS:2000, V.3.4.]) *An integral domain* R *is an EDR if and only if every finitely presented* R*-module decomposes* (*uniquely*) *as a direct sum of cyclic* R*-modules.*

Remark 195. By [FS:2000, Corollary III.6.6.], the following are equivalent for a semilocal integral domain R :

$$R \text{ is Prüfer } \Leftrightarrow R \text{ is Bézout } \Leftrightarrow R \text{ is an EDR.}$$

Valuation Rings

Definition 196. A commutative ring R is said to be a **valuation ring**, iff the ideals of R form a chain, i.e. for any ideals $I, J \lhd R$ we have $I \subseteq J$ or $J \subseteq I$.

Theorem 197. ([Kap:1974, Theorem 63]) *For an integral domain* R *the following are equivalent:*

1. R *is a valuation domain;*

2. *For every* $r, s \in R$ *we have* $s \in Rr$ *or* $r \in Rs$;

3. *For every* $r, s \in R$ *we have* $r \mid s$ *or* $s \mid r$;

4. *For any* $0 \neq q \in \mathbf{Q}$ *we have* $q \in R$ *or* $q^{-1} \in R$;

5. *R is Bézout and local.*

Definition 198. A valuation domain R is said to be

1. **discrete rank** 1 (i.e. **DVR**), iff R is Noetherian;
2. **discrete**, iff no branched prime ideal $L \lhd R$ is idempotent;
3. **strongly discrete**, iff no non-zero prime ideal $L \lhd R$ is idempotent.

Definition 199. Let R be valuation commutative ring with maximal ideal $\mathfrak{m}$. A valuation ring $\widetilde{R}$ is said to be an **intermediate extension** of R, iff

1. $R \subseteq \widetilde{R}$ is a subring with the same unity;
2. The correspondences
$$I \mapsto \widetilde{R}I \text{ and } J \mapsto J \cap R$$
are inverse to each other and establish a bijection between the set of ideals of R and the set of ideals of $\widetilde{R}$;
3. The have a canonical isomorphism $R/\mathfrak{m} \overset{\phi}{\simeq} \widetilde{R}/\widetilde{R}\mathfrak{m}$.

Commutative Gorenstein Rings

Lemma 200. ([EJ:2000, Theorem 9.3.3.]) *The following are equivalent for a commutative Noetherian ring R :*

1. $\mathrm{E}({}_R R)$ *is flat;*
2. $R_{\mathfrak{p}}$ *is a Gorenstein ring of Krull dimension* 0 *for all* $\mathfrak{p} \in \mathrm{Ass}(R)$;
3. $\mathrm{E}(N)$ *is flat for every flat R-module N*;
4. $\mathrm{F}(U)$ *is injective for every injective R-module U (where* $\mathrm{F}(U) \to U$ *is the* flat cover[1] *of* U_R);

[1] See [EJ:2000] for the definition of *flat covers*

5. $U \otimes_R \widetilde{U}$ *is injective for all injective R-modules $U, \widetilde{U}$;*

6. $R_{R^{\mathrm{reg}}}$ *is injective.*

Proposition 201. ([Bas:1963]) *Let R be a commutative n-Gorenstein ring. Every minimal injective coresolution of R is of the form*

$$0 \to R \to E_0 \to ... \to E_n \to 0,$$

where

$$E_h := \sum_{\mathrm{ht}(\mathfrak{p})=h} \mathrm{E}(R/\mathfrak{p}) \text{ for all } h \geq 0.$$

Dedekind Domains

Definition 202. An integral domain R is called **Dedekind**, iff every non-zero ideal $I \lhd R$ is invertible.

Proposition 203. ([Wis:1991, 40.5.]) *The following are equivalent for an integral domain R :*

1. *R is Dedekind;*

2. *R is Noetherian and Prüfer;*

3. *${}_RR$ is hereditary;*

4. *every divisible R-module is injective;*

5. *every cyclic R-module is a direct summand of a direct sum of cyclically presented modules.*

Proposition 204. ([Wis:1991, 40.6.]) *Let R be a Dedekind domain with quotient field $\mathbf{Q}$. Then:*

1. *Every non-zero prime ideal of R is maximal.*

2. *Every ideal in R is a product of prime ideals.*

3. *For every ideal $I \neq R$, we have $R/I \simeq \prod_{i \leq n} R/\mathfrak{p}_i^{k_i}$ with $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ is a set of distinct prime ideals in R and $k_i < \infty$ for $i = 1, ..., n$.*

4. *R is a 1-Gorenstein ring.*

5. *We have an isomorphism of R-modules*

$$\mathbf{Q}/R \simeq \bigoplus_{0 \neq \mathfrak{p} \in \mathrm{Spec}(R)} \mathrm{E}(R/\mathfrak{p}) = \bigoplus_{\mathfrak{p} \in \mathrm{Max}(R)} \mathrm{E}(R/\mathfrak{p}).$$

6. *R has a minimal injective coresolution*

$$0 \to R \to \mathbf{Q} \xrightarrow{\pi} \bigoplus_{\mathfrak{p} \in \mathrm{Max}(R)} \mathrm{E}(R/\mathfrak{p}) \to 0.$$

Proposition 205. *Let R be a Dedekind domain with quotient field $\mathbf{Q}$. An R-module M is cotorsion if and only if $\mathrm{Ext}^1_R(\mathbf{Q}, M) = 0$.*

Abelian Groups ($\mathbb{Z}$-Modules)

Proposition 206. *The ring of integers $\mathbb{Z}$ is a Dedekind domain.*

Definition 207. The **torsion-subgroup** of an Abelian group G is

$$\tau(G) := \{g \in G \mid ng = 0 \text{ for some } n \in \mathbb{N}\}.$$

Definition 208. An Abelian group G is called
torsion, iff $\tau(G) = G$;
torsion-free, iff $\tau(G) = 0$;

Definition 209. An Abelian group G is called
cotorsion, iff $\mathrm{Ext}^1_R(\mathbb{Q}, G) = 0$;
cotorsion-free, iff G has no cotorsion subgroups.

Definition 210. Let p be a prime positive integer and consider the *multiplicatively closed set*

$$\mathbf{S}(p) = \{p^k \mid k \in \mathbb{N}\}.$$

An Abelian group G is called

a p**-group**, iff G is an $\mathbf{S}(p)$*-torsion* $\mathbb{Z}$-module (equivalently, iff for every $g \in G$ there exists some $k \in \mathbb{N}$ with $p^k g = 0$);

a p**-torsion group**, iff G is an $\mathbf{S}(p)$*-torsion* $\mathbb{Z}$-module;

a p**-torsion-free group**, iff G is an $\mathbf{S}(p)$*-torsion-free* $\mathbb{Z}$-module.

Definition 211. Let p be a prime integer and G an Abelian group. The p**-component** of G is

$$p(G) = \{g \in G | \ \exists \ k \in \mathbb{N} \text{ such that } p^k g = 0\}.$$

Definition 212. Let p be a prime integer. The p-component of the Abelian group $\mathbb{Q}/\mathbb{Z}$ is called the **Prüfer** p**-group** and is given by denoted by $\mathbb{Z}_{p^\infty}$:

$$\mathbb{Z}_{p^\infty} := \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = \{q + \mathbb{Z} \mid q \in \mathbb{Q} \text{ and } p^k q \in \mathbb{Z} \text{ for some } k \in \mathbb{N}\}. \tag{3.1}$$

The ring

$$\mathbb{J}_p := \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}) \simeq$$

is called the ring of p**-adic integers**.

Lemma 213. *For suitable direct systems of Abelian groups*

$$\begin{array}{lcll} \mathbb{Z}_{p^\infty} & = & \varinjlim_{p \in \mathbb{P}} & \{\mathbb{Z}_{p^k} \mid k \in \mathbb{N}\} \\ \mathbb{Q}/\mathbb{Z} & = & \varinjlim & \{\mathbb{Z}_n \mid n \in \mathbb{N}\}. \\ \mathbb{Q} & = & \varinjlim & \{\frac{1}{n}\mathbb{Z} \mid n \in \mathbb{N}\}. \end{array}$$

Proposition 214. *Let p be a prime integer.*

1. *We have*

$$\mathbb{Z}_{p^\infty} = \{q + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid p^k q \in \mathbb{Z} \textit{ for some } k \in \mathbb{N}\}.$$

2. *For a suitable inverse system of Abelian groups we have*

$$\mathbb{J}_p := \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}) \simeq \varprojlim \{\mathbb{Z}_{p^k} \mid k \in \mathbb{Z}\}.$$

Proposition 215. ([Wis:1991, 15.10.]) *Let p be a prime positive integer.*

1. *Every torsion Abelian group G is a direct sum of its p-components, i.e. $G = \bigoplus_{p\in\mathbb{P}} p(G)$. In particular, we have $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_{p^\infty}$.*

2. *We have*

$$\{G \mid G \text{ is a } p\text{-torsion Abelian group}\} == \sigma[\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p^n}].$$

   In $\sigma[\mathbb{Z}_{p^\infty}]$, $\mathbb{Z}_{p^\infty}$ is a cogenerator and $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p^n}$ is a generator.

3. *We have*

$$\{G \mid G \text{ is a torsion Abelian group}\} = \sigma[\mathbb{Q}/\mathbb{Z}] = \sigma[\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_n].$$

   In $\sigma[\mathbb{Q}/\mathbb{Z}]$, $\mathbb{Q}/\mathbb{Z}$ is a cogenerator and $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p^n}$ is a generator.

Part II

Tilting and Cotilting Modules

Chapter 4

Tilting (Cotilting) Modules over Associative Rings

In this chapter, we introduce some of the main results on tilting (cotilting) modules over arbitrary associative rings. Throughout this chapter, R denotes a (not necessarily commutative) associative ring with $1_R \neq 0_R$.

4.1 Tilting Modules - Basics

In what follows we present the main definitions and results from the theory of tilting modules that will be used in the sequel.

Definition 216. An R-module M is called a **κ-splitter** for some cardinal κ, iff $\mathrm{Ext}^1_R(M, M^\kappa) = 0$. We call ${}_RM$ a **splitter** , iff M is κ-splitter for every κ.

Definition 217. An R-module T is called a **tilting module**, provided:

1. proj.dim.$(T) < \infty$;

2. $\mathrm{Ext}^i_R(T, T^{(\Lambda)}) = 0$ for every index set Λ and all $i \geq 1$ *(*i.e. ${}_RT$ is a splitter*)*;

3. There exist $T_0, ..., T_k \in \mathrm{Add}(T)$ fitting in an exact sequence of R-modules

$$0 \to R \to T_0 \to T_1 \to ... \to T_k \to 0. \tag{4.1}$$

A tilting R-module with projective dimension at most n is called **n-tilting.**

Definition 218. A left R-module ${}_RT$ is said to be **classical tilting**, iff ${}_RT$ is tilting and $T \in {}_R\mathcal{P}^{<\omega}$. A classical tilting R-module with projective dimension at most n is said to be **classical n-tilting**.

Proposition 219. ([GT:2006, Page 189]) *An R-module T is classical tilting (classical n-tilting for some $n \in \mathbb{N}$), iff*

1. $T \in {}_R\mathcal{P}^{<\omega}$ $(T \in {}_R\mathcal{P}_n^{<\omega})$;
2. $\mathrm{Ext}_R^i(T, T) = 0$ *for all* $i \geq 1$;
3. *There exist* $T_0, ..., T_k \in \mathrm{add}(T)$ *fitting in an exact sequence of R-modules*

$$0 \to R \to T_0 \to T_1 \to ... \to T_k \to 0.$$

Definition 220. A ring S is said to be **tilted from the ring** R, iff there exists a classical tilting R-module T such that $S \simeq \mathrm{End}({}_RT)^{op}$.

Definition 221. Two tilting R-modules T_1, T_2 are said to be **equivalent**, iff $T_1^{\perp_\infty} = T_2^{\perp_\infty}$.

Theorem 222. ([Baz:2004(b), Proposition 3.5., Theorem 3.11.]) *Let $n \geq 1$. The following are equivalent for an R-module T :*

1. *T is an n-tilting R-module;*
2. $\mathrm{Gen}_n(T) = T^{\perp_\infty}$;
3. *The following conditions are satisfied*
   (a) $\mathrm{proj.dim.}(T) \leq n$;
   (b) $\mathrm{Ext}_R^i(T, T^{(\Lambda)}) = 0$ *for every index set Λ and any* $1 \leq i \leq n$;

(c) there exists proj.dim.$(T) \leq k \leq n$ *and* $T_0, ..., T_k \in \mathrm{Add}(T)$ *fitting in an exact sequence of R-modules*

$$0 \to R \to T_0 \to ... \to T_k \to 0.$$

Proposition 223. ([Baz:2004(b), Proposition 3.6.]) *Let* $n \geq 1$. *If* ${}_RT$ *is* n*-tilting, then for every* $l \geq 1$ *we have*

$$\bigcap_{i=1}^{n} T^{\perp_i} = T^{\perp_\infty} = \mathrm{Gen}_n(T) = \mathrm{Gen}_{n+l}(T) = \mathrm{Gen}_\infty(T);$$

whence ${}_RT$ *is* $n+l$ *tilting for every* $l \geq 1$.

Remarks 224.

1. Tilting modules generalize projective generators. Obviously, every projective generator is 0-tilting. In particular, ${}_RT$ is a progenerator if and only if ${}_RT$ is classical 0-tilting.

2. The notion of a tilting module was generalized several times till the above *general* definition was introduced. Still different authors use different notions of tilting modules. For example, R. Colpi and K. Fuller (e.g. [CF:2004]) mean by a *tilting module* a classical 1-tilting module, while they call infinitely generated tilting modules of projective dimension at most 1 *generalized tilting modules.*

Partial Tilting Modules

Definition 225. Let T be an R-module. We call T' :

pre-partial tilting module, iff proj.dim.$(T') < \infty$ and $\mathrm{Ext}^i_R(T', T'^{(\Lambda)}) = 0$ for every index set Λ and all $i \geq 1$;

partial tilting module, iff ${}_RT'$ is *pre-partial tilting* and $T'^{\perp_\infty}$ is closed under direct sums.

A (pre-)partial tilting R-module with projective dimension at most n is called (**pre-**)**partial** n**-tilting** .

Remarks 226. ([Trl:2007, 2.10.])

1. Several authors (e.g. [AC:2001], [AC:2002]) call *pre-partial tilting* modules *partial tilting*. We follow [Trl:2007] in assuming that a partial tilting module T' should satisfy the extra condition "$T'^{\perp_\infty}$ *is closed under direct sums*" to guarantee that partial tilting modules admit tilt-complement. Although our terminology is not standard, we add the prefix "pre" to distinguish between the two different classes of modules and to avoid confusion.

2. Not all pre-tilting modules (or direct summands of tilting modules) are partial tilting. For example, consider $R = \mathbb{Z}$ and $T' = \mathbb{Q}$. As pointed out in [CT:1995, 1.5.], $\mathbb{Q}^{\perp_\infty}$ (which coincides with the class of *cotorsion Abelian groups*[1]) is not closed under arbitrary direct sums, whence $\mathbb{Q}$ is not partial tilting according to our definition). However, $\mathbb{Q}$ is a pre-partial tilting R-module and admits a tilt-complement since $T := \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is a 1-tilting Abelian group.

Definition 227. We call an R-module T' :

classical (**pre-**)**partial tilting** , iff T' is a (pre-)partial tilting R-module and $T' \in {}_R\mathcal{P}^{<\omega}$;

classical (**pre-**)**partial** n**-tilting** , iff T is a (pre-)partial tilting R-module and $T \in {}_R\mathcal{P}^{<\omega}_n$.

Definition 228. A class $\mathcal{U}$ of R-modules is said to be

(n**-**)**tilting**, iff $\mathcal{U} = T^{\perp_\infty}$ for some (n-)tilting R-module T;

(**pre-**)**partial** (n-)**tilting,** iff $\mathcal{U} = T^{\perp_\infty}$ for some (pre-)partial (n-)tilting R-module T.

[1] An Abelian group G is **cotorsion**, iff $\mathrm{Ext}^1_{\mathbb{Z}}(F, G) = 0$ for every torsion free (flat) Abelian group F;

Definition 229. We say an R-module T' admits a **tilt-complement**, iff there exists an R-module T'' such that $T := T' \oplus T''$ is a tilting R-module.

Theorem 230. ([AC:2001, Theorem 4.4.], [AC:2002, Theorem 2.1.])

1. *An R-module T is tilting if and only if ${}_RT$ is pre-partial tilting and $T^{\perp_\infty} \subseteq \mathrm{Gen}({}_RT)$.*

2. *A pre-partial tilting R-module T' is partial tilting if and only if T' admits a tilt-complement T'' which is* Ext*-projective in ${T'}^{\perp_\infty}$.*

Theorem 231. ([AC:2002, Corollary 2.2.]) *Let T' be pre-partial tilting R-module.*

1. *If R is left coherent and ${}_RT'$ is finitely presented, then ${}_RT'$ is partial tilting.*

2. *If R is an Artin algebra and ${}_RT'$ is finitely generated, then ${}_RT'$ is partial tilting.*

4.2 Self-Tilting Modules & Star Modules

Inspired by the notion of (self-small) n-star modules introduced and investigated recently by J. Wei et. al., e.g. ([HHTW:2003]) [Wei:2005b], [Wei:2005a], we introduce the following definitions which extends (to $n > 1$) the corresponding notions in [MO:1989], [CDT:1997] and [Wis:1998]:

Definition 232. We call a (*possibly infinitely generated*) R-module $_RT$ an:

n**-star module**, iff $_RT$ is $(n+1)$-$\sum$-quasi-projective and $\mathrm{Pres}_n(T) = \mathrm{Gen}_n(T)$;

$*^n$**-module**, iff $_RT$ is a self-small n-star module[2];

weak $*^n$**-module**, iff $_RT$ is self-small and $(n+1)$-$\sum$-quasi-projective;

classical $*^n$**-module**, iff $_RT \in {}_R\mathcal{P}_n^{<\omega}$ and T is an n-star module.

n**-quasi-tilting**, iff $\mathrm{Pres}_n(T) = \mathrm{Gen}_n(T) \subseteq T^{\perp_1}$;

n**-self-tilting** , iff $_RT$ is projective on $\mathrm{Gen}_n(T)$ and $\mathrm{Pres}_n(T) = \mathrm{Gen}_n(T)$.

To be consistent with the general definition of a tilting module we adapted in this report, we introduce the following notions:

Definition 233. We say an R-module $_RT$ is a

star module , iff $_RT$ is an n-star module for some $n \geq 1$;

$*$**-module** , iff $_RT$ is a $*^n$-module for some $n \geq 1$;

weak $*$**-module** , iff $_RT$ is a weak $*^n$-module for some $n \geq 1$;

classical $*$**-module** , iff $_RT$ is a classical $*^n$-module for some $n \geq 1$;

quasi-tilting , iff $_RT$ is n-quasi-tilting for some $n \geq 1$;

self-tilting , iff $_RT$ is n-self-tilting for some $n \geq 1$.

Remarks 234.

1. The reader should be warned that the each of the notions in Definition 233 was used initially to denote the corresponding ones in Definition 232 with $n = 1$. As we did with tilting modules, we set $n = 1$ whenever we use the initial notions.

2. The $*^1$-modules (which are by our definition self-small) were introduced by C. Menini and A. Orsatti in [MO:1989] as modules inducing the equivalence of categories in Proposition 245 below.

[2] as introduced originally in [HHTW:2003]

3. The $*^1$-modules are necessarily finitely generated as shown by J. Trlifaj in [Trl1994]. For $n \geq 2$, the (self-small) $*^n$-modules are not necessarily finitely generated as the examples in [Wei:2006] show.

Theorem 235. ([Wei:2005a, Proposition 3.6.]) *The following are equivalent for an R-module T :*

1. ${}_RT$ *is n-quasi-tilting;*

2. ${}_RT$ *is an n-star module and* $\mathrm{Gen}_n({}_RT)$ *is closed under extensions.*

Theorem 236. ([Wei:2005a, Proposition 3.6.]) *The following are equivalent for an R-module T :*

1. ${}_RT$ *is n-tilting* (*i.e.* $\mathrm{Gen}_n({}_RT) = T^{\perp_\infty}$);

2. $\mathrm{Gen}_n({}_RT) = T^{\perp_{1\leq i\leq n}}$;

3. ${}_RT$ *is an n-star module and* ${}_R\mathcal{INJ} \subseteq \mathrm{Gen}_n({}_RT)$;

4. ${}_RT$ *is n-quasi-tilting and* ${}_R\mathcal{INJ} \subseteq \mathrm{Gen}_n({}_RT)$ (*i.e.* ${}_R\mathcal{INJ} \subseteq \mathrm{Gen}_n({}_RT) = \mathrm{Gen}_{n+1}({}_RT) = T^{\perp_1}$).

Theorem 237. ([HHTW:2003, Theorems 2.8., 2.10.]) *The following are equivalent for a left R-module T with* $S := \mathrm{End}({}_RT)^{op}$:

1. ${}_RT$ *is a* $*^n$*-module;*

2. ${}_RT$ *is self-small and for any exact sequence*

$$0 \to M \to N \to L \to 0$$

in ${}_R\mathbb{M}$ *with* $N, L \in \mathrm{Gen}_n({}_RT)$, *we have* $\mathrm{Gen}_n({}_RT)$ *if and only if the following induced sequence is exact:*

$$0 \to \mathrm{Hom}_R(T, M) \to \mathrm{Hom}_R(T, N) \to \mathrm{Hom}_R(T, L) \to 0.$$

3. *We have an equivalence of categories:*

$$\mathrm{Gen}_n({}_RT) \underset{T\otimes_S -}{\overset{\mathrm{Hom}_R(T,-)}{\approx}} T^{\intercal_\infty} := \{{}_SN \mid \mathrm{Tor}_i^S(T, N) = 0 \textit{ for all } i \geq 1\}.$$

Theorem 238. ([HHTW:2003, Theorem 3.5.]) *The following are equivalent for an R-module T :*

1. *${}_RT$ is a self-small n-tilting module;*
2. *${}_RT$ is a $*^n$-module and ${}_R\mathcal{INJ} \subseteq \mathrm{Gen}_n({}_RT)$.*

Theorem 239. *The following are equivalent for $T \in R\mathrm{mod}$:*

1. *${}_RT$ is a (classical) n-tilting module;*
2. *$\mathrm{Gen}_n({}_RT) = T^{\perp_\infty}$ and $\mathrm{Gen}_n({}_RT) \subseteq T^{\perp_n}$;*
3. *${}_RT$ is a $*^n$-module, ${}_R\mathcal{INJ} \subseteq \mathrm{Gen}({}_RT) \subseteq T^{\perp_n}$.*

1-Self-Tilting Modules & Star Modules

Theorem 240. ([CF:2004, 2.4.5., 2.4.6., 2.4.7.]) *The following are equivalent for a left R-module ${}_RT$ with $S := \mathrm{End}({}_RT)^{op}$:*

1. *${}_RT$ is a weak $*^1$-module (i.e. ${}_RT$ is self-small and w-$\sum$-quasi-projective);*
2. *We have an equivalence of categories*
$$\mathrm{Pres}({}_RT) \underset{T\otimes_S -}{\overset{\mathrm{Hom}({}_RT,-)}{\approx}} \mathrm{Cogen}({}_S^dT).$$
3. *For every left R-module ${}_RN$, the following canonical map is surjective:*
$$\eta^l_{T,N} : N \to \mathrm{Hom}_R(T, T\otimes_S N),\ n \mapsto [t \mapsto t\otimes_S n].$$

241. Let ${}_RU$ be a left R-module. By [Wis:1991], the Abelian category $\sigma[{}_RU]$ of U-subgenerated left R-modules has enough injectives and so every $B \in \sigma[{}_RU]$ has an *injective coresolution*, i.e. a long exact sequence

$$0 \to B \overset{\iota}{\to} E^0 \overset{\delta^1}{\to} ... \to E^{n-1} \overset{\delta^n}{\to} E^n \overset{\delta^{n+1}}{\to} E^{n+1} \to ...$$

with E^n injective in $\sigma[{}_RU]$ for each $n \geq 0$. For any $A \in \sigma[{}_RU]$, applying $\mathrm{Hom}_R(A,-)$ to the *cochain complex*

$$\mathbf{E}_{\widetilde{B}} :\ 0 \overset{\delta^0}{\to} E^0 \to ... \to E^{n-1} \overset{\delta^n}{\to} E^n \overset{\delta^{n+1}}{\to} E^{n+1} \to ...$$

yields a *cochain complex* of Abelian groups

$$... \to \mathrm{Hom}_R(A, E^{n-1}) \overset{\delta^n\circ -}{\to} \mathrm{Hom}_R(A, E^n) \overset{\delta^{n+1}\circ -}{\to} \mathrm{Hom}_R(A, E^{n+1}) \to ... \quad (4.2)$$

So, one can define for any $A \in \sigma[{}_RU]$ the additive covariant functors

$$\overline{\mathrm{Ext}}^n_U(A, \bullet) : \sigma[{}_RU] \to \mathbf{Ab},\ B \mapsto H^n(\mathrm{Hom}_R(A, \mathbf{E}_{\widetilde{B}})),$$

where

$$H^n(\mathrm{Hom}_R(A, \mathbf{E}_{\widetilde{B}})) := \mathrm{Ker}((\delta^{n+1}\circ -))/\mathrm{Im}((\delta^n \circ -)) \text{ for } n \geq 1$$

is the *nth-cohomology group* of the cochain complex (4.2).

For $n \geq 1$, we set

$$A^{\perp_n}_{\sigma[{}_RU]} := \{Y \in \sigma[{}_RU] \mid \overline{\mathrm{Ext}}^n_U(A, Y) = 0\}.$$

One may refer to [Wis:1998] for more details.

Proposition 242. ([Wis:1998, 4.2.]) *The following are equivalent for an R-module ${}_RT$:*

1. ${}_RT$ *is* 1-*self-tilting* (*i.e. projective on* $\mathrm{Gen}({}_RT)$) *and* $\mathrm{Pres}({}_RT) = \mathrm{Gen}({}_RT)$);

2. $\mathrm{Pres}({}_RT) = \mathrm{Gen}({}_RT)$ *and* ${}_RT$ *is w-*$\sum$*-quasi-projective;*

3. $\mathrm{Pres}({}_RT) = \mathrm{Gen}({}_RT) \subseteq T^{\perp_1}_{\sigma[{}_RT]}$;

4. $\mathrm{Gen}({}_RT) = T^{\perp_1}_{\sigma[{}_RT]}$;

5. *The following conditions are satisfied:*

   (a) $\overline{\mathrm{Ext}}^2_T(T, N) = 0$ *for all* $N \in \sigma[{}_RT]$;
   (b) $\overline{\mathrm{Ext}}^1_T(T, T^{(\Lambda)}) = 0$ *for every index set* Λ;
   (c) $\mathrm{Ker}(\mathrm{Hom}_R(T, -)) \cap T^{\perp_1}_{\sigma[{}_RT]} = 0$.

   If $\sigma[{}_RT]$ *has a progenerator* G, *then* (*c*) *can be replaced by:*

   (*c′*) *there exist* ${}_RT_1$, ${}_RT_2 \in \mathrm{Add}({}_RT)$ *fitting in a short exact sequence of R-modules*
   $$0 \to G \to T_1 \to T_2 \to 0.$$

Proposition 243. ([Wis:1998, 4.2.]) *If* ${}_RT$ *is* 1-*self-tilting, then* ${}_RT$ *is closed under extensions and products in* $\sigma[{}_RT]$.

Remarks 244.

1. By definition, a 1-self-tilting *module* is (roughly speaking) an R-module T that is 1-tilting in the Grothendieck category $\sigma[{}_RT] = \overline{\mathrm{Gen}({}_RT)}$.

2. A *different definition* of tilting modules in*Abelian* categories see [HRS1996].

3. The 1-quasi-tilting modules may have projective dimension $n \geq 2$ as shown by [CDT:1997, Example 5.4.].

4. Condition "2" in Proposition 242 shows that the class of 1-*self-tilting modules* coincides with the class of 1-*star modules.* It's not clear, whether for a given $n \geq 2$, the class of n-self-tilting modules coincides with the class of n-star modules.

As reported in [Wis:1998], the following characterizations of $*^1$-modules were obtained in [Fol:1997, 3.12.]:

Proposition 245. ([Wis:1998, 5.2.]) *The following are equivalent for a left R-module ${}_RT$:*

1. *${}_RT$ is a $*^1$-module (i.e. a self-small 1-star module);*
2. *${}_RT$ is a self-small 1-self-tilting R-module;*
3. *We have an equivalence of categories*

$$\mathrm{Gen}({}_RT) \underset{T\otimes_S -}{\overset{\mathrm{Hom}({}_RT)}{\approx}} \mathrm{Cogen}({}_S^dT).$$

4. *${}_RT$ satisfies the following conditions:*

   (a) $\overline{\mathrm{Ext}}_T^2(T,N) = 0$ *for every* $N \in \sigma[{}_RT]$;

   (b) *${}_RT$ is finitely presented in $\sigma[{}_RT]$ and* $\overline{\mathrm{Ext}}_T^1(T,T) = 0$;

   (c) $\mathrm{Ker}(\mathrm{Hom}_R(T,-)) \cap T_{\sigma[{}_RT]}^{\perp_1} = 0.$

   If $\sigma[{}_RT]$ has a progenerator G, then (c) can be replaced by:

   (c′) *there exist T_1, $T_2 \in \mathrm{add}({}_RT)$ fitting in a short exact sequences of R-modules:*

$$0 \to G \to T_1 \to T_2 \to 0.$$

Under these conditions, $\mathrm{Hom}_R(G,T)$ is finitely presented in $\mathbb{M}_S$.

Remark 246. Let ${}_RT$ be a left R-module. By Lemma 101, $\mathrm{Stat}({}_RT) \subseteq \mathrm{Gen}({}_RT)$ and $\mathrm{Adstat}({}_RT) \subseteq \mathrm{Cogen}({}_S^dT)$. On the otherhand, $\mathrm{Stat}({}_RT) \subseteq {}_R\mathbb{M}$ and $\mathrm{Adstat}({}_RT) \subseteq {}_S\mathbb{M}$ are (by definition) the largest subcategories, between which the adjoint pair of covariant functors $(- \otimes_S T, \mathrm{Hom}_R(T,-))$ incudes an equivalence.

In the light of Remarks 246 (and 104), we conclude

Proposition 247. ([Xin:1999, Lemma 2.3.]) *For a left R-module ${}_RT$ with $S := \mathrm{End}({}_RT)^{op}$, we have*

${}_RT$ *is a* $*^1$ *-module* $\Leftrightarrow \mathrm{Stat}({}_RT) = \mathrm{Gen}({}_RT)$ *and* $\mathrm{Adstat}({}_RT) = \mathrm{Cogen}({}_S^dT)$;

Theorem 248. ([CDT:1997, Proposition 2.1.]) *The following are equivalent for an R-module* ${}_RT$:

1. ${}_RT$ *is a self-small* 1*-quasi-tilting (i.e.* $\mathrm{Pres}({}_RT) = \mathrm{Gen}({}_RT) \subseteq T^{\perp_1}$*);*
2. ${}_RT$ *is a* $*^1$*-module and* $\mathrm{Gen}({}_RT)$ *is a torsion class;*
3. ${}_RT$ *is a* $*^1$*-module and* $\mathrm{Gen}({}_RT) \subseteq T^{\perp_1}$;
4. ${}_RT$ *is finitely generated and* $\mathrm{Gen}({}_RT) = T^{\perp_1} \cap \sigma[{}_RT]$.

Characterizations of Tilting Modules

In view of Lemma 58, we get from [CT:1995, Proposition 1.3.], [Wis:1998, 4.4.], [CF:2004, Theorem 3.1.5.] the following characterizations of 1-*tilting modules*:

Theorem 249. *The following are equivalent for an R-module T with $S := \mathrm{End}({}_RT)^{op}$:*

1. ${}_RT$ *is* 1-*tilting;*
2. *The following conditions are satisfied:*
   (a) $\mathrm{proj.dim.}({}_RT) \leq 1$;
   (b) $\mathrm{Ext}^1_R(T, T^{(\Lambda)}) = 0$ *for every index set* Λ;
   (c) $\mathrm{Ker}(\mathrm{Hom}_R(T, -)) \cap T^{\perp_1} = 0$.
3. $\mathrm{Gen}({}_RT) = T^{\perp_1}$;
4. T *is* 1-*self-tilting,* ${}_RT$ *is faithful and* T_S *is finitely generated.*
5. T *is* 1-*self-tilting and* $\sigma[{}_RT] = {}_R\mathbb{M}$;
6. T *is* 1-*self-tilting with* $\mathrm{E}({}_RR) \in \mathrm{Gen}({}_RT)$;
7. T *is* 1-*self-tilting and* ${}_R\mathcal{INJ} \subseteq \mathrm{Gen}({}_RT)$;
8. T *is* 1-*self-tilting and* $R \hookrightarrow T^k$ *for some* $k \in \mathbb{N}$;
9. T *is* 1-*self-tilting and* $\{L \mid L <_R T^{(\mathbb{N})}$ *is a cyclic* R-*module*$\}$ *is a set of generators in* ${}_R\mathbb{M}$.

Lemma 250. ([CF:2004, Propositions 1.1.1., 1.1.2., 1.1.3.], [Trl:1992, Lemma 1.1.2.]) *Let T be an R-module. Then*

1. $\mathrm{proj.dim.}({}_RT) \leq 1$ *if and only if* $T^{\perp_1}$ *is closed under factors.*
2. *If* ${}_RT$ *is finitely presented, then* $\mathrm{Ext}^1_R(T, -)$ *commutes with direct sums (whence* $T^{\perp_1}$ *is closed under direct sums).*

3. *If ${}_RT$ is finitely generated and $T^{\perp_1}$ is closed under direct sums and factor modules, then ${}_RT$ is finitely presented.*

4. *The following are equivalent for ${}_RT$:*

   (a) *${}_RT$ is finitely presented and* proj.dim.$({}_RT) \leq 1$;

   (b) *${}_RT$ is small and $T^{\perp_1}$ is closed under direct sums and factors.*

In view of Lemmas 58 and 250, we get from [Col:1993, Theorem 3], [CT:1995, Proposition 1.3.], [CDT:1997, Proposition 2.3.], [Wis:1998, 4.4.] and [CF:2004, Theorem 3.2.1.] the following characterizations of *classical* 1-*tilting modules*:

Theorem 251. *For a left R-module ${}_RT$ with $S := \mathrm{End}({}_RT)^{op}$ the following are equivalent:*

1. *${}_RT$ is a classical* 1*-tilting R-module;*

2. *${}_RT$ is* 1*-tilting R-module and ${}_RT$ is self-small* (*respectively, small, finitely generated, finitely presented*)*;*

3. *${}_RT$ is small,* $\mathrm{Gen}({}_RT) \subseteq T^{\perp_1}$ *and $T^{\perp_1}$ is a torsion class;*

4. $\mathrm{Gen}({}_RT) = T^{\perp_1}$ *and ${}_RT$ is self-small* (*respectively, small, finitely generated, finitely presented*)*;*

5. *The following conditions are satisfied:*

   (a) *${}_RT$ is finitely presented and* proj.dim.$({}_RT) \leq 1$;

   (b) $\mathrm{Ext}^1_R(T, T) = 0$;

   (c) *There exist $T_0, T_1 \in \mathrm{add}({}_RT)$ fitting in a short exact sequence of R-modules $0 \to R \to T_0 \to T_1 \to 0$.*

6. *The following conditions are satisfied:*

   (a) *${}_RT$ is small and* $\mathrm{Gen}({}_RT)$ *is closed under direct sums and factor modules;*

   (b) $\mathrm{Ext}^1_R(T, T) = 0$;

(c) *There exist $T_0, T_1 \in \mathrm{add}({}_RT)$ fitting in a short exact sequence of R-modules $0 \to R \to T_0 \to T_1 \to 0$.*

7. *The following conditions are satisfied:*

   (a) *${}_RT$ is finitely presented and* $\mathrm{proj.dim.}({}_RT) \leq 1$;

   (b) $\mathrm{Ext}^1_R(T, T) = 0$;

   (c) $\mathrm{Ker}(\mathrm{Hom}_R(T, -)) \cap T^{\perp_1} = 0$.

8. *${}_RT$ is a $*^1$-module (respectively self-small 1-self-tilting, 1-quasi-tilting) and $\sigma[{}_RT] = {}_R\mathbb{M}$;*

9. *${}_RT$ is a $*^1$-module (respectively self-small 1-self-tilting, 1-quasi-tilting) with $\mathrm{E}({}_RR) \in \mathrm{Gen}({}_RT)$;*

10. *${}_RT$ is a $*^1$-module (respectively self-small 1-self-tilting, 1-quasi-tilting) and ${}_R\mathcal{INJ} \subseteq \mathrm{Gen}({}_RT)$;*

11. *${}_RT$ is a $*^1$-module (respectively self-small 1-self-tilting, 1-quasi-tilting) and $R \hookrightarrow T^k$ for some $k \in \mathbb{N}$;*

12. *${}_RT$ is a $*^1$-module (respectively self-small 1-self-tilting, 1-quasi-tilting) and $\{L \mid L <_R T^{(\mathbb{N})}$ is a cyclic R-module$\}$ is a set of generators in ${}_R\mathbb{M}$.*

13. *${}_RT$ is a $*^1$-module (respectively self-small 1-self-tilting, 1-quasi-tilting), ${}_RT$ is faithful and T_S is finitely generated;*

14. *${}_RT$ is 1-quasi-tilting, ${}_RT$ is faithful and $\mathrm{Gen}({}_RT)$ is closed under direct products;*

15. *${}_RT$ is 1-quasi-tilting and $\mathrm{Gen}({}_RT)$ is a tilting torsion class.*

252. For an R-module T we have

$$ {}_R\mathcal{Q}\text{-}\mathcal{PROG} = {}_R\mathcal{STAR}_1^{s.s.} \cap {}_R\mathcal{S}\text{-}\mathcal{G}. $$

It follows then that

$$
\begin{aligned}
{}_R\mathcal{PROG} &= {}_R\mathcal{Q}\text{-}\mathcal{PROG} \cap {}_R\mathcal{SUBGEN} \\
&= ({}_R\mathcal{STAR}_1^{s.s.} \cap {}_R\mathcal{S}\text{-}\mathcal{G}) \cap {}_R\mathcal{SUBGEN} \\
&= {}_R\mathcal{STAR}_1^{s.s.} \cap ({}_R\mathcal{S}\text{-}\mathcal{G} \cap {}_R\mathcal{SUBGEN}) \\
&= {}_R\mathcal{STAR}_1^{s.s.} \cap {}_R\mathcal{GEN} \\
&= ({}_R\mathcal{STAR}_1^{s.s.} \cap {}_R\mathcal{S}\text{-}\mathcal{G}) \cap ({}_R\mathcal{STAR}_1^{s.s.} \cap {}_R\mathcal{SUBGEN}) \\
&= {}_R\mathcal{Q}\text{-}\mathcal{PROG} \cap {}_R\mathcal{C}\text{-}\mathcal{TILT}_1.
\end{aligned}
$$

Combining 252 with [Col:1990, Theorem 4.7.], [CF:2004, Ch. 2] yields

Theorem 253. *Let $_RT$ be a left R-module and $S := \mathrm{End}(_RT)^{op}$.*

1. *The following are equivalent:*

   (a) *$_RT$ is a quasi-progenerator;*

   (b) *$\sigma[_RT] \approx {_S\mathbb{M}}$;*

   (c) *$\mathrm{Gen}(_RT) \approx {_S\mathbb{M}}$ and $\mathrm{Gen}(_RT) = \sigma[_RT]$;*

   (d) *$_RT$ is a $*^1$-module and*

      i. $\mathrm{Gen}(_RT) = \sigma[_RT]$; or

      ii. *$_RT$ is a self-generator;* or

      iii. *$_RT$ is quasi-projective.*

2. *The following are equivalent:*

   (a) *$_RT$ is a progenerator;*

   (b) *$_R\mathbb{M} \approx {_S\mathbb{M}}$;*

   (c) *$_RT$ is a $*^1$-module and a generator;*

   (d) *$_RT$ is a quasi-progenerator and a (sub)generator;*

   (e) *$_RT$ is a quasi-progenerator and a classical 1-tilting R-module;*

Corollary 254. ([CF:2004, Corollary 2.4.12.]) *Let $_RR$ be finitely cogenerated (e.g. R is left Artinian). Then $_RT$ is classical 1-tilting if and only if $_RT$ is a faithful $*^1$-module.*

Corollary 255. ([CF:2004, Corollary 2.4.13.]) *Let R be left Artinian, $_RT$ be a left R-module and $\overline{R} := R/\mathrm{ann}_R(T)$. Then $_RT$ is classical 1-tilting if and only if $_{\overline{R}}T$ is a $*^1$-module.*

Definition 256. A non-empty class of R-modules $\mathcal{U} \neq \varnothing$ is said to be **closed under n-images**, iff for every exact sequence of R-modules

$$U_n \to U_{n-1} \to \ldots \to U_1 \to N \to 0,$$

with $U_j \in \mathcal{U}$ $(j = 1, \ldots, n)$, also $N \in \mathcal{U}$.

For any R-module ${}_RU$, the class $\mathrm{Gen}({}_RU)$ is obviously closed under 1-images. It's not known, if $\mathrm{Gen}_n({}_RU)$ is closed under n-images for $n > 1$. However, we have the following partial results:

Lemma 257. ([Wis:1998, 3.2.(2, i)]) *For any R-module ${}_RU$, we have:*

1. *If ${}_RU$ is 2-$\sum$-quasi-projective, then $\mathrm{Gen}_2({}_RU)$ is closed under 2-images;*
2. *If ${}_RU$ is n-star R-module, then $\mathrm{Gen}_n({}_RU)$ is closed under n-images.*

4.3 Cotilting Modules - Basics

Definition 258. An R-module C is called a **cotilting module**, provided:

1. $\text{inj.dim.}({}_RC) < \infty$;

2. $\text{Ext}^i_R(C^\Lambda, C) = 0$ for every index set Λ and all $i \geq 1$;

3. There exists an injective cogenerator ${}_R\mathbf{I}$ and $C_0, ..., C_k \in \text{Prod}({}_RC)$ fitting in an exact sequence of R-modules

$$0 \to C_k \to C_{k-1} \to ... \to C_0 \to \mathbf{I} \to 0. \tag{4.3}$$

A cotilting R-module with injective dimension at most n is called n-**cotilting**.

Theorem 259. ([Baz:2004(b), Proposition 3.5., Theorem 3.11.]) *Let $n \geq 1$. The following are equivalent for an R-module C :*

1. $\text{Cogen}_n({}_RC) = {}^{\perp_\infty}C$;

2. C is n-cotilting;

3. The following conditions are satisfied:

   (a) $\text{inj.dim.}({}_RC) \leq n$;
   (b) $\text{Ext}^i_R(C^\Lambda, C) = 0$ *for every index set Λ and any $1 \leq i \leq n$;*
   (c) there exists an injective cogenerator ${}_R\mathbf{I}$, an integers $\text{inj.dim.}({}_RC) \leq k \leq n$ *and $C_0, ..., C_k \in \text{Prod}({}_RC)$ fitting in an exact sequence of R-modules*
   $$0 \to C_k \to ... \to C_0 \to \mathbf{I} \to 0.$$

Proposition 260. ([Baz:2004(b), Proposition 3.6.]) *Let $n \geq 1$. If C is an n-cotilting R-module, then for every $l \geq 1$ we have*

$$\bigcap_{i=1}^{n} {}^{\perp_i}C = {}^{\perp_\infty}C = \text{Cogen}_n({}_RC) = \text{Cogen}_{n+l}({}_RC) = \text{Cogen}_\infty({}_RC).$$

In particular, ${}_RC$ is $n+l$-cotilting for every $l \geq 1$.

Remarks 261.

1. Cotilting modules can be considered as a generalization of injective cogenerators. Obviously, every injective cogenerator is 0-cotilting.
2. The notion of a cotilting module was generalized several times till the above *general* definition was introduced.

Partial Cotilting Modules

Definition 262. We call an R-module C' a

pre-partial cotilting module, iff inj.dim.$({}_RC') < \infty$ and $\mathrm{Ext}^i_R(C'^{\Lambda}, C') = 0$ for every index set Λ and all $i \geq 1$;

partial cotilting module, iff C' is pre-partial cotilting and ${}^{\perp_\infty}C'$ is closed under direct products.

A (pre-)partial cotilting R-module with injective dimension at most n is called **(pre-) partial n-cotilting**.

Definition 263. A class $\mathcal{U}$ of R-modules is said to be

(n-)cotilting, iff $\mathcal{U} = {}^{\perp_\infty}C$ for some left (n-)cotilting R-module C;

(pre-)partial cotilting, iff $\mathcal{U} = {}^{\perp_\infty}C'$ for some (pre-)partial left (n-)cotilting R-module C'.

Definition 264. Two cotilting R-modules C_1, C_2 are said to be **equivalent**, iff ${}^{\perp_\infty}C_1 = {}^{\perp_\infty}C_2$.

Definition 265. We say an R-module C' admits a **cotilt-complement**, iff there exists an R-module C'', such that $C := C' \oplus C''$ is a cotilting R-module.

Theorem 266. ([AC:2001, Page 249], [AC:2002, Page 93])

1. *An R-module C is cotilting if and only if ${}_RC$ is pre-partial cotilting and* ${}^{\perp_\infty}C \subseteq \mathrm{Cog}({}_RC)$.

2. *Let C' be a pure-injective pre-partial cotilting R-module. Then ${}_RC'$ is partial cotilting if and only if C' admits a cotilt-complement C'' which is* Ext*-injective in* ${}^{\perp_\infty}C'$.

Remarks 267. ([Trl:2007, 3.12.])

1. Several authors (e.g. [AC:2001]) don't assume the extra condition "${}^{\perp_\infty}C'$ *is closed under direct products*" in their definitions of a partial-cotilting module. We follow [Trl:2007] in assuming this extra condition to guarantee that *partial cotilting modules* admit cotilt-complements (whence pure injective). We add the prefix "pre-" to distinguish between the two different classes of modules and to avoid confusion.

2. If R is left Artinian and ${}_RC'$ is a *pure-injective* pre-partial 1-cotilting R-module, then C' admits a cotilt-complement.

Self-Cotilting Modules

Definition 268. (Compare [Wis:2002], [CDT:1997]) We call ${}_RC$:

1. **self-1-cotilting**, iff ${}_RC$ is Cogen$({}_RC)$-injective and Cogen$({}_RC)$ = Copres$({}_RC)$.
2. **f-cotilting,** iff C is cogen$({}_RC)$-injective and cogen$({}_RC)$ = copres$({}_RC)$.

269. Let C be an R-module, $\overline{R} = R/\mathrm{ann}_R(C)$ and denote with $\mathrm{Ext}^n_{\pi[{}_RC]}(\bullet,\bullet) := \mathrm{Ext}^i_{\overline{R}}(\bullet,\bullet)$ the nth-Ext-bifunctor associated to $\pi[{}_RC] = {}_{\overline{R}}\mathbb{M}$. We set

$$^{\perp_n}_{\pi[{}_RC]}C := \{N \in \pi[{}_RC] \mid \mathrm{Ext}^n_{\pi[{}_RC]}(N,C) = 0\} = \ ^{\perp_n}({}_{\overline{R}}C).$$

For more details, see [Wis:2000].

Proposition 270. ([Wis:2002, 3.6.]) *For an R-module C and $\overline{R} = R/\mathrm{ann}_R(C)$ the following are equivalent:*

1. *C is self-1-cotilting;*
2. Copres$({}_RC)$ = Cogen$({}_RC)$ *and C is w-$\prod$-quasi-injective;*
3. *${}_{\overline{R}}C$ is 1-cotilting (i.e. C is cotilting in ${}_{\overline{R}}\mathbb{M} = \pi[{}_RC]$);*
4. Cogen$({}_RC) = {}^{\perp_1}_{\pi[{}_RC]}C$;
5. *The following conditions are satisfied:*
   (a) $\mathrm{Ext}^2_{\pi[{}_RC]}(N,C) = 0$ *for each $N \in \pi[{}_RC]$;*
   (b) $\mathrm{Ext}^1_{\pi[{}_RC]}(C^\Lambda, C) = 0$ *for any index set Λ;*
   (c) $\mathrm{Ker}(\mathrm{Hom}_{\overline{R}}(-,C)) \cap {}^{\perp_1}_{\pi[{}_RC]}C = 0$.
6. *The following conditions are satisfied:*
   (a) $\mathrm{Ext}^2_{\pi[{}_RC]}(N,C) = 0$ *for each $N \in \pi[{}_RC]$;*
   (b) $\mathrm{Ext}^1_{\pi[{}_RC]}(C^\Lambda, C) = 0$ *for any index set Λ;*
   (c) *there exists an injective cogenerator $\mathbf{U} \in \pi[{}_RC]$ and $C_0, C_1 \in$ Prod$({}_RC)$ fitting in a short exact sequence of left R-modules*

$$0 \to C_1 \to C_0 \to \mathbf{U} \to 0.$$

Remark 271. By $1 \Leftrightarrow 2$ in the previous theorem, a *self-1-cotilting R-module* is (roughly speaking) an R-module ${}_RC$ that is 1-cotilting in $\pi[{}_RC] = \overline{{}_R\mathbb{M}}$.

Corollary 272. ([Wis:2002, Corollary 3.8.]) *For a cotilting R-module C we have*

$${}_RC \text{ is injective } \Leftrightarrow {}_RC \text{ is a cogenerator.}$$

Theorem 273. ([ATT:2001], [Wis:2002, 3.6., 3.7.], [CF:2004, Propositions 5.2.5, 5.2.6.]) *The following are equivalent for an R-module C :*

1. *C is 1-cotilting;*
2. $\mathrm{Cogen}({}_RC) = {}^{\perp_1}C$.
3. *The following conditions are satisfied:*
   (a) $\mathrm{inj.dim.}(C) \leq 1$;
   (b) $\mathrm{Ext}^1_R(C^\Lambda, C) = 0$ *for every set* Λ;
   (c) *the exist an injective cogenerator ${}_R\mathbf{I}$ and $C_0, C_1 \in \mathrm{Prod}(C)$ fitting in a short exact sequence of R-modules:*
   $$0 \to C_1 \to C_0 \to \mathbf{I} \to 0.$$
4. *The following conditions are satisfied:*
   (a) $\mathrm{inj.dim.}(C) \leq 1$;
   (b) $\mathrm{Ext}^1_R(C^\Lambda, C) = 0$ *for every set* Λ;
   (c) $\mathrm{Ker}(\mathrm{Hom}_R(-, C)) \cap {}^{\perp_1}C = 0$.
5. *C is self-1-cotilting and $\pi[{}_RC] = {}_R\mathbb{M}$;*
6. *C is self-1-cotilting and ${}_RC$ is faithful.*

Proposition 274. ([CTT:1997, Proposition 2.4.]) *Let C be an R-module.*

1. *Assume ${}_RC$ to be injective. Then ${}_RC$ is 1-cotilting if and only if ${}_RC$ is a cogenerator.*
2. *Let C be 1-cotilting. Then $\mathrm{E}(C) \oplus \mathrm{E}(C)/C$ is an injective cogenerator.*

Proposition 275. ([CTT:1997, Lemma 2.6.]) *Let C be an R-module and assume* $\mathrm{Cogen}({}_RC) \subseteq {}^{\perp_1}C$ *and* ${}^{\perp_1}C$ *is a torsion-free class. Then*

1. $\mathrm{Cogen}({}_RC)$ *is a torsion-free class.*

2. *C is a partial 1-cotilting R-module.*

Lemma 276. ([CF:2004, Proposition 5.2.1.]) *Let C be an R-module. Then* $\mathrm{inj.dim.}({}_RC) \leq 1$ *if and only if* ${}^{\perp_1}C$ *is closed under submodules.*

Definition 277. ([Ang:2000]) A left R-module ${}_RC$ with $S := \mathrm{End}({}_RC)^{op}$ is called **finitely cotilting**, iff

1. $\mathrm{inj.dim.}({}_RC) \leq 1$;

2. $\mathrm{Ext}^1_R(C, C) = 0$;

3. $\mathrm{Ker}(\mathrm{Hom}_R(-, C)) \cap \mathrm{Ker}(\mathrm{Ext}^1_R(-, C)) = 0$;

4. ${}_RC$ is finitely generated and $\mathrm{Hom}_R({}^{f.g.}_{R}\mathbb{M}, C) \subseteq \mathbb{M}^{f.g.}_S$.

Theorem 278. ([Ang:2000, Proposition 2.2.]) *Let C be an R-module.*

1. *Assume ${}_RC$ to be finitely generated and product complete. Then ${}_RC$ is finitely cotilting if and only if ${}_RC$ is 1-cotilting.*

2. *Assume ${}_RC$ to be of finite length (equivalently, ${}_RC$ is Noetherian and Artinian). Then ${}_RC$ is finitely cotilting if and only if ${}_RC$ is 1-cotilting such that* $\mathrm{Hom}_R(-, C)({}^{f.g.}_{R}\mathbb{M}) \subseteq \mathbb{M}^{f.g.}_S$.

4.4 Cotorsion Pairs

Cotorsion pairs were introduced by L. Salce [Sal:1979] in the context of Abelian groups under the name "*cotorsion theories*" and can be considered as dual to the classical "*torsion pairs*" (or "*torsion theories*"). The interest in cotorsion pairs revived recently as they were used in proving the "flat cover conjecture" by L. Bican et. al. [BEE:2001]. Cotorsion pairs proved also to be an important tool in studying tilting (cotilting) classes and approximations of modules (e.g. [GT:2006]).

Definition 279. A pair $\mathfrak{C} := (\mathcal{A}, \mathcal{B})$ of classes of R-modules is said to be a **cotorsion pair,** iff $\mathcal{A} := {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. We call $\mathcal{A} \cap \mathcal{B}$ the **kernel** of $\mathfrak{C}$.

Example 280. Every non-empty class $\mathcal{U} \neq \varnothing$ of R-modules **generates** a cotorsion pair $({}^{\perp_1}\mathcal{U}, ({}^{\perp_1}\mathcal{U})^{\perp_1})$ and **cogenerates** a cotorsion pair $({}^{\perp_1}(\mathcal{U}^{\perp_1}), \mathcal{U}^{\perp_1})$.

Definition 281. A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is said to be
hereditary, iff $\mathcal{A} := {}^{\perp_\infty}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_\infty}$;
complete, iff for every R-module Z there exists $A \in \mathcal{A}$ and $B \in \mathcal{B}$ fitting in a short exact sequence of R-modules

$$0 \to B \to A \to Z \to 0.$$

Definition 282. A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is said to be n**-tilting** (n-**cotilting**), iff there exists an n-tilting R-module ${}_RT$ (an n-cotilting R-module ${}_RC$), such that $\mathcal{B} = T^{\perp_\infty}$ ($\mathcal{A} = {}^{\perp_\infty}C$).

283. The class of all cotorsion pairs over R is *partially ordered* by inclusion in the first component (i.e. $(\mathcal{A}, \mathcal{B}) \leq (\mathcal{A}', \mathcal{B}')$, iff $\mathcal{A} \subseteq \mathcal{A}'$). The $\leq$*-least* cotorsion pair is $({}_R\mathcal{P}_0,\ {}_R\mathbb{M})$ and the $\leq$*-greatest* is $({}_R\mathbb{M},\ {}_R\mathcal{INJ})$.

Theorem 284. ([ST, Theorem 2]) *Let* $0 \leq n < \infty$ *and* $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ *be a cotorsion pair. Then* $\mathfrak{C}$ *is* n*-tilting if and only if*

1. $\mathfrak{C}$ *is hereditary;*
2. $\mathcal{A} \subseteq {}_R\mathcal{P}_n$;
3. $\mathcal{B}$ *is closed under arbitrary direct sums.*

Theorem 285. ([ST:2006, Theorem 2.4.]) *Let* $0 \leq n < \infty$ *and* $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ *be a cotorsion pair. Then* $\mathfrak{C}$ *is* n*-cotilting if and only if*

1. $\mathfrak{C}$ *is hereditary;*
2. $\mathcal{B} \subseteq {}_R\mathcal{I}_n$*;*
3. $\mathcal{A}$ *is closed under arbitrary direct products.*

Theorem 286. ([AC:2001], [Trl:2007, Lemam 2.7.]) *Let* $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ *be an* n*-tilting cotorsion pair and* T *the corresponding* n*-tilting module (for which* $\mathcal{B} = T^{\perp_\infty}$*). Then*

1. $\mathfrak{C}$ *is hereditary, complete and* $\mathfrak{C} \leq ({}_R\mathcal{P}_n, ({}_R\mathcal{P}_n)^{\perp_\infty})$ *(i.e.* $\mathcal{A} \subseteq {}_R\mathcal{P}_n$*);*
2. $\mathrm{Ker}(\mathfrak{C}) := \mathcal{A} \cap \mathcal{B} = \mathrm{Add}({}_RT)$*;*
3. $\mathcal{A}$ *coincides with the class of all* R*-modules* A *possessing an* n*-coresolution*
$$0 \to A \to T_0 \to ... \to T_n \to 0$$
with $T_i \in \mathrm{Add}(T)$ *for all* $i \leq n$*;*
4. $\mathcal{B}$ *coincides with the class of all* R*-modules* B *possessing an infinite resolution*
$$... \to T_{i+1} \to T_i \to ... \to T_0 \to B \to 0$$
with $T_i \in \mathrm{Add}(T)$ *for all* $0 \leq i < \infty$*. Hence,* $\mathcal{B}$ *is closed under arbitrary direct sums.*

Theorem 287. ([AC:2001], [Trl:2007, Lemam 3.9.]) *Let* $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ *be an* n*-cotilting cotorsion pair and* C *the corresponding* n*-cotilting module (for which* $\mathcal{A} = {}^{\perp_\infty}C$*). Then*

1. $\mathfrak{C}$ *is hereditary, complete and* $({}^{\perp_\infty}({}_R\mathcal{I}_n), {}_R\mathcal{I}_n) \leq \mathfrak{C}$ *(i.e.* ${}^{\perp_\infty}({}_R\mathcal{I}_n) \subseteq \mathcal{A}$*);*
2. $\mathrm{Ker}(\mathfrak{C}) := \mathcal{A} \cap \mathcal{B} = \mathrm{Prod}({}_RC)$*;*
3. $\mathcal{B}$ *coincides with the class of all* R*-modules* B *possessing an* n*-resolution*
$$0 \to C_n \to ... \to C_0 \to B \to 0$$
with $C_i \in \mathrm{Prod}(C)$ *for all* $0 \leq i \leq n$*;*

4. *$\mathcal{A}$ coincides with the class of all R-modules A possessing an infinite coresolution*

$$0 \to A \to C_0 \to ... \to C_i \to C_{i+1} \to ...$$

where $C_i \in \mathrm{Prod}(C)$ for all $0 \leq i < \infty$. Hence, $\mathcal{A}$ is closed under arbitrary direct products.

Tilting and cotilting classes are characterized as follows

Theorem 288. (*Compare* [AC:2001, 4.2. & 4.3.]) *Let $n \in \mathbb{N}$.*

1. *A class of modules $\mathcal{T}$ is n-tilting if and only if*

   (a) *$({}^{\perp_\infty}\mathcal{T}, \mathcal{T})$ is a complete cotorsion pair;*

   (b) *$\mathcal{T}$ is closed under arbitrary direct sums; and whenever there exists a short exact sequence of R-modules*

   $$0 \to X \to Y \to Z \to 0$$

   with $X, Y \in \mathcal{T}$, then $Z \in \mathcal{T}$;

   (c) *${}^{\perp_\infty}\mathcal{T} \subseteq {}_R\mathcal{P}_n$.*

2. *A class $\mathcal{C}$ of R-modules is n-cotilting if and only if*

   (a) *$(\mathcal{C}, \mathcal{C}^{\perp_\infty})$ is a complete cotorsion pair;*

   (b) *$\mathcal{C}$ is closed under arbitrary direct products; and whenever there exists a short exact sequence*

   $$0 \to X \to Y \to Z \to 0$$

   with $Y, Z \in \mathcal{C}$, then $X \in \mathcal{C}$;

   (c) *$\mathcal{C}^{\perp_\infty} \subseteq ({}_R\mathcal{I}_n)$.*

4.5 Classes of Finite (Cofinite) Type

Definition 289. ([Trl:2007]) A class $\mathcal{U}$ of left R-modules is said to be
of finite type (**of countable type**), iff there exists $n < \infty$ and a subset $\mathcal{S} \subseteq {}_R\mathcal{P}_n^{<\omega}$ ($\mathcal{S} \subseteq {}_R\mathcal{P}_n^{\leq\omega}$), such that $\mathcal{U} = \mathcal{S}^{\perp_\infty}$;
of cofinite type, iff there exists $n < \infty$ and a subset of right R-modules $\mathcal{S} \subseteq (\mathcal{P}_n^{<\omega})_R$, such that $\mathcal{U} = \mathcal{S}^{\intercal_\infty}$;
definable, iff $\mathcal{U}$ is closed under arbitrary direct products, direct limits and pure submodules.

Definition 290. A subclass $\mathcal{R} \subseteq {}_R^{f.g.}\mathbb{M}$ is said to be **resolving**, iff the following conditions are satisfied:

1. $\mathcal{R}$ contains all finitely generated projective left R-modules;
2. $\mathcal{R}$ is closed under direct summands and extensions;
3. whenever there exists a short exact sequence of left R-modules
$$0 \to X \to Y \to Z \to 0$$
with $Y, Z \in \mathcal{R}$, then $X \in \mathcal{R}$.

Remark 291. ([Trl:2007, 4.11.]) A subclass $\mathcal{R} \subseteq {}_R\mathcal{P}_1^{<\omega}$ is resolving if and only if $\mathcal{R}$ is closed under extensions and summands, and $R \in \mathcal{R}$.

Definition 292. An R-module U is said to be **definable** (respectively **of finite type**, **of countable type**), iff $U^{\perp_\infty}$ is definable (respectively of finite type, of countable type). We say U is **of cofinite type**, iff ${}^{\perp_\infty}U$ is of cofinite type.

Theorem 293. ([AHT:2006]) *Let $\mathcal{U}$ be a class of R-modules that is of finite type. Then $\mathcal{U}$ is tilting (and definable).*

All tilting modules known so far were noticed to be of finite type. Recently S. Bazzoni and J. Šťovíček proved

Theorem 294. ([BS, Theorem 4.2.]) *Let R be an arbitrary ring, $n \geq 0$ and T an n-tilting R-module. Then T is of finite type.*

Corollary 295. *Let $\mathcal{U}$ be a class of R-modules. Then*

$$\mathcal{U} \text{ is of finite type } \Leftrightarrow \mathcal{U} \text{ is tilting (and definable)}.$$

Remark 296. The expected dual of Theorem 294 does not hold: S. Bazzoni gave in [Baz, Proposition 4.5.] a class of cotilting modules over *non strongly discrete valuation domains* that are not of cofinite type (see Proposition 382).

The following result shows that there is a correspondence between the n-tilting left R-modules (which are now known to be of finite type by Theorem 294) and the right n-cotilting R-modules of cofinite type:

Theorem 297. ([AHT:2006, Proposition 2.3.]) *Let $n < \infty$ and denote with $(\mathcal{F}_0)_R$ the class of flat right R-modules.*

1. *If ${}_RT$ is an n-tilting left R-module, then ${}^dT_R$ is an n-cotilting right R-module of cofinite type and, moreover, ${}^{\perp_\infty}({}^dT_R) = (\mathcal{F}_0)_R = {}^{\intercal_\infty}T$.*

2. *${}_RM \in T^{\perp_\infty}$ if and only if ${}^dM_R \in {}^{\perp_\infty}({}^dT_R)$ for every left R-module M.*

Theorem 298. ([Trl:2007, Theorem 4.14.], [GT:2006, Theorem 8.2.8.]) *Let R be right Noetherian. Assume also that ${}_R\mathcal{F}_1 = {}_R\mathcal{P}_1$ (e.g. R is left perfect, or left hereditary, or 1-Iwangsawa-Gorenstein). Then every 1-cotilting left R-module is of cofinite type.*

The following results gives examples, where the assumptions (whence the results) of Theorem 298 hold:

Corollary 299. ([Trl:2007], [GT:2006]) *All 1-cotilting left R-modules are equivalent to duals of 1-tilting right R-modules, if (for example):*

1. *R is right Artinian;*

2. *R is right Noetherian and left hereditary;*

3. *R is 1-Gorenstein;*

Theorem 300. ([AHT:2006, Theorem 2.2.]) *Let $n < \infty$.*

1. *There are bijective correspondences:*

$$\{T^{\perp_\infty} \mid {}_RT \text{ is } n\text{-tilting}\} \quad \overset{^{\perp_\infty}(-)\cap R\mathrm{mod}}{\underset{(-)^{\perp_\infty}}{\longleftrightarrow}} \quad \{\mathcal{R} \subseteq {}_R\mathcal{P}_n^{<\omega} \text{ resolving}\};$$

$$\{^{\perp_\infty}C \mid C_R \text{ is } n\text{-cotilting of c.t.}^3\} \quad \overset{(-)^{\intercal_\infty}\cap R\mathrm{mod}}{\underset{^{\intercal_\infty}(-)}{\longleftrightarrow}} \quad \{\mathcal{R} \subseteq {}_R\mathcal{P}_n^{<\omega} \text{ resolving}\}.$$

2. *Let $\varnothing \neq \mathcal{S} \subseteq {}_R\mathcal{P}_n^{<\omega}$ be a non-empty subclass. Then $\mathcal{S}^{\perp_\infty}$ is an n-tilting class (of finite type). Moreover, ${}^{\intercal_\infty}(\mathcal{S}^{\perp_\infty}) \subseteq \mathbb{M}_R$ is the corresponding n-cotilting class of cofinite type, i.e.*

$$^{\intercal_\infty}(\mathcal{S}^{\perp_\infty}) = {}^{\intercal_\infty}({}^{\perp_\infty}(\mathcal{S}^{\perp_\infty}) \cap R\mathrm{mod}).$$

Theorem 301. ([GT:2006, Theorem 8.1.2.]) *Let $n \geq 0$ and ${}_RT$ be an n-tilting left R-module. Then T_R^d is an n-cotilting right R-module with n-cotilting class given by*

$$\mathcal{C} := {}^{\intercal_\infty}T = \{N_R \mid \mathrm{Tor}_i^R(N,T) = 0 \text{ for all } i \leq n\}.$$

Moreover, if $\mathcal{X} \subseteq {}_R\mathcal{P}_n^{<\omega}$ is such that $T^{\perp_\infty} = \mathcal{X}^{\perp_\infty}$, then $\mathcal{C} = {}^{\intercal_\infty}\mathcal{X}$.

[3] i.e. of *cofinite type*

4.6 Tilting Modules over Gorenstein Rings

In this section we consider the tilting modules over Iwangsawa-Gorenstein rings.

Theorem 302. ([AHT:2006, Proposition 3.6.]) *The following are equivalent for a Noetherian ring* R :

1. *R is Gorenstein (i.e. R is Noetherian,* $\text{inj.dim.}({}_RR) < \infty$ *and* $\text{inj.dim.}(R_R) < \infty$*);*
2. *The Gorenstein injective modules in ${}_R\mathbb{M}$ and in $\mathbb{M}_R$ form a tilting class;*
3. *The Gorenstein injective modules in $\mathbb{M}_R$ form a tilting class and the class of Gorenstein flat modules in $\mathbb{M}_R$ form a cotilting class;*
4. *The Gorenstein injective modules in ${}_R\mathbb{M}$ form a tilting class and the class of Gorenstein flat modules in ${}_R\mathbb{M}$ form a cotilting class.*

Theorem 303. ([AHT:2006, Proposition 3.6.]) *Let R be a Gorenstein ring. The following are equivalent:*

1. *${}_RR$ (or R_R) is Artinian;*
2. *$(\mathcal{GP}, \mathcal{P})$ is a cotilting cotorsion pair;*
3. *${}_RR$ is a cotilting module and ${}^{\perp_\infty}({}_RR) = \mathcal{GP}(R)$.*

Corollary 304. ([AHT:2006, Corollary 3.8.]) *An Artin algebra R is Gorenstein if and only if there is a cotilting module $C \in R\text{mod}$, such that $\mathcal{GP}(R) = {}^{\perp_\infty}C$.*

Lemma 305. ([AHT:2006, Lemma 3.9.]) *Let R be an Artin algebra with a duality $D : R\text{mod} \to \text{mod}R$. The following are equivalent:*

1. $\text{inj.dim.}({}_RR) < \infty$;
2. *$D({}_RR)^{\perp_\infty}$ is a tilting class in $\mathbb{M}_R$;*
3. *${}^{\perp_\infty}({}_RR)$ is a cotilting class in ${}_R\mathbb{M}$.*

Theorem 306. ([AHT:2006, Proposition 3.10.]) *Let R be an Artin algebra such that* inj.dim.$({}_RR) < \infty$ *with a duality $D : R\text{mod} \to \text{mod}R$. Then the following are equivalent:*

1. *R is Gorenstein;*
2. *$D({}_RR)^{\perp_\infty} = (\mathcal{P}_R^{<\infty})^{\perp_\infty}$;*
3. *The pre-partial tilting module $D({}_RR)$ admits a tilt-complement of finite injective dimension;*
4. *The pre-partial cotilting module ${}_RR$ admits a cotilt-complement of finite projective dimension;*
5. *Every (finitely presented) module in $D({}_RR)^{\perp_\infty}$ is generated by $D({}_RR)$;*
6. *Every (finitely presented) module in ${}^{\perp_\infty}({}_RR)$ is cogenerated by ${}_RR$;*
7. *$\mathcal{GP}(R) = {}^{\perp_\infty}({}_RR)$.*

Theorem 307. ([Trl:2007, Example 2.6.]) *Let R be an n-Gorenstein ring and consider a minimal injective coresolution of ${}_RR$*

$$0 \to {}_RR \to I_0 \to I_1 \to \ldots \to I_n \to 0.$$

Then $T_{IG} := \bigoplus_{k=0}^{n} I_k$ is an n-tilting left R-module and

$$T_{IG}^{\perp_\infty} = \{{}_RM \mid M \text{ is Gorenstein injective}\}.$$

Theorem 308. ([Trl:2007, Theorem 4.14.]) *Let R be a Gorenstein ring of injective dimension ≤ 1. Then all cotilting R-modules are of cofinite type.*

4.7 Special Constructions

Theorem 309. ([Fuj:1992]) *Let R be a basic semiperfect Noetherian ring and consider the quotient ring $\overline{R} := R/\mathrm{Jac}(R)$. Let e be a primitive idempotent of R such that*

$$\mathrm{Ext}^1_R(\overline{e}\overline{R}, \overline{e}\overline{R}) = 0 \text{ and } \mathrm{Ext}^i_R(\overline{e}\overline{R}, R) = 0 \text{ for } 0 \leq i < l.$$

Then for each i, $1 \leq i \leq l$, there exists a left R-module Y_i with local endomorphism ring such that $R(1-e) \oplus Y_i$ is an i-tilting left R-module.

Lemma 310. *If R is a von Neumann regular ring, then every strongly finitely presented left (right) R-module is projective.*

Proposition 311. ([AHT:2006, Example 2.4. (ii)]) *Let R be von Neumann regular.*

1. *A left R-module ${}_RT$ is (classical) 1-tilting if and only if ${}_RT$ is a projective generator (a progenerator). Moreover, ${}_R\mathbb{M}$ is the only tilting class in ${}_R\mathbb{M}$.*

2. *A left R-module ${}_RC$ is 1-cotilting of cofinite type if and only if ${}_RC$ is an injective cogenerator. Moreover, ${}_R\mathbb{M}$ is the only cotilting class in ${}_R\mathbb{M}$ of cofinite type.*

Theorem 312. ([CF:2004, Theorems 2.4.6, 2.4.7.], [CDT:1997, Theorem 1.5.]) *For an* (R,S)*-bimodule* ${}_RP_S$ *we have:*

1. *The following are equivalent:*

   (a) ${}_RP$ *is a progenerator;*

   (b) ${}_R\mathbb{M} \underset{P\otimes_S -}{\overset{\mathrm{Hom}_R(P,-)}{\approx}} {}_S\mathbb{M}$ *and* $S := \mathrm{End}({}_RP)^{op}$;

   (c) $\mathbb{M}_S \underset{-\otimes_R P}{\overset{\mathrm{Hom}_S(P,-)}{\approx}} \mathbb{M}_R$ *and* $R \simeq \mathrm{End}(P_S)$;

   (d) P_S *is a progenerator.*

2. *The following are equivalent:*

   (a) ${}_RP$ *is a quasi-progenerator* (*i.e.* P *is progenerator in* $\sigma[{}_RP]$);

   (b) $\sigma[{}_RP] \underset{P\otimes_S -}{\overset{\mathrm{Hom}_R(P,-)}{\approx}} {}_S\mathbb{M}$ *and* $S := \mathrm{End}({}_RP)^{op}$ (*and* $\mathrm{Gen}({}_RP) = \sigma[{}_RP]$);

   (c) $\sigma[P_S] \underset{-\otimes_R P}{\overset{\mathrm{Hom}_S(P,-)}{\approx}} \mathbb{M}_R$ *and* $R := \mathrm{End}(P_S)$ (*and* $\mathrm{Gen}(P_S) = \sigma[P_S]$);

   (d) P_S *is a quasi-progenerator* (*i.e.* P *is progenerator in* $\sigma[P_S]$).

3. *The following are equivalent for an* (R,S)*-bimodule* ${}_RP_S$:

   (a) ${}_RT$ *is classical* 1*-tilting and* $S = \mathrm{End}({}_RT)^{op}$;

   (b) *There exists a torsion class* ${}_R\mathcal{X} \subseteq {}_R\mathbb{M}$ *and a torsion-free class* ${}_S\mathcal{Y} \subseteq {}_S\mathbb{M}$, *such that*

   $${}_R\mathcal{INJ} \subseteq {}_R\mathcal{X} \underset{T\otimes_S -}{\overset{\mathrm{Hom}_R(T,-)}{\approx}} {}_S\mathcal{Y} \supseteq {}_S\mathcal{P}_0;$$

   (c) *There exists a torsion class* $\mathcal{X}_S \subseteq \mathbb{M}_S$ *and a torsion-free class* $\mathcal{Y}_R \subseteq \mathbb{M}_R$, *such that*

   $$(\mathcal{INJ})_S \subseteq \mathcal{X}_S \underset{-\otimes_R T}{\overset{\mathrm{Hom}_S(T,-)}{\approx}} \mathcal{Y}_R \supseteq (\mathcal{P}_0)_R;$$

   (d) T_S *is classical* 1*-tilting and* $R \simeq \mathrm{End}(T_S)$.

4.8 Tilting (Cotilting) Theorems

In this section we show how tilting modules generalize classical progenerators and include some of the main *tilting theorems*.

To the end of this section, we keep the following setting: ${}_RT$ is an R-module, $S := \mathrm{End}({}_RT)^{op}$ and we consider the covariant functors

$$\mathrm{Hom}_R(T,-): {}_R\mathbb{M} \to {}_S\mathbb{M} \text{ and } T \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}.$$

In order that tilting (cotilting) modules induce equivalences (dualities) between full subcategories of module categories, suitable *finiteness conditions* should be assumed.

Tilting Theorems

The following is called the **Brenner-Butler Theorem** or the (**Fundamental**) **Tilting Theorem** :

Theorem 313. ([BB:1980], [HR:1982]) *If ${}_RT$ is a classical 1-tilting left R-module, then*

1. $({}_RT^{\perp_1}, \mathrm{Ker}(\mathrm{Hom}_R(T,-)))$ *and* $(\mathrm{Ker}(T \otimes_S -), {}^{\perp_1}T_S)$ *are torsion pairs.*
2. T *induces a pair of category equivalences*

$$\mathrm{Ker}(\mathrm{Ext}^1_R(T,-)) \underset{T\otimes_S -}{\overset{\mathrm{Hom}_R(T,-)}{\approx}} \mathrm{Ker}(\mathrm{Tor}^S_1(T,-));$$

$$\mathrm{Ker}(\mathrm{Hom}_R(T,-)) \underset{\mathrm{Tor}^S_1(-,T)}{\overset{\mathrm{Ext}^1_R(T,-)}{\approx}} \mathrm{Ker}(T \otimes_S -).$$

The following is a generalization of the *fundamental tilting theorem* to the case $n > 1$:

Theorem 314. ([Miy:1986, Theorem 1.16.]) *If $_RT$ is classical n-tilting, then for all $0 \leq i \leq n$ there are equivalences of categories*

$$\bigcap_{j \leq n, j \neq i} \mathrm{Ker}(\mathrm{Ext}_R^j(T,-)) \underset{\mathrm{Tor}_i^S(-,T)}{\overset{\mathrm{Ext}_R^i(T,-)}{\approx}} \bigcap_{j \leq n, j \neq i} \mathrm{Ker}(\mathrm{Tor}_j^S(-,T)).$$

Theorem 315. ([Wis:1998, 5.5.]) *For an R-module T with $S := \mathrm{End}(_RT)^{op}$, we have*

$$_RT \text{ is self-small } 1\text{-self-tilting}^4 \Leftrightarrow \mathrm{Gen}(_RT) \underset{T \otimes_S -}{\overset{\mathrm{Hom}_R(T,-)}{\approx}} \mathrm{Cogen}(^d_ST).$$

Cotilting Theorems

In what follows, we include some of the main *cotilting theorems* that can be considered (in some sense) as dual to the *fundamental tilting theorem* 313.

Theorem 316. ([Wis:2002, 4.10.]) *For an R-module C with $S := \mathrm{End}({}_RC)^{op}$ we have*

$$ {}_RC \textit{ is f-cotilting } \Leftrightarrow \mathrm{cogen}({}_RC) \underset{\mathrm{Hom}_S(-,C)}{\overset{\mathrm{Hom}_R(-,C)}{\circlearrowleft}} \mathbb{M}_S^{f.g.} \cap \mathrm{Cogen}(C_S). $$

To the end of this section, we keep the following setting: ${}_RC_S$ is a bimodule and we consider the functors

$$
\begin{array}{llll}
\Delta_C^R := \mathrm{Hom}_R(-,C) & : {}_R\mathbb{M} \to \mathbb{M}_S & ; & \Delta_C^S := \mathrm{Hom}_S(-,C) \quad : \mathbb{M}_S \to {}_R\mathbb{M}; \\
\Gamma_C^R := \mathrm{Ext}_R^1(-,C) & : {}_R\mathbb{M} \to \mathbb{M}_S & ; & \Gamma_C^S := \mathrm{Ext}_S^1(-,C) \quad : \mathbb{M}_S \to {}_R\mathbb{M}.
\end{array}
$$

Moreover, we set

$$
\begin{array}{lllllll}
{}_R\mathcal{X} & := & \mathrm{Ker}(\mathrm{Hom}_R(-,C)) \cap {}_R^{f.g.}\mathbb{M}; & \quad & {}_R\mathcal{Y} & := & \mathrm{Cogen}({}_RC) \cap {}_R^{f.g.}\mathbb{M}; \\
\mathcal{X}_S & := & \mathrm{Ker}(\mathrm{Hom}_S(-,C)) \cap \mathbb{M}_S^{f.g.}; & & \mathcal{Y}_S & := & \mathrm{Cogen}(C_S) \cap \mathbb{M}_S^{f.g.}.
\end{array}
$$

Definition 317. ([Ang:2000]) We call an R-module C a **Colby-module**, iff

1. ${}_R^{f.p.}\mathbb{M} \subseteq \mathrm{Ker}(\mathrm{Ext}_R^2(-,C))$;
2. $\mathrm{Ext}_R^1(C,C) = 0$;
3. $\mathrm{Ker}(\mathrm{Hom}_R(-,C)) \cap \mathrm{Ker}(\mathrm{Ext}_R^1(-,C)) \cap {}_R^{f.p.}\mathbb{M} = 0$;
4. ${}_RC$ is finitely generated and the functor $\mathrm{Hom}_R(-,C) : {}_R\mathbb{M} \to \mathbb{M}_{\mathrm{End}({}_RC)^{op}}$ restricts to a functor

$$ \mathrm{Hom}_R(-,C) : {}_R^{f.g.}\mathbb{M} \to \mathbb{M}^{f.g.}_{\mathrm{End}({}_RC)^{op}}. $$

Definition 318. ([Ang:2000]) We call an (R,S)-bimodule ${}_RC_S$ a **Colby-bimodule**, iff ${}_RC_S$ is faithfully balanced and ${}_RC, C_S$ are Colby-modules.

Remark 319. Let R be left Noetherian and S be right Noetherian. The Colby-bimodules defined above coincide with the cotilting bimodules in the sense of [Col:1989].

Proposition 320. ([Ang:2000, Proposition 4.3.]) *The following are equivalent for an (R,S)-bimodule ${}_RC_S$:*

1. *${}_RC_S$ is a Colby-(R,S)-bimodule;*
2. *R is left coherent, ${}_RC$ is finitely presented, ${}_RC$ is a Colby-module and $S \simeq \mathrm{End}({}_RC)^{op}$.*

In view of Remark 319, the following cotilting theorem generalizes that of Colby [Col:1989]:

Theorem 321. ([Ang:2000, Theorem 4.4.]) *Let ${}_RC_S$ be a Colby-bimodule.*

1. *$({}_R\mathcal{X}, {}_R\mathcal{Y})$ and $(\mathcal{X}_S, \mathcal{Y}_S)$ are torsion theories.*
2. *${}_R\mathcal{Y} \subseteq {}^{f.p.}_{R}\mathbb{M}$, $\mathcal{Y}_S \subseteq \mathbb{M}^{f.p.}_S$ and we have dualities*
$$ {}_R\mathcal{Y} \overset{\mathrm{Hom}_R(-,C)}{\underset{\mathrm{Hom}_S(-,C)}{\circlearrowleft}} \mathcal{Y}_S \text{ and } {}_R\mathcal{X} \cap {}^{f.p.}_{R}\mathbb{M} \overset{\mathrm{Ext}^1_R(-,C)}{\underset{\mathrm{Ext}^1_S(-,C)}{\circlearrowleft}} \mathcal{X}_S \cap \mathbb{M}^{f.p.}_S. $$
3. *We have*
$$ \begin{array}{rcccl} (\Gamma^S_C \circ \Delta^R_C)({}^{f.g.}_{R}\mathbb{M}) & = & 0 & = & (\Gamma^R_C \circ \Delta^S_C)({}^{f.g.}_{S}\mathbb{M}); \\ (\Delta^S_C \circ \Gamma^R_C)({}^{f.p.}_{R}\mathbb{M}) & = & 0 & = & (\Delta^R_C \circ \Gamma^S_C)({}^{f.p.}_{S}\mathbb{M}). \end{array} $$

Definition 322. Let ${}_RC_S$ be an (R,S)-bimodule. We say

a left R-module ${}_RM$ is C**-reflexive**, iff $M \simeq \mathrm{Hom}_{-S}(\mathrm{Hom}_{R-}(M,C),C)$ canonically;

a right S-module N_S is C**-reflexive**, iff $N \simeq \mathrm{Hom}_{R-}(\mathrm{Hom}_{-S}(N,C),C)$ canonically.

Definition 323. A bimodule ${}_RC_S$ is said to define a **finitistic generalized Morita duality**, iff

1. ${}_RR$ and S_S are C-reflexive;
2. ${}^R_C({}^{f.g.}_{R}\mathbb{M}) \subseteq \mathbb{M}^{f.g.}_R$ and $\Delta^S_C(\mathbb{M}^{f.g.}_R) \subseteq {}^{f.g.}_{R}\mathbb{M}$;
3. Every finitely generated submodule of a finitely generated C-reflexive module is C-reflexive;

4. Any extension of a finitely generated C-reflexive module by a finitely generated C-reflexive module is C-reflexive.

Theorem 324. ([Ang:2000, Theorem 4.6.]) *The following are equivalent for an (R, S)-bimodule ${}_RC_S$:*

1. *${}_RC_S$ is a Colby-bimodule;*

2. *${}_RC_S$ is faithfully balanced and satisfies the properties 1-3 in Theorem 321.*

3. *${}_RC_S$ defines a finitistic generalized Morita duality.*

Morita Rings

Definition 325. R is said to be a **left Morita ring** (a **right Morita ring**), iff R is left Artinian and the minimal injective cogenerator ${}_R\mathbf{I}$ is finitely generated (R is right Artinian and the minimal injective cogenerator $\mathbf{I}_R$ is finitely generated).

Theorem 326. ([Ang:2000, Theorem 3.3.]) *Let R, A be rings and consider an (R, A)-bimodule ${}_R\mathbf{I}_A$, which is faithfully balanced, an injective cogenerator on both sides, and induces a duality*

$$ {}_R^{f.g.}\mathbb{M} \overset{\mathrm{Hom}_R(-,\mathbf{I})}{\underset{\mathrm{Hom}_A(-,\mathbf{I})}{\circlearrowright}} \mathbb{M}_A^{f.g.}. $$

If R is a left Morita ring, then the following are equivalent for a left R-module C :

1. *There is a classical 1-tilting right S-module T_A such that ${}_RC \simeq D(T_A)$;*
2. *${}_RC$ is finitely generated and the following conditions are satisfied*
   (a) $\mathrm{inj.dim.}({}_RC) \leq 1$;
   (b) $\mathrm{Ext}^1_R(C, C) = 0$;
   (c) There exist $C_0, C_1 \in \mathrm{add}({}_RC)$ fitting in a short exact sequence of left R-modules
   $$0 \to C_1 \to C_2 \to W \to 0.$$
3. *${}_RC$ is a Colby-module;*
4. *${}_RC$ is finitely cotilting.*

Under these conditions, C_S is a finitely presented finitely cotilting S-module, where $S := \mathrm{End}(T_A)$.

Theorem 327. ([Ang:2000, Proposition 3.5.]) *Let R be a hereditary left Morita ring. The following are equivalent for a left R-module C with $S := \mathrm{End}({}_RC)^{op}$:*

1. *${}_RC$ is finitely cotilting;*
2. *${}_RC$ is 1-cotilting with ${}_RC$ and C_S finitely generated;*
3. *${}_RC$ is a classical 1-tilting module.*

Chapter 5

Tilting (Cotilting) Modules over Commutative Rings

In this chapter we restrict our attention to the structure of tilting (cotilting) modules over commutative rings. Our main references for the theory of modules over commutative rings, in addition to the classical books in Commutative Algebra, are [FS:2000] and [Mat:2004].

Throughout this chapter, R denotes a commutative ring with $1_R \neq 0_R$.

5.1 Finitely generated tilting modules over commutative rings

Over commutative rings, $*^1$-modules turn out to be quasi-progenerators and classical tilting modules are progenerators.

Lemma 328. ([CM:1993], [Trl1994, Theorem 1, Corollary 5]) *Let R be a commutative ring. An R-module T is a $*^1$-module if and only if ${}_RT$ is a quasi-progenerator.*

It was pointed out by Y. Miyashita in the introduction of his paper [Miy:1992] that it is a *"fact that usual tilting modules of finite projective dimension over a commutative ring are necessarily projective"*.

However, we could not find any explicit proof for this *fact* in the *published literature*!! I thank both Prof. J. Trlifaj and Prof. S. Bazzoni for providing me with *two* proofs of this **fact**[1]:

Proposition 329. *Let R be a commutative ring. If ${}_RT$ is a finitely generated tilting R-module, then ${}_RT$ is projective.*

Combining Propositions 312, 329 with Lemma 328, we get

Proposition 330. *Let R be a commutative ring. For an R-module T the following are equivalent:*

1. *${}_RT$ is classical n-tilting for some $n \in \mathbb{N}$;*
2. *${}_RT$ is classical 1-tilting;*
3. *${}_RT$ is a faithful $*^1$-module;*
4. *${}_RT$ is faithful and a quasi-progenerator;*
5. *${}_RT$ is faithful, finitely generated and projective;*
6. *${}_RT$ is a progenerator.*

Remark 331. The assumption that the ring R in Proposition 330 is commutative is essential. In fact, R. Colpi et. al. gave in [CDT:1997] examples of faithful $*^1$-modules (over non-commutative algebras) that are neither *quasi-progenerators* nor *classical* 1-*tilting.*

Proposition 332. ([AHT:2006, Example 2.4. (ii)]) *Let R be a (commutative) von Neumann regular ring.*

1. *An R-module ${}_RT$ is 1-tilting if and only if ${}_RT$ is a projective generator. Moreover, ${}_R\mathbb{M}$ is the only tilting class in ${}_R\mathbb{M}$.*
2. *An R-module ${}_RC$ is 1-cotilting of cofinite type if and only if ${}_RC$ is an injective cogenerator. Moreover, ${}_R\mathbb{M}$ is the only cotilting class in ${}_R\mathbb{M}$ of cofinite type.*

[1] in December 2006.

Corollary 333. ([Trl:2007], [GT:2006]) *Let R be a commutative ring. All* 1*-cotilting R-modules are equivalent to duals of* 1*-tilting R-modules, if (for example):*

1. *R is Artinian;*
2. *R is Noetherian and hereditary (e.g. R is a Dedekind integral domain);*
3. *R is* 1*-Iwangsawa-Gorenstein;*
4. *R is a strongly discrete valuation integral domain.*

5.2 Fuchs-Salce Divisible Modules

The following construction was introduced by L. Fuchs and L. Bazzoni [FS:2000] as a generalization of the divisible **Fuchs module** $\partial := \partial_{<R^\times>}$ introduced in [Fuc:1984] for an integral domain R.

334. Let R be an integral domain and $\mathbf{S} \subseteq R^\times$ an admissible multiplicatively closed set. The **Fuchs-Salce S-divisible module** $\partial_{<\mathbf{S}>}$ is the *free* R-module with *free basis* the set of all tuples

$$\{(\ ), (s_1, ..., s_k) \mid k \geq 1,\ s_i \in \mathbf{S} \text{ for } i = 1, ..., k\},$$

where $\varpi := (\ )$ denotes the empty tuple, and with defining relations

$$(s_1, ..., s_k)s_k := (s_1, ..., s_{k-1}),\ k \geq 1 \text{ and } (s)s := \varpi.$$

Proposition 335. *Let R be an integral domain, $\mathbf{S} \subseteq R^\times$ an admissible multiplicatively closed set and $\partial_{<\mathbf{S}>}$ the corresponding Fuchs-Salce $\mathbf{S}$-divisible module.*

1. proj.dim.$(\partial_{<\mathbf{S}>}) = 1$;

2. *the torsion submodule $\tau(\partial_{<\mathbf{S}>})$ is $\mathbf{S}$-torsion and fits in a short exact sequence of R-modules*

$$0 \to \tau(\partial_{<\mathbf{S}>}) \to \partial_{<\mathbf{S}>} \to R_{\mathbf{S}} \to 0; \tag{5.1}$$

3. *the sequence (5.1) splits if and only if* proj.dim.$(R_{\mathbf{S}}) \leq 1$;

4. $\mathrm{Gen}(\partial_{<\mathbf{S}>}) = \mathcal{DI}(\mathbf{S})$;

5. *The factor module $\partial_{<\mathbf{S}>}/R\varpi$ is isomorphic to a direct summand of $\partial_{<\mathbf{S}>}$;*

6. *An R-module M is $\mathbf{S}$-divisible if and only if $\mathrm{Ext}^1_R(\partial_{<\mathbf{S}>}, M) = 0$, i.e. $\partial^{\perp_1}_{<\mathbf{S}>} = \mathcal{DI}(\mathbf{S})$.*

7. *$\partial_{<\mathbf{S}>}$ is a 1-tilting R-module.*

Theorem 336. ([Fac:1988, Theorems 2.3., 3.6.]) *Let R be an integral domain, $\partial := \partial_{<R^{\times}>}$ be the Fuchs module over R, $E := \mathrm{End}(\partial_R)$, $Z(E)$ the center of E and $\partial^{\circ} := \mathrm{Ext}^1_R(\partial, R)$ Then*

$$Z(E) \simeq R \simeq \mathrm{End}(\partial^{\circ}_E).$$

Definition 337. Let R be an integral domain, $\partial := \partial_{<R^{\times}>}$ the Fuchs module, $E := \mathrm{End}(\partial_R)$ and $I := \{f \in E \mid f(\varpi) = 0\} \lhd E$.

1. Let M be a right E-module and consider the canonical morphism

$$\psi_M : M \otimes_E I \to M, \; m \otimes_E f \mapsto f(m).$$

   The right E-module M is called:

   (a) I**-torsion-free**, iff ψ_M is injective;

   (b) I**-divisible**, iff ψ_M is surjective;

2. a right E-module N is called

   (a) I**-reduced**, iff N is cogenerated by ∂^d_E;

   (b) I**-cotorsion**, iff $\mathrm{Hom}_E(M, N) = 0 = \mathrm{Ext}^1_E(M, N)$ for every I-divisible I-torsion-free right E-module M.

Notation. Let R be an integral domain, $\partial := \partial_{<R^{\times}>}$ the Fuchs module, $E := \mathrm{End}(\partial_R)$ and $I := \{f \in E \mid f(\varpi) = 0\} \lhd E$. Moreover, we set

$$\begin{array}{lllllll} \mathcal{DI}(I) & := & \{M_E \mid M \text{ is } I\text{-divisible}\}; & \mathcal{TF}(I) & := & \{M_E \mid M \text{ is } I\text{-torsion-free}\}. \\ \mathcal{R}(I) & := & \{N_E \mid N \text{ is } I\text{-reduced}\}; & \mathcal{C}(I) & := & \{N_E \mid N \text{ is } I\text{-cotorsion}\}; \end{array}$$

Theorem 338. ([Fac:1987, Page 2237], [Fac:1988]) *Let R be an integral domain. Consider the divisible Fuchs module $\partial := \partial_{<R^{\times}>}$, $E := \mathrm{End}({}_R\partial)$ and $I := \{f \in E \mid f(\varpi) = 0\}$.*

1. *The functors $\mathrm{Hom}_R(\partial, -)$ and $- \otimes_E \partial$ induce an equivalence of categories*
$${}_R\mathcal{DI} \approx \mathcal{R}(I) \cap \mathcal{C}(I).$$

2. *The functors $\mathrm{Ext}^1_R(\partial_R, -)$ and $\mathrm{Tor}^E_1(-, \partial_R)$ induce an equivalence of categories*
$${}_R\mathcal{R} \approx \mathcal{DI}(I) \cap \mathcal{C}(I).$$

5.3 Tilting (Cotilting) Modules & Localization

Tilting modules

Theorem 339. ([AHT:2006, Proposition 4.3.]) *Let R be a commutative ring, T an n-tilting R-module, and $\mathcal{T} := T^{\perp_\infty}$ be the corresponding n-tilting class.*

1. *Let $\mathbf{S} \subseteq R^{\mathrm{reg}}$ be an admissible multiplicatively closed set. Then $T_{\mathbf{S}}$ is an n-tilting $R_{\mathbf{S}}$-module and the corresponding n-tilting class of $R_{\mathbf{S}}$-modules is*

$$T_{\mathbf{S}}^{\perp_\infty} = \mathcal{T} \cap {}_{R_{\mathbf{S}}}\mathbb{M} = \mathcal{T}_{\mathbf{S}} := \{M_{\mathbf{S}} \simeq M \otimes_R R_{\mathbf{S}} \text{ for some } M \in \mathcal{T}\}.$$

2. *Let $N \in {}_R\mathbb{M}$. Then $N \in \mathcal{T}$ if and only if $N_{\mathfrak{m}} \in \mathcal{T}_{\mathfrak{m}}$ for every $\mathfrak{m} \in$* $\mathrm{Max}(R)$.

Example 340. ([Sal:2004, Page 3]) Let R be a Matlis domain. Then for any cardinal numbers γ and δ we have a 1-tilting module (called the **Matlis tilting module**):

$$T := \mathbf{Q}^{(\gamma)} \oplus (\mathbf{Q}/R)^{(\delta)}.$$

Definition 341. Let R be a commutative ring and $\mathbf{S} \subseteq R^{\mathrm{reg}}$ be an admissible multiplicatively closed set. Then $R_{\mathbf{S}}$ is called a **Matlis localization**, iff $\mathrm{proj.dim.}(R_{\mathbf{S}}) \leq 1$.

The following result characterizes *Matlis localizations* using tilting modules:

Theorem 342. ([AHT:2005, Theorem 1.1.]) *Let R be a commutative ring and $\mathbf{S} \subseteq R^{\mathrm{reg}}$ an admissible multiplicatively closed set. Then the following are equivalent:*

1. $\mathrm{proj.dim.}(R_{\mathbf{S}}) \leq 1$ *(i.e. $R_{\mathbf{S}}$ is a Matlis localization);*
2. *$T := R_{\mathbf{S}} \oplus R_{\mathbf{S}}/R$ is a 1-tilting module;*
3. $\mathrm{Gen}(R_{\mathbf{S}}) = (R_{\mathbf{S}}/R)^{\perp_1}$.

4. $\operatorname{Gen}(R_{\mathbf{S}}) = \mathcal{DI}(\mathbf{S})$.

5. *$R_{\mathbf{S}}/R$ decomposes into a direct sum of countably presented R-submodules.*

6. *$R_{\mathbf{S}}/R$ decomposes into a direct sum of countably generated modules $\{M_\lambda \mid \lambda < \kappa\}$ such that for each $\alpha < \kappa$ we have $\oplus_{\beta<\alpha} M_\beta = R_{\mathbf{S}_\alpha}/R$ for a submonoid $\mathbf{S}_\alpha \subseteq \mathbf{S}$.*

7. *R has an $\mathbf{S}$-divisible envelope.*

Cotilting modules

Proposition 343. ([Baz, Proposition 2.2.]) *Let $f : R \to \widetilde{R}$ be a morphism of commutative rings. If ${}_RC$ is 1-cotilting with $\operatorname{Ext}^1_R(\widetilde{R}, C) = 0$, then $\operatorname{Hom}_R(\widetilde{R}, C)$ is pure-injective as an R-module and 1-cotilting as an $\widetilde{R}$-module.*

The following result of S. Bazzoni shows that, *up to equivalence*, the study of 1-cotilting modules over integral domains can be restricted to the local case.

Theorem 344. ([Baz, Theorem 2.4.]) *Let R be an integral domain. If C is a 1-cotilting R-module, then*

1. *for every maximal ideal $\mathfrak{m} \lhd R$, the $R_{\mathfrak{m}}$-module $C^{\mathfrak{m}} := \operatorname{Hom}_R(R_{\mathfrak{m}}, C)$ is 1-cotilting.*

2. $\displaystyle\prod_{\mathfrak{m}\in\operatorname{Max}(R)} C^{\mathfrak{m}}$ *is a cotilting R-module that is equivalent to C.*

5.4 Tilting (Cotilting) Modules over Prüfer Domains

The following two results play an *essential role* by characterizing tilting (cotilting) modules over Prüfer domains.

Theorem 345. ([Sal:2005, Theorem 1.8]) *Let R be a Prüfer domain and T a tilting R-module. Then $\mathcal{D}(T)$ is a finitely generated localizing system.*

Theorem 346. ([Baz, Proposition 7.4.], [BGS:2005, Theorem 3.3.]) *Let R be a Prüfer domain. Then every tilting (cotilting) R-module is* 1*-tilting* (1*-cotilting*).

Tilting modules

Theorem 347. ([BET:2005, Theorem 4.4.]) *Let R be a Prüfer domain and $\mathcal{T}$ a tilting torsion class. Then $\mathcal{T} = \mathcal{S}^{\perp_1}$, where*

$$\mathcal{S} = \{{}_R M \mid \ {}_R M \textit{ is cyclic}\} \cap \ {}_R^{f.p.}\mathbb{M} \cap \ {}^{\perp_1}\mathcal{T}.$$

Theorem 348. ([Sal:2005, Corollary 2.2.]) *Let R be a Prüfer domain and $\mathcal{T} = T^{\perp_1}$ be a tilting torsion class. Then*

$$\mathcal{T} = \{{}_R M \mid IM = M \textit{ for all } I \in \mathcal{D}(T)\}.$$

Theorem 349. ([Sal:2005, Theorem 2.5., Corollary 2.6.]) *Let R be a Prüfer domain.*

1. *If T is a tilting R-module, then $\overline{T} := T/\tau(T)$ is a projective $R_{\mathcal{D}(T)}$-module.*
2. *If T is a torsion-free tilting R-module, then $R_{\mathcal{D}(T)} = R$; in particular ${}_R T$ is projective.*

Theorem 350. ([Sal:2005, Proposition 2.7.]) *Let R be a Prüfer domain and $R \subseteq \widetilde{R} \subseteq \mathbf{Q}$ be an overring of R. If* proj.dim.$({}_R\widetilde{R}) \leq 1$, *then $\widetilde{R} \oplus \widetilde{R}/R$ is tilting.*

The Fuchs-Salce divisible module $\partial_{<\mathbf{S}>}$ associated to an admissible multiplicatively closed set $\mathbf{S} \subseteq R^{\times}$ (of an integral domain R) was generalized by L. Salce [Sal:2005] as follows:

351. ([Sal:2005], [GT:2006]) Let R be a Prüfer domain

Proposition 352. ([Sal:2005]) *Let R be a Prüfer domain, $\mathfrak{F}$ a finitely generated localizing filter of R-ideals and consider the R-module $\partial_{\mathfrak{F}}$. Then*

1. $\text{proj.dim.}(\partial_{\mathfrak{F}}) \leq 1$;
2. *$\partial_{\mathfrak{F}}$ is $\mathfrak{F}$-divisible;*
3. $\mathcal{DI}(\mathfrak{F}) \subseteq \text{Gen}(\partial_{\mathfrak{F}})$;
4. $\mathcal{DI}(\mathfrak{F}) \subseteq \partial_{\mathfrak{F}}^{\perp_1}$;
5. *$R_{\mathfrak{F}}$ is an epic image of $\partial_{\mathfrak{F}}$;*
6. *$\partial_{\mathfrak{F}}/\partial_0$ is isomorphic to a direct summand of $\partial_{\mathfrak{F}}$;*
7. *$\partial_{\mathfrak{F}}$ is a tilting R-module.*

Theorem 353. ([Sal:2005, Theorem 2.8.], [GT:2006, Theorems 6.2.15., 6.2.19.]) *Let R be a Prüfer domain.*

1. *There is a bijective correspondence*

   $$\{\mathcal{T} := T^{\perp_1} \mid {}_RT \text{ is tilting}\} \longleftrightarrow \{\mathfrak{F} \mid \mathfrak{F} \text{ is a f.g. localizing system of } R\},$$

   given by

   $$\begin{array}{lll} \mathcal{T} & \mapsto & \{J \lhd R \mid \exists I \lhd R \text{ s.t. } 0 \neq I \subseteq J \text{ and } R/I \in ({}^{\perp_1}\mathcal{T})^{<\omega}\}; \\ \mathcal{L} & \mapsto & \{{}_RM \mid IM = M \text{ for all } I \in \mathcal{L}\}. \end{array}$$

2. *The set of* Salce tilting modules

   $$\{\delta_{\mathcal{L}} \mid \mathcal{L} \text{ is a finitely generated localizing system of ideals of } R\}$$

   is a representative set (up to equivalence) of the class of all tilting R-modules.

Theorem 354. ([Sal:2005, Corollary 2.9.]) *Let R be a Prüfer domain. If T is a tilting R-module, then T is a direct summand of the $\mathcal{D}(T)$-divisible R-module $\partial_{\mathcal{D}(T)}$.*

Cotilting modules

Since all cotilting modules over a Prüfer domain are 1-cotilting by Theorem 346, we restate Theorem 344 for Prüfer domains:

Theorem 355. ([Baz, Corollary 2.6.]) *Let R be a Prüfer domain. If C is a cotilting R-module, then*

1. *for every maximal ideal $\mathfrak{m} \lhd R$, the $R_{\mathfrak{m}}$-module $C^{\mathfrak{m}} := \mathrm{Hom}_R(R_{\mathfrak{m}}, C)$ is cotilting .*
2. $\prod_{\mathfrak{m} \in \mathrm{Max}(R)} C^{\mathfrak{m}}$ *is a cotilting R-module equivalent to C.*

Remark 356. Theorem 355 means that the study of cotilting modules over Prüfer domains can be reduced to the case of *valuation domains*.

Theorem 357. ([Baz, Theorem 6.10.]) *Over Prüfer domains, every cotilting module is equivalent to a cotilting module that is a direct product of indecomposable pure-injective modules.*

Theorem 358. ([GT:2006, Corollary 8.2.12.]) *Let R be a Prüfer domain.*

1. *There is a bijective correspondence*

   $$\{\mathcal{C} \mid \mathcal{C} \textit{ is cotilting of cofinite type}\} \longleftrightarrow \{\mathfrak{F} \mid \mathfrak{F} \textit{ is a f.g. L.S.}^2 \textit{ of } R\},$$

   given by

   $$\begin{array}{rcl} \mathcal{C} & \mapsto & \{J \lhd R \mid \exists I \lhd R \textit{ s.t. } 0 \neq I \subseteq J \textit{ and } R/I \in ({}^{\intercal_1}\mathcal{C})^{<\omega}\}; \\ \mathcal{L} & \mapsto & \{{}_R M \mid M \textit{ is } I\textit{-torsion-free for all } I \in \mathcal{L} \cap \mathrm{mod}R\}. \end{array}$$

2. *Up to equivalence, cotilting R-modules of cofinite type are the duals of the Salce tilting modules $\{\delta_{\mathcal{L}} \mid \mathcal{L}$ is a finitely generated localizing system of ideals of $R\}$.*

Remark 359. Theorem 358 does not classify all cotilting classes and modules over Prüfer domains, as there exist valuation domains with cotilting modules not of cofinite type. An example of such valuation domains is due to S. Bazzoni (see Proposition 382).

Proposition 360. ([AHT:2006, Example 2.4 (ii)]) *Let R be an elementary divisor domain (e.g. a semilocal Prüfer domain).*

1. *A class $\mathcal{T}$ of R-modules is tilting (of finite type) if and only if there exists $\mathbf{S} \subseteq R$ such that $\mathcal{T} = \mathcal{DI}(\mathbf{S})$;*

2. *A class $\mathcal{C}$ of R-modules is cotilting of cofinite type if and only if there exists a set $\mathbf{S} \subseteq R$ such that $\mathcal{C} = \mathcal{TF}(\mathbf{S})$.*

5.5 Tilting (Cotilting) Modules over Gorenstein Rings

In this section we consider the results that are known - so far - about the structure of 1-tilting (1-cotilting) modules over commutative Gorenstein rings.

Proposition 361. ([Baz, Proposition 7.2.]) *Let R be an integral domain. If* $\mathrm{inj.dim.}(R) = 1$, *then every tilting R-module is* 1*-tilting.*

In what follows, we set of any commutative 1-Gorenstein ring R :

$$\mathbf{P}_0 := \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathrm{ht}(\mathfrak{q}) = 0\} \text{ and } \mathbf{P}_1 := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{ht}(\mathfrak{p}) = 1\}.$$

Setting $\widetilde{Q} := \bigoplus_{\mathfrak{q}\in\mathbf{P}_0} \mathrm{E}(R/\mathfrak{q})$, it follows by Proposition 201 that R has a minimal injective coresolution

$$0 \to R \to \widetilde{Q} \xrightarrow{\pi} \bigoplus_{\mathfrak{p}\in\mathbf{P}_1} \mathrm{E}(R/\mathfrak{p}) \to 0.$$

Moreover, for any subset $A \subseteq \mathbf{P}_1$ we define

$$F_{(A)} := \bigoplus_{\mathfrak{p}\in A} \mathrm{E}(R/\mathfrak{p}),\ G_{(A)} := \bigoplus_{\mathfrak{p}\in \mathbf{P}_1\backslash A} \mathrm{E}(R/\mathfrak{p}) \text{ and } R_{(A)} := \pi^{-1}(F_{(A)}) \subseteq \widetilde{Q}. \tag{5.2}$$

Lemma 362. ([AHT:2006, 4.1.], [EJ:2000, 9.1.10., 9.3.3.]) *Let R be a commutative* 1*-Gorenstein ring and $A \subseteq \mathbf{P}_1$.*

1. *We have a short exact sequence of R-modules*
$$0 \to R \to \widetilde{Q} \xrightarrow{\pi} \bigoplus_{\mathfrak{p}\in\mathbf{P}_1} \mathrm{E}(R/\mathfrak{p}) \to 0.$$

2. *We have a short exact sequence of R-modules*
$$0 \to R \to R_{(A)} \to F_{(A)} \to 0.$$

3. *We have a short exact sequence of R-modules*
$$0 \to R_{(A)} \to \widetilde{Q} \to G_{(A)} \to 0.$$

4. $\mathrm{flat.dim.}(G_{(A)}) \leq 1$ *and* ${}_R\widetilde{Q}$ *is flat, hence* $R_{(A)}$ *is also flat.*

Tilting modules

363. ([AHT:2006, Section 4]) For any 1-Gorenstein commutative ring R we have a tilting R-module (called the **Bass tilting module**) given by:

$$T := \bigoplus_{\mathfrak{q}\in\mathbf{P}_0} \mathrm{E}(R/\mathfrak{q}) \oplus \bigoplus_{\mathfrak{p}\in\mathbf{P}_1} \mathrm{E}(R/\mathfrak{p}).$$

Theorem 364. ([AHT:2006, 4.1.]) *Let R be a commutative 1-Gorenstein ring, $A \subseteq \mathbf{P}_1$ and consider the* injective cogenerator $\mathbf{I} := \bigoplus_{\mathfrak{p}\in\mathbf{P}_0\cup\mathbf{P}_1} \mathrm{E}(R/\mathfrak{p})$.

1. *We have a 1-tilting R-module*

   $$T_{(A)} := R_{(A)} \oplus \bigoplus_{\mathfrak{p}\in A} \mathrm{E}(R/\mathfrak{p}).$$

   Moreover, the corresponding 1-tilting class is

   $$T_{(A)}^{\perp_1} = \{{}_RM \mid \mathrm{Ext}^1_R(\mathrm{E}(R/\mathfrak{p}), M) = 0 \textit{ for every } \mathfrak{p} \in A\}.$$

2. *We have a torsion class of R-modules*

   $$\mathrm{Gen}(R_{(A)}) = \mathrm{Gen}(T_{(A)}) = T_{(A)}^{\perp_1} = (\bigoplus_{\mathfrak{p}\in A} \mathrm{E}(R/\mathfrak{p}))^{\perp_1}.$$

Cotilting modules

Theorem 365. ([AHT:2006, 4.1.]) *Let R be a commutative 1-Gorenstein ring, $A \subseteq \mathbf{P}_1$ and consider the injective cogenerator $\mathbf{I} := \bigoplus_{\mathfrak{p}\in\mathbf{P}_0\cup\mathbf{P}_1} \mathrm{E}(R/\mathfrak{p})$. Then we have a 1-cotilting R-module*

$$C_{(A)} := \ T_{(A)}^d = R_{(A)}^d \oplus \prod_{\mathfrak{p}\in A} \mathbf{J}_{\mathfrak{p}},$$

where

$$\mathbf{J}_{\mathfrak{p}} := \mathrm{End}_R(\mathrm{E}(R/\mathfrak{p}))$$

is the $\mathfrak{p}$-adic module *and*

$$R_{(A)}^d \simeq \bigoplus_{\mathfrak{p}'\in\mathbf{P}_1\setminus A} \mathrm{E}(R/\mathfrak{p}') \oplus (\bigoplus_{\mathfrak{q}\in\mathbf{P}_0} \mathrm{E}(R/\mathfrak{q})^{(\alpha_{\mathfrak{q}})}) \textit{ for some set } \{\alpha_{\mathfrak{q}} \mid \alpha_{\mathfrak{q}} \geq 1\}_{\mathfrak{q}\in\mathbf{P}_0}.$$

5.6 Tilting (Cotilting) Modules over Dedekind Domains

In this section we consider the structure of tilting (cotilting) modules over arbitrary Dedekind domains.

Tilting modules

Tilting modules over *small*[3] Dedekind domains were completely characterized by J. Trlifaj and S. Wallutis in [TW:2002, Theorem 12] and [TW:2003, Proposition 4] assuming *Gödel's Axiom of Constructibility* $\mathcal{V} = \mathcal{L}$. However, it has been shown recently by S. Bazzoni et. al. that this set theoretic assumption can be removed:

Theorem 366. ([BET:2005, Theorem 5.3.]) *Let R be an arbitrary Dedekind domain and consider the minimal injective coresolution of R*

$$0 \to R \to \mathbf{Q} \xrightarrow{\pi} \mathbf{Q}/R \to 0.$$

1. *For every $A \subseteq \mathrm{Max}(R)$ we have a tilting R-module*

$$T_{(A)} := R_{(A)} \oplus \bigoplus_{\mathfrak{p} \in A} \mathrm{E}(R/\mathfrak{p}),$$

where

$$R_{(A)} := \pi^{-1}(\bigoplus_{\mathfrak{p} \in A} \mathrm{E}(R/\mathfrak{p})) \subseteq \mathbf{Q}.$$

2. *We have*

$$\begin{aligned} T_{(A)}^{\perp_\infty} &= (\bigoplus_{\mathfrak{p} \in A} \mathrm{E}(R/\mathfrak{p}))^{\perp_\infty} \\ &= \{{}_RM \mid \mathfrak{p}M = M \text{ for each } \mathfrak{p} \in A\}. \end{aligned}$$

[3] An integral domain R is called **small**, iff $|R| \leq 2^\omega$ and $|\mathrm{Spec}(R)| \leq \omega$.

Theorem 367. ([BET:2005, Theorem 5.3.], [GT:2006]) *Let R be a Dedekind domain.*

1. *There is a bijective correspondence*

$$\{\mathcal{T} \mid \mathcal{T} \subseteq {}_R\mathbb{M} \textit{ is tilting}\} \longleftrightarrow \mathcal{PS}(\mathrm{Max}(R)),$$

   given by

$$\begin{array}{lcl} \mathcal{T} & \mapsto & \{P \in \mathrm{Max}(R) \mid PM = M \textit{ for all } M \in \mathcal{T}\}; \\ A & \mapsto & \{{}_RM \mid PM = M \textit{ for all } P \in A\}. \end{array}$$

2. *The set of all tilting R-modules (up to equivalence) is given by the* Bass tilting modules

$$\{T_{(A)} \mid A \subseteq \mathrm{Max}(R)\}.$$

Cotilting modules

Theorem 368. ([ET:2000, Theorem 16]) *Let R be a Dedekind domain and $\mathfrak{C}$ be a class of cotorsion R-modules.*

1. *There is a set $A_{\mathfrak{C}} \subseteq \mathrm{Spec}(R)$ such that*

$$^{\perp_1}\mathfrak{C} = \{{}_RM \mid \forall\ \mathfrak{p} \in A_{\mathfrak{C}},\ R/\mathfrak{p} \nsubseteqq A\}.$$

   In fact

$$A_{\mathfrak{C}} = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \exists\ C \in \mathfrak{C} \textit{ such that } R/\mathfrak{p} \notin {}^{\perp_1}C\}.$$

2. *There exists a class $\mathfrak{U}$ of pure-injective R-modules such that $^{\perp_1}\mathfrak{C} = {}^{\perp_1}\mathfrak{U}$.*

3. *$^{\perp_1}\mathfrak{C}$ is a cotilting torsion-free class and every R-module M has a $^{\perp_1}\mathfrak{C}$-cover*[4].

Theorem 369. ([ET:2000], [Trl:2007], [Baz]) *Let R be a Dedekind domain.*

[4]see [GT:2006] for the definition of (*pre-*)*covers*

1. *For every $A \subseteq \mathrm{Max}(R)$, we have a cotilting R-module*

$$C_{(A)} := \mathbf{Q} \oplus \bigoplus_{\mathfrak{p} \in \mathrm{Max}(R) \setminus A} \mathrm{E}(R/\mathfrak{p}) \oplus \prod_{\mathfrak{p} \in A} \mathbf{J}_{\mathfrak{p}},$$

   where $\mathbf{J}_{\mathfrak{p}} := \mathrm{End}(\mathrm{E}(R/\mathfrak{p})_R)$.

2. *We have*

$$^{\perp_\infty} C_{(A)} = \bigcap_{\mathfrak{p} \in A} \{ {}_R M \mid \mathrm{Tor}_1^R(C, R/\mathfrak{p}) = 0 \}.$$

Theorem 370. ([Baz], [GT:2006, Theorem 8.2.9.]) *Let R be a Dedekind domain.*

1. *A class $\mathcal{C}$ of R-modules is cotilting if and only if there exists a set of maximal R-ideals $A \subseteq \mathrm{Max}(R)$, such that*

$$\mathcal{C} = \{ {}_R M \mid \mathrm{Tor}_1^R(R/\mathfrak{m}, M) = 0 \text{ for all } \mathfrak{m} \in A \}.$$

2. *The set of all cotilting R-modules (up to equivalence) is given by the*

$$\{ C_{(A)} \mid A \subseteq \mathrm{Max}(R) \}.$$

5.7 Tilting (Cotilting) Modules over Valuation Domains

In this section we consider the structure of tilting (cotilting) modules over arbitrary Dedekind domains.

Tilting modules

Proposition 371. ([Sal:2004, 4.1.-4.5. & 4.13.]) *Let R be a valuation domain, T a tilting R-module, $\overline{T} := T/\tau(T)$ and $\mathbf{S} := D_R(T)$ be the divisibility set of T.*

1. *$\tau(T)$ is $\mathbf{S}$-divisible and $\mathbf{S}$-torsion;*
2. *$T^{\#} = \overline{T}^{\#}$;*
3. *$\mathrm{Ext}^1_R(T, \overline{T}^{(\kappa)}) = 0$ for all cardinals κ;*
4. *$\overline{T} \simeq T \otimes_R R_{T\#}$; whence $\mathrm{proj.dim.}({}_{R_{T\#}}\overline{T}) \leq 1$;*
5. *If $T \simeq R_{T\#}^{(\gamma)} \oplus (R_{T\#}/R)^{(\delta)}$ for some $\gamma, \delta \neq 0$, then ${}_RT$ is tilting.*

Theorem 372. ([Sal:2004, 3.4., 3.6, 3.8., 4.6., 4.13.]) *Let R be a valuation domain, T a tilting R-module and $\overline{T} := T/\tau(T)$. Assume moreover that ${}_RT$ is of countable rank, or that $\mathcal{V} = \mathcal{L}$ and $\left|\widehat{R}\right| \leq 2^{\aleph_0}$.*

1. *Let ${}_RT$ be torsion-free.*
   (a) *If $\mathrm{proj.dim.}({}_RT) \leq 1$ and N is an R-module with $T^{\#} \subseteq N^{\#}$, then*
   $$\mathrm{Ext}^1_R(T, N^{(\omega)}) = 0 \Leftrightarrow T \text{ is free as an } R_{N\#}\text{-module.}$$
   (b) *${}_RT$ is tilting if and only if ${}_RT$ is free.*
2. *If $\mathrm{proj.dim.}(R_{T\#}) \leq 1$, then ${}_RT$ is tilting if and only if*
   $$T \simeq R_{T\#}^{(\gamma)} \oplus (R_{T\#}/R)^{(\delta)} \text{ for some non-zero cardinals } \gamma, \delta.$$

3. *If ${}_RT$ is tilting, then:*

   (a) *$\overline{T}$ is free as an $R_{T^\#}$-module;*

   (b) $\mathrm{Gen}({}_RT) = \mathrm{Gen}({}_R\partial)$.

Theorem 373. ([Sal:2004, Theorem 4.7.]) *Let R be a valuation domain, ∂ be the Fuchs divisible R-module and T a divisible R-module with* $\mathrm{proj.dim.}({}_RT) = 1$. *Then*

1. *${}_RT$ is tilting if and only if ${}_RT$ is mixed.*

2. *If ${}_RT$ is tilting, then* $\mathrm{Gen}({}_R\partial) = \mathrm{Gen}({}_RT)$.

Definition 374. A non-empty subset $\varnothing \neq \mathbf{A} \subseteq R$ is said to be **saturated**, iff

$$aa' \in \mathbf{A} \Rightarrow a \in \mathbf{A} \text{ or } a' \in \mathbf{A} \text{ for all } a, a' \in R.$$

Lemma 375. ([GT:2006, Lemma 6.2.20.]) *Let R be a valuation domain. A subset $\mathbf{A} \subseteq R$ is a saturated and an admissible multiplicatively closed set if and only if $R \backslash \mathbf{A} \in \mathrm{Spec}(R)$.*

Theorem 376. ([GT:2006, Theorem 6.2.21.]) *Let R be a valuation domain.*

1. *There is a bijection*

$$\{\mathcal{T} \mid \mathcal{T} \subseteq {}_R\mathbb{M} \textit{ is tilting}\} \longleftrightarrow \mathrm{Spec}(R),$$

   given by

$$\begin{array}{rcccl} \mathcal{T} & \mapsto & \bigcup\limits_{M \in \mathcal{T}} M^\# & := & \{s \in R \mid sM \subsetneqq M \textit{ for some } M \in \mathcal{T}\}; \\ P & \mapsto & \mathcal{DI}(R \backslash P) & := & \{{}_RM \mid sM = M \textit{ for every } s \in R \backslash P\}. \end{array}$$

2. *The set of Fuchs-Salce tilting R-modules $\{\delta_{R\backslash P}\}_{P \in \mathrm{Spec}(R)}$ is a representative set (up to equivalence) of the class of all tilting R-modules.*

Cotilting modules

377. Let R be an integral domain and C a 1-cotilting R-module. We associate to C the set

$$\mathcal{G} = \mathcal{G}(C) := \{I \lhd R \mid R/I \in {}^{\perp_\infty}C\} = \{I \lhd R \mid R/I \in \mathrm{Cogen}({}_RC)\}$$

and

$$\mathcal{G}' := \mathcal{G} \cap (\mathrm{Spec}(R) \backslash \{0\}).$$

For $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{G}'$, we say $\mathfrak{p}_2$ covers $\mathfrak{p}_1$ in $\mathcal{G}$ (and write $\mathfrak{p}_2 \gtrdot \mathfrak{p}_1$), iff $\mathfrak{p}_2 \supsetneqq \mathfrak{p}_1$ and there is no $\mathfrak{p} \in \mathcal{G}'$ lying properly between $\mathfrak{p}_2$ and $\mathfrak{p}_1$.

378. Let R be a valuation domain, C a cotilting R-module and $\mathcal{G} := \mathcal{G}(C)$. For every non-empty subclass $\varnothing \neq \mathcal{H} \subseteq \mathcal{G}$ we set

$$\sup \mathcal{H} := \sum_{I \in \mathcal{H}} I \text{ and (in case } \mathcal{H} \neq \{0\}) \ \inf \mathcal{H} := \bigcap_{0 \neq I \in \mathcal{H}} I$$

and define

$$\begin{array}{ccccccc} \phi & : & \mathcal{G}' & \to & \mathcal{G}, & \mathfrak{p} & \mapsto & \inf\{N \in \mathcal{G}' \mid R_N/\mathfrak{p} \in {}^{\perp_\infty}C\}; \\ \psi & : & \mathcal{G}' & \to & \mathcal{G}', & \mathfrak{p} & \mapsto & \sup\{N \in \mathcal{G}' \mid R_{\phi(\mathfrak{p})}/N \in {}^{\perp_\infty}C\}. \end{array}$$

Remark 379. Let R be a valuation domain and C a cotilting R-module. By [Baz, Lemma 3.3.], $\sup \mathcal{G}(C) = C_\#$ and $\inf \mathcal{G}(C)$ are *idempotent* prime ideals. For every $\mathfrak{p} \in \mathcal{G}'$, $\phi(\mathfrak{p})$ is an idempotent prime ideal (that might be 0).

Proposition 380. ([Baz, Proposition 4.1.]) *Let R be a valuation domain with maximal ideal $\mathfrak{m}$. If ${}_RC$ is torsion-free and cotilting, then*

1. *${}^{\perp_\infty}C$ is the class of all torsion-free R-modules.*
2. *C is equivalent to $\mathbf{Q} \oplus \widehat{\mathfrak{m}}$;*
3. *${}_RC$ is of cofinite type.*

Proposition 381. ([Baz, Proposition 4.3.]) *Let R be a valuation domain with maximal ideal $\mathfrak{m}$, C a cotilting R-module and $0 \neq W := C_\#$. Then the following are equivalent:*

1. *${}_RC$ is of cofinite type;*

2. ${}^{\perp_\infty}C = \{{}_RM \mid \forall\ 0 \neq m \in M,\ \mathrm{ann}_R(m) \subseteq W\}$;

3. $\mathrm{E}(\mathbf{Q}/W) \oplus \widehat{\mathfrak{m}/W}$ *is a cotilting R-module equivalent to C.*

Proposition 382. ([Baz, Proposition 4.5.]) *Let R be a valuation domain with maximal ideal $\mathfrak{m}$ and $0 \neq \mathfrak{p} \in \mathrm{Spec}(R)$ be idempotent. Then*

1. *The following R-module is cotilting:*

$$C := \mathbf{Q} \oplus \widehat{R_{\mathfrak{p}}} \oplus \widehat{R_{\mathfrak{p}}/\mathfrak{p}} \oplus \widehat{\mathfrak{m}/\mathfrak{p}}.$$

2. *We have*

$${}^{\perp_\infty}C = \{{}_RM \mid \forall\ 0 \neq m \in M,\ \mathrm{ann}_R(m) = 0 \text{ or } \mathrm{ann}_R(m) = \mathfrak{p}\};$$

3. ${}_RC$ *is* ***not*** *of cofinite type.*

Theorem 383. ([Baz, Corollary 4.6.]) *The following are equivalent for a valuation domain R :*

1. *R is strongly discrete;*

2. *All cotilting R-modules are equivalent to duals of tilting R-modules;*

3. *All cotilting R-modules are of cofinite type.*

Proposition 384. ([Baz, Proposition 5.2., Lemma 5.3.]) *Let R be a valuation domain with maximal ideal $\mathfrak{m}$. Let C be a cotilting R-module with associated set $\mathcal{G} := \mathcal{G}(C)$ and $0 \neq \mathfrak{p} := \sup \mathcal{G}$ and $\mathfrak{p}_0 := \inf \mathcal{G}$. Then*

1. *C is equivalent to the cotilting R-module*

$$\mathbf{Q} \oplus \widehat{R_{\mathfrak{p}_0}} \oplus \mathrm{Hom}_R(R_{\mathfrak{p}}/\mathfrak{p}_0, C) \oplus \widehat{\mathfrak{m}/\mathfrak{p}};$$

and moreover, the $R_{\mathfrak{p}}/\mathfrak{p}_0$-module $\mathrm{Hom}_R(R_{\mathfrak{p}}/\mathfrak{p}_0, C)$ is cotilting.

2. *If $\widetilde{R}$ is a maximal immediate extension of R, then $\mathrm{Hom}_R(\widetilde{R}, C)$ is an cotilting R-module and an $\widetilde{R}$-cotilting module; and moreover, the cotilting R-modules C and $\mathrm{Hom}_R(\widetilde{R}, C)$ are equivalent.*

Remark 385. Let R be a valuation domain with maximal ideal $\mathfrak{m}$. According to Proposition 384, to characterize the cotilting R-modules, we may restrict ourselves to those cotilting modules C for which $\sup \mathcal{G}(C) = \mathfrak{m}$ and $\inf \mathcal{G}(C) = 0$; and one may assume (without loss of generality) that R is a *maximal valuation domain*.

Lemma 386. ([Baz, Lemma 6.7.]) *Let R be a maximal valuation domain with maximal ideal $\mathfrak{m}$. Let C be a cotilting R-module and $\mathcal{G} := \mathcal{G}(C)$. For every $N \in \mathcal{G}'$ with $\phi(N) \neq 0$ one and only one of the following is satisfied:*

1. $\phi(N) = \inf \mathcal{G}'$ *(i.e.* $\phi(N) \gtrdot 0$);
2. $\phi(N) = \sup\{\psi(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{G}', \mathfrak{p} \subsetneqq \phi(N)\}$;
3. *there exists* $\mathfrak{p} \in \mathcal{G}'$ *such that* $\phi(N) \gtrdot \psi(\mathfrak{p})$.

Theorem 387. ([Baz, Theorem 6.9.]) *Let R be a maximal valuation domain with maximal ideal $\mathfrak{m}$. Let C be a cotilting R-module, $\mathcal{G} := \mathcal{G}(C)$ and assume* $\sup \mathcal{G} = \mathfrak{m}$ *and* $\inf \mathcal{G} = 0$. *Then C is equivalent to the cotilting R-module*

$$E := \mathbf{Q} \bigoplus \prod_{\phi(\mathfrak{p}) \in \mathcal{G}} \frac{R_{\phi(\mathfrak{p})}}{\psi(\mathfrak{p})} \bigoplus \prod_{\phi(N) \gtrdot \psi(\mathfrak{p})} \frac{R_{\phi(N)}}{\psi(\mathfrak{p})} \bigoplus_{\phi(N) \gtrdot 0} R_{\phi(N)}.$$

5.8 Tilting (Cotilting) Abelian Groups

In this section we consider the structure of tilting (cotilting) Abelian groups, i.e. tilting (cotilting) $\mathbb{Z}$-modules. The structure of tilting (cotilting) Abelian groups was completely described by R. Göbel and J. Trlifaj in [GT:2000] assuming *Gödel's Axiom of Constructibility* $\mathcal{V} = \mathcal{L}$. Since $\mathbb{Z}$ is a Dedekind domain, it follows that the results of Section 5.6 hold for tilting (cotilting) Abelian groups without any extra set theoretic assumptions.

We begin with restating Theorem 346 for $R = \mathbb{Z}$:

Theorem 388. *All tilting* (*cotilting*) *Abelian groups are* 1*-tilting* (1*-cotilting*)*.*

Tilting modules

Theorem 389. ([GT:2000, Theorem 2.3., Corollary 2.4.], [EM:2002, Proposition XVI.1.13.]) *For every $A \subseteq \mathbb{P}$, let $R_{(A)} \subseteq \mathbb{Q}$ be the subring generated by $\mathbb{Z} \cup \{\frac{1}{p} \mid p \in A\}$. Then*

1. *the class of* all *partial tilting Abelian groups is*

$$\{\bigoplus_{p\in A} \mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus R_{(A)}^{(\beta)} \mid A \subseteq \mathbb{P},\ \alpha_p, \beta \text{ are cardinals and } \alpha_p \neq 0\}.$$

2. *The class of* all *tilting Abelian groups is*

$$\{\bigoplus_{p\in A} \mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus R_{(A)}^{(\beta)} \mid A \subseteq \mathbb{P},\ \alpha_p \neq 0, \beta \neq 0 \text{ are cardinals}\}.$$

3. *A torsion class of Abelian groups $\mathcal{T}$ is tilting if and only if there is some $A \subseteq \mathbb{P}$, such that*

$$\mathcal{T} = \{G \mid pG = G \text{ for all } p \in A\}.$$

Cotilting modules

Theorem 390. ([GT:2000, Theorem 2.1., Corollary 2.2.], [EM:2002, Proposition XVI.1.14.]) *Denote by* $\mathbb{P}$ *the set of primes in* $\mathbb{Z}$.

1. *The class of* all *partial cotilting Abelian groups is*

$$\{\bigoplus_{p\in A}\mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus \mathbb{Q}^{(\gamma)} \oplus \prod_{q\in B}\widehat{\mathbb{J}_q^{(\beta_q)}} \mid A, B \subseteq \mathbb{P}, A\cap B = \varnothing,\ \alpha_p, \beta_q \neq 0\}.$$

2. *The class of* all *cotilting Abelian groups is*

$$\{\bigoplus_{p\in A}\mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus \mathbb{Q}^{(\gamma)} \oplus \prod_{q\in \mathbb{P}\setminus A}\widehat{\mathbb{J}_q^{(\beta_q)}} \mid A \subseteq \mathbb{P},\ \alpha_p, \beta_q \neq 0 \ \&\ (\gamma > 0 \ \mathit{or}\ A = \varnothing)\}.$$

3. *A torsion-free class of Abelian groups* $\mathcal{TF}$ *is cotilting if and only if there is* $B \subseteq \mathbb{P}$, *such that*

$$\mathcal{TF} = \{G \mid p(G) = 0 \ \mathit{for\ all}\ p \in B\},$$

   where $p(G)$ *is the* p-component *of* G.

Chapter 6

Historical Remarks

The interest in tilting (cotilting) modules stems mainly from the fact that they allow generalization of the classical Morita equivalences (dualities) as well as being a main tool in the study of the representation theory of *finite dimensional* and *Artin algebras*.

In what follows we present a *brief* literature review of *Tilting and Cotilting Modules* with emphasis on the case where the ground ring is commutative[1]. Although we go through many of the main articles related to the topic of the report, no attempt or claim is made to be encyclopedic.

6.1 Tilting Modules

- *Tilting Theory* originated with the study of reflection functors by I. Bernstein, I. Gelfand and V. Ponomarev [BGP:1973].
- The first instance of a *tilting module* goes back to the work by M. Auslander, M. Platzeck and I. Reiten on a module-theoretic interpretation of Coxeter functors [APR:1979]. These modules (referred to nowadays as the *APR-tilting* modules) were constructed using projective resolutions of a suitable simple modules.

[1] In addition to referring to most of the articles and preprints in the list of references at the end of this report, several parts of this literature review are prepared with the help of reviews for the cited articles in *Mathematical Reviews* and *ZentralBlattMath*.

- The construction of the *APR-tilting modules* was extended by S. Brenner and M. Butler [BB:1980], who introduced, in a rather restrictive way, tilting modules of finite dimensional algebras over base fields. In fact, they were the first to begin an axiomatic approach to the study of tilting modules.

- The theory of *tilting modules* was generalized and extensively developed by D. Happel and C. Ringel [HR:1982], who gave the generally accepted set of axioms of a tilting modules over (finite dimensional, Artin) algebras.

- Direct proofs for the main results on *tilted algebras* obtained by D. Happel and C. Ringel were provided by K. Bongartz [Bon:1981], who gave also various applications of the theory, in particular to *representation-finite algebras*.

- In [Miy:1986], Y. Miyashita considered *strongly finitely presented* tilting modules of *arbitrary finite projective* dimension and studied the equivalences of categories induced by them. Later, Miyashita's construction of strongly finitely presented tilting modules of finite projective dimension was (properly) generalized by H. Fujita [Fuj:1992].

- An *infinitely generated* tilting module of projective dimension 1 over an arbitrary commutative integral domain R is the so called **Fuchs divisible module** ∂ due to L. Fuchs [Fuc:1984]. This module was investigated intensively by A. Facchini in [Fac:1987] and [Fac:1988], who showed that ∂ has the tilting property and proved equivalences between suitable subcategories of $\mathbb{M}_R$ and $\mathbb{M}_{\mathrm{End}({}_R\partial)}$.

- In [Wak:1988], T. Wakamatsu provided a generalization of tilting modules : an R-module ${}_RW$ (possibly with infinite projective dimension) is said to be a *Wakamatsu tilting module*, iff ${}_RW$ is faithfully balanced and
$$\mathrm{Ext}^i_R(W,W) = 0 = \mathrm{Ext}^i_{\mathrm{End}({}_RW)^{op}}(W,W) \text{ for all } i \geq 1.$$
Wakamatsu tilting modules were investigated by several authors (e.g. F. Mantese and I. Reiten in [MR:2004]).

- As a further generalization of progenerators, the notion of $*^1$-modules was introduced by C. Menini and A. Orsatti in [MO:1989] (and named later $*$-*modules* in [Col:1990]). Such modules were shown to be finitely generated by J. Trlifaj in [Trl1994]. For the relation between $*^1$-modules and other classes of modules, one may consult [CF:2004].

- In [Miy:1992], Y. Miyashita extended the tilting theorems he obtained earlier for classical n-tilting modules in [Miy:1986] to more general ones. In particular, the new extended versions contained the theorems in [Mat:1964] and [Fac:1987] on equivalences of categories of modules related to divisible modules as special cases.

- The notion of arbitrary (*infinitely generated*) tilting modules over arbitrary rings was introduced by R. Colpi and J. Trlifaj in [CT:1995] for the one dimensional case.

- The class of (finitely generated) 1-*quasi-tilting modules* was presented by R. Colpi et. al. [CDT:1997] and their relations with $*^1$-modules was clarified.

- R. Wisbauer considered in [Wis:1998] arbitrary (possibly infinitely generated) 1-quasi-tilting modules and called them 1-*self-tilting modules.* A 1-self-tilting R-module T can be thought of (roughly speaking) as a 1-tilting object in the associated Grothendieck category $\sigma[_RT]$ (of T-subgenerated R-modules). In particular, it was pointed out that $*^1$-modules coincide with the self-small 1-self-tilting modules.

- L. Angeleri-Hügel and F.U. Coelho extended in [AC:2001] both the notions of infinitely generated 1-tilting modules and strongly finitely presented n-tilting modules by introducing the notion of *infinitely generated n-tilting modules.*

- R. Göbel and J. Trlifaj gave in [GT:2000] a complete description of (partial) tilting Abelian groups assuming *Gödel's Axiom of Constructability* $\mathcal{V} = \mathcal{L}$.

- In [GT:2001], E. Gregorio and A. Tonolo introduced several notions of finitely generated modules inducing a tilting equivalence (e.g. *weakly tilting modules*, *fc-tilting modules*, *e-tilting modules*).

- In [TW:2002] (and [TW:2003]), J. Trlifaj and S. Wallutis extended in a natural way the characterizations of tilting $\mathbb{Z}$-modules to tilting modules over *small* Dedekind domains assuming *Gödel's Axiom of Constructibility* $\mathcal{V} = \mathcal{L}$. Recently, S. Bazzoni et. al. obtained in [BET:2005] the same characterizations of tilting modules over *arbitrary* Dedekind domains without assuming $\mathcal{V} = \mathcal{L}$.

- In [BS:2001], S. Bazzoni and L. Salce investigated *cotorsion theories* over integral domains. In particular, they showed that there is a strong connection between the *Whitehead modules* over an integral domain R and the cotorsion theory generated by $\mathbf{Q}/R$. In case R is an IC-domain (i.e. an *anti-maximal valuation domain* R such that proj.dim.$({}_R\mathbf{Q}) = 1$ and GL.dim.$(R) = 2$), they showed (assuming $\mathcal{V} = \mathcal{L}$) that the cotorsion theory generated by the *divisible* R-module $\mathbf{Q}/R$ coincides with the cotorsion theory cogenerated by the Fuchs tilting R-module $\partial := \partial_{<R^{\times}>}$.

- In [Sal:2004], L. Salce investigated the structure of tilting modules over valuation domains. Assuming T is a module (of countable type) over a valuation domain R, or that $\left|\widehat{R}\right| \leq 2^{\aleph_0}$ and $\mathcal{V} = \mathcal{L}$, sufficient and necessarily conditions are given for ${}_RT$ to be tilting in case ${}_RT$ is torsion-free or proj.dim.$(R_{T\#}) \leq 1$.

- The relation between classical 1-tilting modules and other classes of modules related to equivalences between (sub)categories of modules (e.g. $*^1$-modules, quasi-progenerators and progenerators) was investigated in details in [CF:2004]. Moreover, the authors established the important fundamental tilting theorem induced by a classical 1-tilting module ${}_RT$ and clarified its consequences on the relations between the global dimensions of the rings R and $S := \mathrm{End}({}_RT)^{op}$ in addition to their Grothendieck groups.

- In [Baz], S. Bazzoni considered tilting modules over Prüfer domains. She showed, in particular, that *all* tilting modules over Prüfer domains are of projective dimension at most 1, i.e. 1-tilting.

- Making use of results in [Baz], the structure of tilting modules over Prüfer domains was investigated by L. Salce [Sal:2005]. In particular, he showed that the tilting torsion classes over a Prüfer domain R correspond bijectively to *finitely generated localizing systems* (i.e. *Gabriel filters*) of R-ideals. For such a system $\mathfrak{F}$, a *generalized Fuchs divisible* *module* $\partial_{\mathfrak{F}}$ is constructed which turns out to be tilting and to generate the corresponding tilting torsion class.

- In [AHT:2005], L. Angeleri Hügel et. al. studied the relation between localizations and tilting modules. In particular, they showed that if R is any ring and $\mathcal{O} \subseteq R^{\text{reg}}$ is a *left Ore set*, then proj.dim.$(\mathcal{O}^{-1}R) \leq 1$ if and only if $\mathcal{O}^{-1}R \oplus \mathcal{O}^{-1}R/R$ is a tilting R-module. To each such subset, they constructed a divisible module $\partial_{\mathcal{O}}$ that extends the Fuchs-Salce divisible module to the case of non-commutative rings.

- So far, all known tilting modules were known to be of finite type. Tilting modules over special classes of ground rings were shown to be of finite type by several authors (e.g. [Baz]). Making use of the crucial reduction to the *countable case* by J. Šťovíček and J. Trlifaj [ST], it was shown recently by S. Bazzoni and J. Šťovíček [BS] that over any ground ring "*all tilting modules are of finite type*".

- In a number of recent papers (e.g. [HHTW:2003], [Wei:2005b]), J. Wei et. al. generalized the notions of (self-small) $*^1$-modules to those of (self-small) n-star modules, where $n > 1$, and clarified their relations with (classical) n-tilting modules.

- In [Wei:2005b] J. Wei generalized the notions of self-small n-star modules (classical n-tilting modules) to self-small $*^\infty$-modules (∞-tilting modules of possibly infinite projective dimension). He showed in particular, that classical n-tilting modules in the sense of Y. Miyashita [Miy:1986] are precisely the ∞-tilting modules of finite projective dimension $\leq n$; and that Miyashita's generalization of the Brenner-Butler's Tilting Theorem holds for ∞-tilting modules.

- In their recent monograph [GT:2006], R. Göbel and J. Trlifaj considered arbitrary tilting modules from the point view of *approximations of modules*. Making use of recent developments in the theory of cotorsion pairs and the key result of S. Bazzoni and J. Šťovíček [BS] that over any ground ring "*all tilting modules are of finite type*", new proofs of several results in the theory of tilting modules are provided. Moreover, making use of results in [Sal:2005], [Baz] and [AHT:2006], a classification (up to equivalence) of all tilting modules over Prüfer domains, Dedekind domains and valuation domains is given.

- The recent monograph [AHK:2007] contains a number of interesting articles on *Tilting Theory* including (in addition to the introduction) an interesting short article by the editors on the "*Basic Results in Tilting Theory*", a very useful survey on *Infinite dimensional tilting modules and cotorsion pairs* by J. Trlifaj and an appendix on the *origin, relevance and future of Tilting Theory* by C.M. Ringel.

6.2 Cotilting Modules

"Cotilting Theory" extends *Morita duality* in analogy to the way that tilting theory extends *Morita equivalences.* In particular, cotilting modules generalize injective cogenerators similarly as tilting modules generalize progenerators.

- Cotilting modules appeared first, as vector space duals of tilting modules over finite dimensional (Artin) algebras, see e.g. [Hap:1988, IV.7.8.].
- A central point in cotilting theory is to obtain a dual of the tilting theorem of S. Brenner and M. Butler [BB:1980]. Several *cotilting theorems* were obtained by different authors using different notions of cotilting modules satisfying *suitable conditions* (e.g. [Col:1989], [CF:1990], [Ang:2000], [Wis:2002]).
- The first work aiming to generalize the cotilting modules in the representation theory of Artin algebras is due to R. Colby [Col:1989]. He also defined a *cotilting bimodule* as a faithfully balanced bimodule ${}_RU_S$ that is cotilting on both sides and obtained a "Cotilting Theorem" dual to that of the Brenner-Butler theorem in case the ground ring is Noetherian.
- A second approach to cotilting modules is that of R. Colpi et. al. [CDT:1997] who gave a definition of 1-cotilting modules by dualizing the definition of 1-tilting modules as given in [CT:1995].
- In [Col:2000], R. Colpi defined a *cotilting bimodule* as a faithfully balanced bimodule ${}_RU_S$ that is cotilting (in the sense of [CDT:1997]) on both sides. Moreover, he investigated dualities induced by such bimodules as a generalization of the classical theory of Morita dualities and obtained some sort of *cotilting theorem.* That theorem was recovered, under weaker conditions, by F. Mantese in [Man:2001].
- In her attempt to bridge the two main approaches to cotilting modules mentioned above, L. Angeleri Hügel [Ang:2000] presented the notions of *finitely cotilting modules* and *Colby-modules* over arbitrary ground rings. She showed that finitely cotilting modules and cotilting modules

in the sense of R. Colby [Col:1989] coincide over *Morita rings*; and that a module ${}_RC$ that is finitely generated and *product complete* is finitely cotilting if and only if ${}_RC$ is cotilting in the sense of [CDT:1997]. Moreover, she obtained a cotilting theorem for Colby-bimodules over arbitrary ground rings (showing that there is no need for the full power of the Noetherian assumption in Colby's cotilting theorem).

- The notion of infinitely generated cotilting modules of *arbitrary* finite injective dimension was introduced by L. Angeleri Hügel and F.U. Coelho [AC:2001] as dual to infinitely generated tilting modules of finite projective dimension presented in the same paper.

- In [GT:2000], R. Göbel and J. Trlifaj gave a complete description of (partial) cotilting Abelian groups assuming Gödel's Axiom of Constructability $\mathcal{V} = \mathcal{L}$. In [CF:2001], R. Colby and K. Fuller introduced the notion of *costar modules* as dual to $*^1$-module (in the sense of [MO:1989]). In case R is a finite-dimensional algebra over a field K, they showed that a finitely generated R-module is a costar module if and only if $\mathrm{Hom}_K(M, K)$ is a $*^1$-module, and that a faithful costar module is a cotilting module. The interplay between *costar modules* and cotilting modules was investigated further by the two authors in [CF:2004].

- In [Wis:2002], R. Wisbauer considered various injectivity and cogenerating conditions for objects in Grothendieck categories which result from dualizing notions of interest in the study of (self)-tilting objects. In particular, for a given module ${}_RC$ and $\overline{R} := R/\mathrm{ann}_R(C)$, he considered cotilting modules in the category $\pi[{}_RM] = {}_{\overline{R}}\mathbb{M}$. Moreover, he introduced the notion of *f-cotilting modules* and showed that such an R-module ${}_RC$ induces a duality between suitable subcategories of ${}_R\mathbb{M}$ and $\mathbb{M}_{\mathrm{End}({}_RC)^{op}}$.

- In [Baz], S. Bazzoni studied cotilting modules over commutative rings and showed that one can restrict himself to cotilting modules over local commutative rings. Moreover, she showed that all cotilting modules over Prüfer domains are 1-cotilting; and that the study of their structure can be reduced to the study of a special class of cotilting modules over *maximal valuation domains*. Moreover, she showed that all cotilting modules over a valuation domain R are of *cofinite type* if and only if R is strongly discrete. For strongly discrete valuation domains and maximal valuation domains, a complete description (up to equivalence) for the structure of cotilting modules was given.

- The relation between cotilting modules and *pure-injectivity* was investigated by several authors. For example, S. Bazzoni showed in [Baz:2003, Theorem 2.8] that 1-cotilting modules over any ground ring are pure-injective; and in [Baz:2004(a)] that all cotilting modules are pure-injective in case the base ring R is a Prüfer domain or is a countable commutative ground ring. Recently, J. Šťovíček [Sto:2006, Sto:2006] proved that over any ground ring "*all cotilting modules are pure-injective*".

- In [GT:2006], R. Göbel and J. Trlifaj suggested two possible directions for the classification of cotilting modules, namely showing that cotilting modules over specific rings are dual to tilting modules (whence of cofinite type), or using known classification of pure-injective modules over particular rings. They presented results in either direction and gave, as an application, a classification (up to equivalence) of all cotilting modules over Dedekind domains.

- The recent monograph [AHK:2007] contains a number of interesting articles on cotilting modules including an interesting article on *cotilting dualities* by R. Colpi and K. Fuller.

Part III

Open Problems

In this part, we include some open problems about the structure of tilting and cotilting modules over commutative rings that we formulated after an intensive literature review. Some of these problems are also highlighted in the literature (e.g. [GT:2006]).

Recently Solved Problems:

First of all, we point out that two of the *long standing* open problems in the theory of tilting (cotilting) modules over arbitrary associative rings were solved recently.

All known tilting modules were noticed to be of finite type and it was *conjectured* that all tilting modules are of finite type. Recently S. Bazzoni and J. Šťovíček proved the conjecture in the affirmative:

First Solved Problem: ([BS, Theorem 4.2.]) Over any associative ring, *all* tilting modules are of finite type.

The relation between cotilting and pure-injective modules was investigated by many authors. Several classes of cotilting modules were known to be pure-injective. Recently, J. Šťovíček settled this problem:

Second Solved Problem: ([Sto:2006, Theorem 13]) Over any associative ring, *all* cotilting modules are pure-injective.

Main Goal and Main Problem:

Main Goal: Developing a "*Multiplicative Ideal Theory*"-approach to study the structure of tilting (cotilting) modules over commutative rings.

Main Problem: Characterizing the tilting (cotilting) modules over different classes of (non Prüfer) integral domains and commutative rings.

Some Observations:

In what follows we answer the question that was the *main motivation* for investigating tilting modules over commutative rings:

> **Characterize the commutative rings R, for which non-zero finitely generated ideals are classical 1-tilting R-modules?**

As a consequence of Proposition 329, we get the following characterization of Dedekind (Prüfer) domains:

Theorem 391. *A commutative ring R is a Dedekind (Prüfer) domain if and only if every non-zero (finitely generated) ideal $0 \neq I \lhd R$ is n-tilting for some $n \in \mathbb{N}$.*

Open Problems on Tilting Modules:

Problem 1: Characterize, *up to equivalence,* the 1-tilting modules over special classes of (*non-Prüfer*) integral domains (e.g. Matlis domains, Krull domains and their generalizations). Extend such characterizations to n-tilting modules for $n > 1$.

Problem 2: Characterize, *up to equivalence,* all 1-tilting modules over commutative 1-Gorenstein rings. Extend such characterizations to n-tilting modules over n-Gorenstein rings for $n > 1$.

Remark 392. By [Baz], for $n \geq 1$, all n-tilting modules over 1-Prüfer domains are 1-tilting, hence completely characterized as in [Sal:2005].

The remark above suggests:

Problem 3: Fine an upper bound for the projective dimension of n-tilting modules over n-Prüfer domains (in the sense of Costa [Cos:1994]). Is n such an upper bound?

Problem 4: For $n \geq 2$, characterize, *up to equivalence,* all n-tilting modules over n-Prüfer domains.

Problem 5: Characterize the commutative rings for which every (finitely generated) ideal is a 1-self-tilting module ($*^1$-modules). Extend such characterizations to $n > 1$.

Remark 393. The class of $*^1$-modules (i.e. self-small 1-star modules) coincides with the class of $*$-modules in the sense of [MO:1989], whence finitely generated as shown by J. Trlifaj in [Trl1994]. For $n \geq 2$, self-small n-star modules are not necessarily finitely generated as shown by J. Wei in [Wei:2006].

By Proposition 245, self-small 1-self-tilting modules coincide with the $*^1$-modules (which are in fact finitely generated). This suggests:

Problem 6: Extend the definition of (self-small) 1-self-tilting modules to (self-small) n-self-tilting modules for $n > 1$ and clarify its relation with (self-small) n-star modules.

The $*^1$-module modules over valuation rings were characterized by P. Zanardo in [Zan:1990]. This suggests:

<u>Problem 7:</u> Characterize, *up to equivalence*, all (self-small) 1-star modules over special classes of commutative rings. Extend such characterizations to (self-small) n-star modules for $n > 1$.

Open Problems on Cotilting Modules:

Being a progenerator in it its own category of modules, every ring R is a tilting left (right) R-module. However, for an arbitrary ring R, the left (right) R-module ${}_RR$ (R_R) is not necessarily an injective cogenerator and not even cotilting in general.

Definition 394. We call the ring R
(n-)**cotilting ring**, iff ${}_RR$ and R_R are (n-)cotilting modules;
partial (n-)**cotilting ring**, iff ${}_RR$ and R_R are partial (n-)cotilting modules;
finitely cotilting ring, iff ${}_RR$ and R_R are finitely cotilting modules;
f-cotilting ring, iff ${}_RR$ and R_R are f-cotilting modules;
Colby-ring, iff ${}_RR_R$ is a Colby-bimodule.

Example 395. By Theorem 303, every (commutative) Artinian n-Gorenstein ring is an n-cotilting ring.

Example 396. (*Trlifaj*, see [Ang:2000, Examples 2.1., 2.4.]) The ring of integers $\mathbb{Z}$ is finitely cotilting that is not cotilting. For a prime number $p \in \mathbb{P}$, the complete discrete valuation ring $\mathbb{J}_p$ of all p-adic integers is a Colby-ring that is not finitely cotilting.

Problem 1. Investigate the *new* classes of rings defined above. In particular, construct examples of (partial) cotilting rings, finitely cotilting rings, f-cotilting rings and Colby-rings.

Problem 2. Give a complete description, *up to equivalence*, for the structure of the 1-cotilting modules over commutative 1-Gorenstein rings. Extend the results to n-Gorenstein rings for $n \geq 2$.

Problem 3. Give a complete description, *up to equivalence,* for the structure of self-cotilting modules over special classes of commutative rings and domains (e.g. Prüfer domains, Dedekind domains, valuation domains).

In light of Theorems 301 and 299 an open problem (raised by J. Trlifaj in [Trl:2007, 4.20.] and [GT:2006]) is:

Problem 4. Characterize the rings over which *all* cotilting modules are equivalent to duals of tilting modules.

<u>Problem 5</u>. ([GT:2006]) Characterize the structure of cotilting modules over Matlis domains.

S. Bazzoni gave in [Baz] a complete description for the structure of cotilting modules *of cofinite type* over Prüfer (valuation) domains. However, not much is known about the structure of n-cotilting modules over *non-Prüfer* domains:

<u>Problem 6</u>. Characterize the 1-cotilting modules (*of cofinite type*) over special classes of *non-Prüfer* commutative rings. Extend the results to n-cotilting modules over for $n > 1$.

Part IV

Appendix: (Co)Homology

Complexes

In this appendix we recall the basic definitions and results concerning the Ext and Tor functors that are needed in this report. Our main reference for the basics of "Homological Algebra" will be [Osb:2000] (in addition to [Rot:1979], [HS:1970] and [Wei-2003]).

Throughout R, S denote associative rings with non-zero units. All modules are assumed to be unitary. With **Ab** we denote the category of Abelian groups, with ${}_R\mathbb{M}, {}_S\mathbb{M}$ the categories of left R-module, left S-modules, respectively, and with $\mathbb{M}_R$, $\mathbb{M}_S$ the categories of right R-modules, right S-modules respectively.

397. With a **degree-k morphism** between $\mathbb{Z}$*-indexed sequences of* R*-modules* $\varphi : \{A_n\}_{n\in\mathbb{Z}} \to \{A'_n\}_{n\in\mathbb{Z}}$ we mean a set of R-linear morphisms $\{\varphi_n : A_n \to A_{n+k}\}_{n\in\mathbb{Z}}$. The class of $\mathbb{Z}$-indexed sequences of R-modules along with degree-0 morphisms between them form an *additive category*, which we denote with $\mathbb{L}_R^{\mathbb{Z}}$.

Chain Complexes:

398. A **chain complex** $\mathbf{C} = (\mathbf{C}, \partial) := \{C_n, \partial_n\}_{n\in\mathbb{Z}}$ over R consists of a set $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules along with a set of R-linear morphisms $\{\partial_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$

$$... \to C_{n+1} \overset{\partial_{n+1}}{\to} C_n \overset{\partial_n}{\to} C_{n-1} \to ...$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. The maps $\{\partial_n\}_{n\in\mathbb{Z}}$ are called **differentials** (or **boundary operators**). For each $n \in \mathbb{Z}$, we set $Z_n := \mathrm{Ker}(\partial_n)$ (called the **nth-cycle**) and $B_n := \mathrm{Im}(\partial_n)$ (called the **nth-boundary**).

399. For two chain complexes $(\mathbf{C}, \partial)$ and $(\widetilde{\mathbf{C}}, \widetilde{\partial})$, a **morphism of chain complexes** $\phi : \mathbf{C} \to \widetilde{\mathbf{C}}$ consists of a class of R-linear morphisms $\{\phi_n : C_n \to \widetilde{C}_n\}$, such that $\phi_{n-1} \circ \partial_n = \widetilde{\partial}_n \circ \phi_n$ for all $n \in \mathbb{Z}$. The class of all chain complexes over R with morphisms of chain complexes form an *Abelian category*, which we denote by $\mathbf{Ch}_R$. Moreover, if R, S are two rings and $F : {}_R\mathbb{M} \to {}_S\mathbb{M}$ is an additive covariant functor, then for every chain complex $\{C_n, \partial_n\}_{n\in\mathbb{Z}}$ over R we have a chain complex over S given by $\{F(C_n), F(\partial_n)\}_{n\in\mathbb{Z}}$.

400. Let $\{C_n, \partial_n\}_{n\in\mathbb{Z}}$ be a chain complex over R. The **homology modules**[2] associated to $\mathbf{C}$ are

$$H_n(\mathbf{C}) := Z_n/B_n = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1}) \text{ for all } n \in \mathbb{Z}, \tag{6.1}$$

and measure the *deviation from exactness* for the chain complex $\mathbf{C}$. It's clear that we have a $\mathbb{Z}$-indexed sequence of R-modules $H(\mathbf{C}) := \{H_n(\mathbf{C})\}_{n\in\mathbb{Z}}$. Every morphism $\phi : \mathbf{C} \to \mathbf{D}$ of chain complexes over R induces a degree-0 morphism of $\mathbb{Z}$-indexed sequences of R-modules $H(\phi) : H(\mathbf{C}) \to H(\mathbf{D})$. Hence, we have a functor, the so called **homology functor**

$$\mathbb{H} : \mathbf{Ch}_R \to \mathbb{L}_R^{\mathbb{Z}},\ \mathbf{C} \mapsto \{H_n(\mathbf{C})\}_{n\in\mathbb{Z}}. \tag{6.2}$$

Definition 401. A chain complex

$$\mathbf{C} :=\ ... \to C_{n+1} \overset{\partial_{n+1}}{\to} C_n \overset{\partial_n}{\to} C_{n-1} \to ... \to C_1 \overset{\partial_1}{\to} C_0 \overset{\partial_0}{\to} \partial \to 0$$

is said to be

projective, iff C_n is projective for each $n \geq 0$;

acyclic, iff $H_n(\mathbf{C}) = 0$ for each $n \geq 1$; equivalently, iff the following long sequence is *exact:*

$$... \to C_{n+1} \overset{\partial_{n+1}}{\to} C_n \overset{\partial_n}{\to} C_{n-1} \to ... \to C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0 \overset{\pi}{\to} H_0(\mathbf{C}) \to 0$$

Cochain Complexes

402. A **cochain complex** $\mathbf{C} = (\mathbf{C}, \delta) := \{C^n, \delta^{n+1}\}_{n\in\mathbb{Z}}$ over R consists of a class of R-modules $\{C^n\}_{n\in\mathbb{Z}}$ and a class of R-linear morphisms $\{\delta^{n+1} : C^n \to C^{n+1}\}_{n\in\mathbb{Z}}$

$$... \to C^{n-1} \overset{\delta^n}{\to} C^n \overset{\delta^{n+1}}{\to} C^{n+1} \to ...$$

[2]called also Homology groups in case $R = \mathbb{Z}$

such that $\delta^{n+1} \circ \delta^n = 0$ for all $n \in \mathbb{Z}$. The maps $\{\delta^n\}_{n\in\mathbb{Z}}$ are called **differentials** (or **coboundary operators**). For each $n \in \mathbb{Z}$ we set $Z^n := \mathrm{Ker}(\delta^{n+1})$ (called the **nth-cocycle**) and $B^n := \mathrm{Im}(\delta^n)$ (called the **nth-coboundary**).

403. For two cochain complexes $(\mathbf{C}, \delta)$ and $(\widetilde{\mathbf{C}}, \widetilde{\delta})$ a **morphism of cochain complexes** $\psi : \mathbf{C} \to \widetilde{\mathbf{C}}$ consists of a class of R-linear morphisms $\{\psi^n : C^n \to \widetilde{C}^n\}$, such that $\psi^{n+1} \circ \delta^{n+1} = \widetilde{\delta}^{n+1} \circ \psi^n$ for all $n \in \mathbb{Z}$. The class of all cochain complexes over R with morphisms of cochain complexes form an Abelian category, which we denote by $\mathbf{CCh}_R$. Moreover, if R, S are two rings and $F : {}_R\mathbb{M} \to {}_S\mathbb{M}$ is an additive covariant functor, then for every cochain complex $\{C^n, \delta^{n+1}\}_{n\in\mathbb{Z}}$ over R we have a cochain complex over S given by $\{F(C^n), F(\delta^{n+1})\}_{n\in\mathbb{Z}}$.

Remark 404. For every chain complex $(\mathbf{C}, \partial) := \{C_n, \partial_n\}_{n\in\mathbb{Z}}$ we can construct a cochain complex $(\mathbf{D}, \boldsymbol{\delta}) := \{D^n, \delta^{n+1}\}_{n\in\mathbb{Z}} = \{C_{-n}, \partial_{-n}\}_{n\in\mathbb{Z}}$ (and vice versa).

405. Let $\{C^n, \delta^{n+1}\}_{n\in\mathbb{Z}}$ be a cochain complex over R. The **cohomology modules**[3] associated to $\mathbf{C}$ are

$$H^n(\mathbf{C}) := Z^n/B^n = \mathrm{Ker}(\delta^{n+1})/\mathrm{Im}(\delta^n) \text{ for all } n \in \mathbb{Z},$$

and *measure the deviation from exactness* of the cochain complex $\mathbf{C}$. It's clear that we have then a $\mathbb{Z}$-indexed sequence of R-modules $H(\mathbf{C}) := \{H^n(\mathbf{C})\}_{n\in\mathbb{Z}}$. Every morphism $\psi : \mathbf{C} \to \mathbf{D}$ of cochain complexes over R induces a degree-0 morphism of $\mathbb{Z}$-indexed sequences of R-modules $H(\psi) : H(\mathbf{C}) \to H(\mathbf{D})$. Hence, we have a functor, the so called **cohomology functor**

$$\mathbb{CH} : \mathbf{CCh}_R \to \mathbb{L}_R^{\mathbb{Z}},\ \mathbf{C} \mapsto \{H^n(\mathbf{C})\}_{n\in\mathbb{Z}}.$$

Definition 406. A cochain complex of R-modules

$$\mathbf{C}:\ 0 \to \delta \xrightarrow{\delta^0} C^0 \xrightarrow{\delta^1} C^1 \xrightarrow{\delta^2} C^2 \to ... \to C^{n-1} \xrightarrow{\delta^n} C^n \xrightarrow{\delta^{n+1}} C^{n+1} \to ...$$

(with $C^n = 0$ for all $n < 0$) is called

injective, iff C^n is injective for all $n \geq 0$;

acyclic, iff the cohomology module $H^n(\mathbf{C}) = 0$ for each $n \geq 0$, equivalently iff the following long sequence is exact

$$0 \to H^0(\mathbf{C}) \xrightarrow{\iota} C^0 \xrightarrow{\delta^1} C^1 \xrightarrow{\delta^2} ... \to C^{n-1} \xrightarrow{\delta^n} C^n \xrightarrow{\delta^{n+1}} C^{n+1} \to ...$$

[3] In case $R = \mathbb{Z}$, these are called the **cohomology groups**

Homological Dimensions

Definition 407. Let A be an R-module. A short exact sequence of R-modules

$$0 \to K \to P \xrightarrow{\pi} A \to 0 \tag{6.3}$$

with P_R free (respectively projective, flat) is called a **free** (respectively **projective, flat**) **presentation** of A.

Lemma 408. *Every R-module has a free (respectively projective, flat) presentation.*

Definition 409. A **projective (flat) resolution** of an R-module A is a *long exact sequence*

$$\mathbf{P}: \ \ldots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\pi} A \to 0, \tag{6.4}$$

where P_n is *projective (flat)* for all $n \geq 0$. We usually write

$$\mathbf{P}_A := \ \ldots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{\partial_1} P_0$$

(so that $\mathbf{P}$ can be rewritten as $\mathbf{P}_A \xrightarrow{\pi} A \to 0$) and

$$\mathbf{P}_{\widetilde{A}} := \mathbf{P}_A \to 0: \ \ldots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} 0.$$

Remark 410. Let A be an R-module with a projective (flat) resolution $\mathbf{P}$ (6.4). Then

$$H_0(\mathbf{P}_{\widetilde{A}}) = \mathrm{Ker}(\partial_0)/\,\mathrm{Im}(\partial_1) = P_0/\mathrm{Ker}(\pi) \simeq A.$$

So a projective (flat) resolution of A is a *projective (flat) acyclic* chain complex

$$\mathbf{C} := \ \ldots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \ldots \to C_1 \xrightarrow{\partial_1} C_0$$

(for which $C_n = 0$ for $n < 0$) together with an isomorphism of R-modules $H_0(\mathbf{C}) \simeq A$.

Lemma 411. *Every R-module A has a free (projective, flat) resolution.*

Definition 412. Let B be an R-module. A short exact sequence of R-modules
$$0 \to B \xrightarrow{\iota} E \to G \to 0 \tag{6.5}$$
with ${}_RE$ injective is called an **injective copresentation** of B.

Lemma 413. *Every R-module has an injective copresentation.*

Definition 414. An **injective coresolution** of an R-module B is a long exact sequence
$$\mathbf{E}:\ 0 \to B \xrightarrow{\iota} E^0 \xrightarrow{\delta^1} E^1 \to \ldots \to E^{n-1} \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} E^{n+1} \to \ldots \tag{6.6}$$
where E^n is injective for all $n \geq 0$. We usually write
$$\mathbf{E}_B:\ E^0 \xrightarrow{\delta^1} E^1 \to \ldots \to E^{n-1} \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} E^{n+1} \to \ldots$$
(so that $\mathbf{E}$ can be rewritten as $0 \to B \xrightarrow{\iota} \mathbf{E}_A$) and
$$\mathbf{E}_{\widetilde{B}} := 0 \to \mathbf{E}_B:\ 0 \xrightarrow{\delta^0} E^0 \xrightarrow{\delta^1} E^1 \xrightarrow{\delta^2} E^2 \to \ldots \to E^n \xrightarrow{\delta^{n+1}} E^{n+1} \to \ldots$$

Remark 415. Let B be an R-module with an injective coresolution $\mathbf{E}$ (6.6). Then
$$H^0(\mathbf{E}_{\widetilde{B}}) = \mathrm{Ker}(\delta^1)/\,\mathrm{Im}(\delta^0) \simeq \mathrm{Ker}(\delta^1) = \mathrm{Im}(\iota) = B. \tag{6.7}$$
So an injective coresolution of B is an *injective acyclic* cochain complex
$$\mathbf{C}:\ C^0 \xrightarrow{\delta^1} C^1 \xrightarrow{\delta^2} C^3 \to \ldots \to C^{n-1} \xrightarrow{\delta^n} C^n \xrightarrow{\delta^{n+1}} C^{n+1} \to \ldots$$
(for which $C^n = 0$ for $n < 0$) together with an isomorphism of R-modules $H^0(\mathbf{C}) \simeq B$.

Lemma 416. *Every R-module B has an injective coresolution.*

Derived Functors

In what follows we introduce the notions of left and right derived functors of *additive functors*. Let R, S be rings. The reader should be warned that different authors interchange the definitions of left and right derived functors (especially those of contravariant functors). In what follows we follow [Osb:2000].

Definition 417. A covariant functor $F : {}_R\mathbb{M} \to {}_S\mathbb{M}$ is called

additive, iff $F(\bigoplus\limits_{j=1}^{n} M_j) = \bigoplus\limits_{j=1}^{n} F(M_j)$ for any finite set of R-modules $\{M_1, ..., M_n\}$;

strongly additive, iff $F(\bigoplus\limits_{\Lambda} M_\lambda) = \bigoplus\limits_{\Lambda} F(M_\lambda)$ for any set of R-modules $\{M_\lambda\}_\Lambda$.

Left Derived Covariant Functors

418. Let $F : {}_R\mathbb{M} \to {}_S\mathbb{M}$ be an *additive covariant functor*. For an R-module M pick a *projective resolution* of M :

$$... \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \overset{\partial_n}{\to} P_{n-1} \to ... \to P_1 \overset{\partial_1}{\to} P_0 \overset{\pi}{\to} M \to 0.$$

Applying F to the chain complex of R-modules

$$\mathbf{P}_{\widetilde{M}} : \ ... \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \overset{\partial_n}{\to} P_{n-1} \to ... \to P_1 \overset{\partial_1}{\to} P_0 \overset{\partial_0}{\to} 0,$$

we get a chain complex of S-modules:

$$... \to F(P_{n+1}) \overset{F(\partial_{n+1})}{\to} F(P_n) \overset{F(\partial_n)}{\to} F(P_{n-1}) \to ... \to F(P_1) \overset{F(\partial_1)}{\to} F(P_0) \overset{F(\partial_0)}{\to} 0$$

Define

$$\mathcal{L}_n F(M) := H_n(F(\mathbf{P}_{\widetilde{M}})) := \mathrm{Ker}(F(\partial_n)) / \mathrm{Im}(F(\partial_{n+1})), \text{ for } n \geq 0.$$

It can be shown that S-modules $\mathcal{L}_n F(M)$ are (up to isomorphism) independent of the projective resolution of M used to compute them; and that, moreover, we have *additive covariant* functors (called the **left derived functors of** F):

$$\mathcal{L}_n F(\bullet) : {}_R\mathbb{M} \to {}_S\mathbb{M}, \text{ for } n \geq 0.$$

Remark 419. Let $F : {}_R\mathbb{M} \to {}_S\mathbb{M}$ be an *additive right exact covariant functor*. If M is an R-module, then applying F to a projective resolution $\mathbf{P} : \mathbf{P}_M \to M \to 0$ of M yields an exact sequence of S-modules

$$F(P_1) \overset{F(\partial_1)}{\to} F(P_0) \overset{F(\pi)}{\to} F(M) \to 0;$$

and applying it to the chain complex of R-modules $\mathbf{P}_{\widetilde{M}} := \mathbf{P}_M \overset{\partial_0}{\to} 0$ yields the chain complex of S-modules

$$... \to F(P_1) \overset{F(\partial_1)}{\to} F(P_0) \overset{F(\partial_0)}{\to} 0.$$

Consequently, we get isomorphisms of S-modules

$$\begin{array}{lclcl} \mathcal{L}_0 F(M) & := & H_0(F(\mathbf{P}_{\widetilde{M}})) & = & \mathrm{Ker}(F(\partial_0))/\mathrm{Im}(F(\partial_1)) \\ & = & F(P_0)/\mathrm{Ker}(F(\pi)) & \simeq & F(M).\blacksquare \end{array} \tag{6.8}$$

Left Derived Contravariant Functors

420. Let $G : {}_R\mathbb{M} \rightarrow {}_S\mathbb{M}$ be an *additive contravariant* functor. For an R-module M pick a *projective resolution* of M :

$$... \rightarrow P_{n+1} \overset{\partial_{n+1}}{\rightarrow} P_n \overset{\partial_n}{\rightarrow} P_{n-1} \rightarrow ... \rightarrow P_1 \overset{\partial_1}{\rightarrow} P_0 \overset{\pi}{\rightarrow} M \rightarrow 0.$$

Applying G to the chain complex

$$\mathbf{P}_{\widetilde{M}} : \ ... \rightarrow P_{n+1} \overset{\partial_{n+1}}{\rightarrow} P_n \overset{\partial_n}{\rightarrow} P_{n-1} \rightarrow ... \rightarrow P_1 \overset{\partial_1}{\rightarrow} P_0 \overset{\partial_0}{\rightarrow} 0,$$

we get a cochain complex of S-modules:

$$0 \overset{G(\partial_0)}{\rightarrow} G(P_0) \overset{G(\partial_1)}{\rightarrow} G(P_1) \rightarrow ... \rightarrow G(P_{n-1}) \overset{G(\partial_n)}{\rightarrow} G(P_n) \overset{G(\partial_{n+1})}{\rightarrow} G(P_{n+1}) \rightarrow ...$$

Define

$$\mathcal{L}^n G(M) := H^n(G(\mathbf{P}_{\widetilde{M}})) := \mathrm{Ker}(G(\partial_{n+1}))/\,\mathrm{Im}(G(\partial_n)), \text{ for } n \geq 0. \quad (6.9)$$

It can be shown that the S-modules $\mathcal{L}^n G(M)$ are (up to isomorphism) independent of the projective resolution of M used to compute them and that, moreover, we have *additive contravariant* functors (called the **left derived functors of** G):

$$\{\mathcal{L}^n G(\bullet) : \ {}_R\mathbb{M} \rightarrow \ {}_S\mathbb{M}, \text{ for } n \geq 0.$$

Remark 421. Let $G : {}_R\mathbb{M} \rightarrow {}_S\mathbb{M}$ be an *additive left exact contravariant functor*. If M is an R-module, then applying G to a projective resolution $\mathbf{P} : \mathbf{P}_M \rightarrow M \rightarrow 0$ of M yields a short exact sequence of R-modules

$$0 \rightarrow G(M) \overset{G(\pi)}{\rightarrow} G(P_0) \overset{G(\partial_1)}{\rightarrow} G(P_1);$$

and applying it to the cochain complex of R-modules $\mathbf{P}_{\widetilde{M}} := \mathbf{P}_M \overset{\partial_0}{\rightarrow} 0$ yields the cochain complex of S-modules

$$0 \overset{G(\partial_0)}{\rightarrow} G(P_0) \overset{G(\partial_1)}{\rightarrow} G(P_1) \rightarrow ...$$

Consequently, we get isomorphisms of S-modules

$$\begin{array}{rclcl} \mathcal{L}^0 G(M) & := & H^0(G(\mathbf{P}_{\widetilde{M}})) & = & \mathrm{Ker}(G(\partial_1))/\mathrm{Im}(G(\partial_0)) \\ & := & \mathrm{Ker}(G(\partial_1)) & = & \mathrm{Im}(G(\pi)) \\ & = & G(M).\blacksquare & & \end{array} \quad (6.10)$$

Right Derived Covariant Functors

422. Let $F : {}_R\mathbb{M} \to {}_S\mathbb{M}$ be an *additive covariant functor*. For an R-module M pick an *injective coresolution*

$$\mathbf{E}: \; 0 \to M \xrightarrow{\iota} E^0 \xrightarrow{\delta^1} E^1 \to ... \to E^{n-1} \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} E^{n+1} \to ...$$

Applying F to the cochain complex of R-modules

$$\mathbf{E}_{\widetilde{M}}: \; 0 \xrightarrow{\delta^0} E^0 \xrightarrow{\delta^1} E^1 \to ... \to E^{n-1} \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} E^{n+1} \to ...,$$

we get a cochain complex of S-modules

$$0 \xrightarrow{F(\delta^0)} F(E^0) \xrightarrow{F(\delta^1)} F(E^1) \to ... \to F(E^{n-1}) \xrightarrow{F(\delta^n)} F(E^n) \xrightarrow{F(\delta^{n+1})} F(E^{n+1}) \to ...$$

Define

$$\mathcal{R}^n F(M) := H^n(F(\mathbf{E}_{\widetilde{M}})) := \operatorname{Ker}(F(\delta^{n+1}))/\operatorname{Im}(F(\delta^n)), \text{ for } n \geq 0. \quad (6.11)$$

It can be shown that the S-modules $\mathcal{R}^n F(M)$ are (up to isomorphism) independent of the injective coresolution of M used to compute them and that, moreover, we have *additive covariant* functors (called the **right derived functors of** F):

$$\mathcal{R}^n F(\bullet): \; {}_R\mathbb{M} \to \; {}_S\mathbb{M}, \text{ for } n \geq 0.$$

Remark 423. Let $F : {}_R\mathbb{M} \to {}_S\mathbb{M}$ be an *additive left exact covariant functor*. If M is an R-module, then applying F to an injective coresolution $\mathbf{E} : 0 \xrightarrow{\iota} M \to \mathbf{E}_M$, yields an exact sequence

$$0 \to F(M) \xrightarrow{F(\iota)} F(E^0) \xrightarrow{F(\delta^1)} F(E^1);$$

and applying it to $\mathbf{E}_{\widetilde{M}} := 0 \xrightarrow{\delta^0} E^0 \xrightarrow{\delta^1} E^1 \to ...$ yields the cochain complex

$$0 \xrightarrow{F(\delta^0)} F(E^0) \xrightarrow{F(\delta^1)} F(E^1) \to ...$$

Consequently, we get isomorphisms of S-modules

$$\begin{array}{rcl} \mathcal{R}_0 F(M) & := & H^0(F(\mathbf{E}_{\widetilde{M}})) \;\; = \;\; \operatorname{Ker}(F(\delta^1))/\operatorname{Im}(F(\delta^0)) \\ & = & \operatorname{Ker}(F(\delta^1)) \;\; = \;\; \operatorname{Im}(F(\iota)) \\ & = & F(M). \blacksquare \end{array}$$

Right Derived Contravariant Functors

424. Let $G : {}_R\mathbb{M} \to {}_S\mathbb{M}$ be an *additive contravariant functor*. For an R-module M pick an *injective coresolution*

$$\mathbf{E}:\ 0 \to M \xrightarrow{\iota} E^0 \xrightarrow{\delta^1} E^1 \to ... \to E^{n-1} \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} E^{n+1} \to ...$$

Applying G to the cochain complex

$$\mathbf{E}_{\widetilde{M}}:\ 0 \xrightarrow{\delta^0} E^0 \xrightarrow{\delta^1} E^1 \to ... \to E^{n-1} \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} E^{n+1} \to ...$$

we get a cochain complex of S-modules

$$G(E^{n+1}) \xrightarrow{G(\delta^{n+1})} G(E^n) \xrightarrow{G(\delta^n)} G(E^{n-1}) \to ... \to G(E^1) \xrightarrow{G(\delta^1)} G(E^0) \xrightarrow{G(\delta^0)} 0$$

Define

$$\mathcal{R}^n G(M) := H^n(G(\mathbf{E}_{\widetilde{M}})) := \mathrm{Ker}(G(\delta^n))/\,\mathrm{Im}(G(\delta^{n+1})),\ \text{for } n \geq 0. \quad (6.12)$$

It can be shown that the S-modules $\mathcal{R}^n G(M)$ are (up to isomorphism) independent of the injective coresolution of M used to compute them and that, moreover, we have *additive contravariant* functors (called the **right derived functors of** G):

$$\mathcal{R}^n G(\bullet):\ {}_R\mathbb{M} \to\ {}_S\mathbb{M},\ \text{for } n \geq 0.$$

Remark 425. Let $G : {}_R\mathbb{M} \to {}_S\mathbb{M}$ be an *additive right exact contravariant functor*. For any ${}_RM$, applying G to an injective coresolution $\mathbf{E} : 0 \to M \xrightarrow{\iota} \mathbf{E}_M$ yields an exact sequence

$$G(E^1) \xrightarrow{G(\delta^1)} G(E^0) \xrightarrow{G(\iota)} G(M) \to 0;$$

and applying it to the cochain complex of R-modules $\mathbf{E}_{\widetilde{M}} := 0 \xrightarrow{\delta^0} E^0 \xrightarrow{\delta^1} E^1 \to ...$ yields the cochain complex of S-modules

$$... \to G(E^1) \xrightarrow{G(\delta^1)} G(E^0) \xrightarrow{G(\delta^0)} 0$$

Consequently, we get isomorphisms of S-modules

$$\begin{array}{rcccl} \mathcal{R}^0 G(M) & := & H^0(G(\mathbf{E}_{\widetilde{M}})) & = & \mathrm{Ker}(G(\delta^0))/\,\mathrm{Im}(G(\delta^1)) \\ & = & G(E^0)/\mathrm{Ker}(G(\iota)) & \simeq & G(M).\blacksquare \end{array}$$

Ext and Tor Functors

For an R-module M we set

$$\begin{array}{lll}
\mathbf{P}_{\widetilde{M}} & := & ... \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \overset{\partial_n}{\to} P_{n-1} \to ... \to P_1 \overset{\partial_1}{\to} P_0 \longrightarrow 0 \\
 & & \text{(obtained from a } \textit{projective resolution } \mathbf{P}_{\widetilde{M}} \to M \to 0); \\
\mathbf{F}_{\widetilde{M}} & := & ... \to F_{n+1} \overset{\partial_{n+1}}{\to} F_n \overset{\partial_n}{\to} F_{n-1} \to ... \to F_1 \overset{\partial_1}{\to} F_0 \longrightarrow 0; \\
 & & \text{(obtained from a } \textit{flat resolution } \mathbf{F}_{\widetilde{M}} \to M \to 0); \\
\mathbf{E}_{\widetilde{M}} & := & 0 \to E^0 \overset{\delta^1}{\to} E^1 \to ... \to E^{n-1} \overset{\delta^n}{\to} E^n \overset{\delta^{n+1}}{\to} E^{n+1} \to ... \\
 & & \text{(obtained from an } \textit{injective coresolution } 0 \to M \to \mathbf{E}_{\widetilde{M}}); \\
\mathfrak{P}_M & := & 0 \to X \to P \overset{\pi}{\to} M \to 0 \text{ (a projective presentation of } {}_R M); \\
\mathfrak{F}_M & := & 0 \to X \to F \overset{\pi}{\to} M \to 0 \text{ (a flat presentation of } {}_R M); \\
\mathfrak{E}_M & := & 0 \to M \overset{\iota}{\to} E \to Z \to 0 \text{ (an injective copresentation of } {}_R M);
\end{array}$$

The following table provides a summary of the definitions presented in the sequel:

$$\begin{array}{llllll}
\mathrm{Ext}_R^n(\bullet, B) & := & (\mathcal{L}^n \mathrm{Hom}_R(-, B))(\bullet), & \mathrm{Ext}_R^n(A, B) & = & H^n(\mathrm{Hom}_R(\mathbf{P}_{\widetilde{A}}, B)); \\
\overline{\mathrm{Ext}}_R^n(A, \bullet) & := & (\mathcal{R}_n \mathrm{Hom}_R(A, -))(\bullet), & \overline{\mathrm{Ext}}_R^n(A, B) & = & H^n(\mathrm{Hom}_R(A, \mathbf{E}_{\widetilde{B}})); \\
 & & & & & \\
\mathrm{Tor}_n^R(\bullet, B) & := & (\mathcal{L}^n(- \otimes_R B))(\bullet), & \mathrm{Tor}_n^R(A, B) & = & H_n(\mathbf{P}_{\widetilde{A}} \otimes_R B)); \\
\overline{\mathrm{Tor}}_n^R(A, \bullet) & := & (\mathcal{L}^n(A \otimes_R -))(\bullet), & \overline{\mathrm{Tor}}_n^R(A, B) & = & H_n(A \otimes_R \mathbf{P}_{\widetilde{B}})); \\
 & & & & & \\
\mathrm{tor}_n^R(A, \bullet) & := & H_n(\mathbf{F}_{\widetilde{A}} \otimes_R -)), & \mathrm{tor}_n^R(A, B) & = & H_n(\mathbf{F}_{\widetilde{A}} \otimes_R B)); \\
\overline{\mathrm{tor}}_n^R(\bullet, B) & := & H_n(A \otimes_R \mathbf{F}_{\widetilde{B}})), & \overline{\mathrm{tor}}_n^R(A, B) & = & H_n(A \otimes_R \mathbf{F}_{\widetilde{B}}));
\end{array} \tag{6.13}$$

The ext-functors

The Functor $\mathrm{ext}_R(A, \bullet)$:

426. Let A be an R-module. For any R-module B, applying $\mathrm{Hom}_R(-, B)$ to an arbitrary *projective presentation* of A :

$$0 \longrightarrow K \xrightarrow{\iota} P \xrightarrow{\pi} A \to 0.$$

we get the exact sequence of Abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(A, B) \xrightarrow{-\circ\pi} \mathrm{Hom}_R(P, B) \xrightarrow{-\circ\iota} \mathrm{Hom}_R(K, B).$$

We define

$$\mathrm{ext}_R(A, B) := \mathrm{coker}(-\circ\iota) = \mathrm{Hom}_R(K, B)/\mathrm{Im}(-\circ\iota), \tag{6.14}$$

i.e. $\mathrm{ext}_R(A, B)$ is the *Abelian group* that yields an exact sequence of Abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(A, B) \xrightarrow{-\circ\pi} \mathrm{Hom}_R(P, B) \xrightarrow{-\circ\iota} \mathrm{Hom}_R(K, B) \longrightarrow \mathrm{ext}_R(A, B) \longrightarrow 0. \tag{6.15}$$

It can be shown that Abelian groups $\mathrm{ext}_R(A, B)$ are (up to isomorphism) independent of the projective resolution of ${}_RA$ used to compute them; and that, moreover, we have an additive covariant functor

$$\mathrm{ext}_R(A, \bullet) : \ {}_R\mathbb{M} \to \mathbf{Ab}.$$

Remark 427. Let A, B be R-modules and pick a projective presentation of A :

$$0 \to K \xrightarrow{\iota} P \xrightarrow{\pi} A \longrightarrow 0.$$

Define an equivalence relation on $\mathrm{Hom}_R(K, B)$:

$$\varphi_1 \sim \varphi_2 \Leftrightarrow \varphi_1 - \varphi_2 = \psi \circ \iota \text{ for some } \psi \in \mathrm{Hom}_R(P, B).$$

Then $\mathrm{ext}_R(A, B)$ can be shown to be isomorphic to the Abelian group of *equivalence classes* $\mathrm{Hom}_R(K, B)/\sim$, i.e.

$$\mathrm{ext}_R(A, B) \simeq \mathrm{Hom}_R(K, B)/\sim \ = \{[\varphi] \mid \varphi \in \mathrm{Hom}_R(K, B) \ \}.$$

The Functor $\overline{\text{ext}}_R(A, \bullet)$:

428. Let B be an R-module and

$$0 \longrightarrow B \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 0$$

be an injective copresentation. For an R-module A, applying $\text{Hom}_R(A, -)$ yields an exact sequence of Abelian groups

$$0 \longrightarrow \text{Hom}_R(A, B) \xrightarrow{\iota\circ-} \text{Hom}_R(A, E) \xrightarrow{\pi\circ-} \text{Hom}_R(A, G)$$

We define

$$\overline{\text{ext}}_R(A, B) := \text{coker}(\pi \circ -) = \text{Hom}_R(A, G)/\text{Im}(\pi \circ -); \tag{6.16}$$

i.e. $\overline{\text{ext}}_R(A, B)$ is the *Abelian group* that yields an exact sequence of Abelian groups

$$0 \longrightarrow \text{Hom}_R(A, B) \xrightarrow{\iota\circ-} \text{Hom}_R(A, E) \xrightarrow{\pi\circ-} \text{Hom}_R(A, G) \longrightarrow \overline{\text{ext}}_R(A, B) \longrightarrow 0. \tag{6.17}$$

It can be shown that Abelian groups $\overline{\text{ext}}_R(A, B)$ are (up to isomorphism) independent of the injective copresentation of ${}_RB$ used to compute then; and that, moreover, we have an additive covariant functor

$$\overline{\text{ext}}_R(\bullet, B) : \ {}_R\mathbb{M} \to \mathbf{Ab}.$$

Ext and Extensions

Definition 429. Let A, B be R-modules. An **extension of** B **by** A[4] is a short exact sequence of R-modules

$$0 \to B \to E \to A \to 0.$$

430. Two extensions $0 \to B \to E \to A$ and $0 \to B \to E' \to A \to 0$ of B by A are **equivalent**, iff there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & B & \xrightarrow{\iota} & E & \xrightarrow{\pi} & A & \to & 0 \\ & & \| & & \downarrow\simeq & & \| & & \\ 0 & \to & B & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & A & \to & 0 \end{array} \tag{6.18}$$

With $\mathrm{Ext}_R(A, B)$ we denote the set of extensions of B by A.

Definition 431. An extension of B by A is said to be **split**, iff it's equivalent to the *canonical extension*

$$0 \to B \xrightarrow{\iota_B} A \oplus B \xrightarrow{\pi_A} A \to 0. \tag{6.19}$$

With $S(A, B)$ we denote the class of split extensions of B by A.

Lemma 432. *If A, B are R-modules with $\mathrm{Ext}_R(A, B) = 0$, then every extension of B by A splits.*

Theorem 433. *For any R-modules A, B we have a bijection*

$$\mathrm{Ext}_R(A, B) \overset{\Theta}{\longleftrightarrow} \mathrm{ext}_R(A, B). \tag{6.20}$$

Corollary 434. *Given two R-modules A, B, the set $\mathrm{Ext}_R(A, B)$ has the structure of an Abelian group with neutral element $S(A, B) \overset{\Theta}{\longleftrightarrow} 0_{\mathrm{ext}_R(A,B)}$.*

[4] Some authors would say, E is an **extension of** A **by** B

Theorem 435. *For any R-modules A, B we have isomorphisms of Abelian groups*

$$\mathrm{Ext}_R(A,B) \simeq \mathrm{ext}_R(A,B) \simeq \overline{\mathrm{ext}}_R(A,B).$$

Lemma 436. *Let A, B be R-modules and $\{M_\lambda\}_\Lambda$ be a class of R-modules. For every $n \geq 0$, we have isomorphisms of Abelian groups*

$$\mathrm{Ext}_R(\bigoplus_{\lambda\in\Lambda} M_\lambda, B) = \prod_{\lambda\in\Lambda} \mathrm{Ext}_R(M_\lambda, B) \text{ and } \mathrm{Ext}_R(A, \prod_{\lambda\in\Lambda} M_\lambda) = \prod_{\lambda\in\Lambda} \mathrm{Ext}_R(A, M_\lambda). \tag{6.21}$$

Theorem 437. ([HS:1970, Theorem III.5.2.]) *Consider an exact sequence of R-modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \to 0.$$

1. *For every R-module A, we have a* connecting morphism $\beta : \mathrm{Hom}_R(A,N) \dashrightarrow \mathrm{Ext}_R(A,L)$ *such that the following sequence of Abelian groups is exact*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(A,L) & \longrightarrow & \mathrm{Hom}_R(A,M) & \longrightarrow & \mathrm{Hom}_R(A,N) \\ & \overset{\beta}{\dashrightarrow} & \mathrm{Ext}_R(A,L) & \longrightarrow & \mathrm{Ext}_R(A,M) & \longrightarrow & \mathrm{Ext}_R(A,N) \end{array}$$

2. *For every R-module B, we have a* connecting morphism $\beta : \mathrm{Hom}_R(A,N) \dashrightarrow \mathrm{Ext}_R(A,L)$ *such that the following sequence of Abelian groups is exact*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(N,B) & \longrightarrow & \mathrm{Hom}_R(M,B) & \longrightarrow & \mathrm{Hom}_R(L,B) \\ & \overset{\gamma}{\dashrightarrow} & \mathrm{Ext}_R(N,B) & \longrightarrow & \mathrm{Ext}_R(M,B) & \longrightarrow & \mathrm{Ext}_R(L,B). \end{array}$$

Theorem 438. *The following are equivalent for an R-module A :*

1. *A_R is projective;*
2. $\mathrm{Hom}_R(A,-)$ *is (right) exact;*
3. $\mathrm{Ext}_R(A,M) = 0$ *for every R-module M.*

Theorem 439. *The following are equivalent for an R-module B :*

1. *B is injective;*
2. $\mathrm{Hom}_R(-,B)$ *is (right) exact;*
3. $\mathrm{Ext}_R(M,B) = 0$ *for every R-module M.*

The Functors $\mathrm{Ext}_R^n(\bullet, B)$:

440. Let B be an R-module and consider the *additive left exact contravariant functor* $\mathrm{Hom}_R(-, B) : {}_R\mathbb{M} \to \mathbf{Ab}$. Associated to B is a class of additive contravariant functors

$$\mathrm{Ext}_R^n(\bullet, B) := \mathcal{L}^n \mathrm{Hom}_R(-, B)(\bullet),\ n \geq 0. \tag{6.22}$$

As clarified above, to find $\mathrm{Ext}_R^n(A, B) := \mathcal{L}^n \mathrm{Hom}_R(-, B)(A)$ for some R-module A, we pick a *projective resolution* $\mathbf{P}_A$ of A and compute the *cohomology groups*

$$\mathrm{Ext}_R^n(A, B) := H^n(\mathrm{Hom}_R(\mathbf{P}_{\widetilde{A}}, B)),\ n \geq 0.$$

It follows directly from Remark 421 that we have

Proposition 441. *Consider an R-module B. For every R-module A we have a natural isomorphism of Abelian groups* $\mathrm{Ext}_R^0(A, B) \simeq \mathrm{Hom}_R(A, B)$.

The Functors $\overline{\mathrm{Ext}}_R^n(A, \bullet)$:

442. Let A be an R-module and consider the *additive left exact covariant functor* $\mathrm{Hom}_R(A, -) : {}_R\mathbb{M} \to \mathbf{Ab}$. Associated to A is a set of additive covariant functors

$$\overline{\mathrm{Ext}}_R^n(A, \bullet) := \mathcal{R}_n \mathrm{Hom}_R(A, -)(\bullet),\ n \geq 0. \tag{6.23}$$

As clarified above, to compute $\overline{\mathrm{Ext}}_R^n(A, B)$ for some R-module B, pick an injective coresolution $\mathbf{E}_B$ of B and find the *cohomology groups*

$$\overline{\mathrm{Ext}}_R^n(A, B) := H^n(\mathrm{Hom}_R(A, \mathbf{E}_{\widetilde{B}})),\ n \geq 0.$$

It follows directly from Remark 423 that we have

Proposition 443. *Let A be an R-module. For any R-module B we have a natural isomorphism of Abelian groups* $\overline{\mathrm{Ext}}_R^0(A, B) \simeq \mathrm{Hom}_R(A, B)$.

Proposition 444. *For any R-modules A, B, we have natural isomorphisms of Abelian groups*

$$\overline{\mathrm{Ext}}_R^n(A, B) \simeq \mathrm{Ext}_R^n(A, B), \text{ for } n \geq 0. \tag{6.24}$$

Lemma 445. *Let A, B be R-modules and $\{M_\lambda\}_\Lambda$ be a class of R-modules. For every $n \geq 0$, we have isomorphisms of Abelian groups*

$$\mathrm{Ext}_R^n(\bigoplus_{\lambda\in\Lambda} M_\lambda, B) = \prod_{\lambda\in\Lambda} \mathrm{Ext}_R^n(M_\lambda, B) \text{ and } \mathrm{Ext}_R^n(A, \prod_{\lambda\in\Lambda} M_\lambda) = \prod_{\lambda\in\Lambda} \mathrm{Ext}_R^n(A, M_\lambda). \tag{6.25}$$

Proposition 446. *For any R-modules A, B, we have natural isomorphisms of Abelian groups*

$$\begin{array}{ccccc} \mathrm{Ext}_R^1(A, B) & \simeq & \mathrm{Ext}_R(A, B) & \simeq & \overline{\mathrm{Ext}}_R^1(A, B) \\ & \simeq & \mathrm{ext}_R(A, B) & \simeq & \overline{\mathrm{ext}}_R(A, B). \end{array}$$

Theorem 447. *Consider a short exact sequence of R-modules*

$$0 \to L \to M \to N \to 0.$$

1. *For every R-module A, we have a sequence of* connecting morphisms *of Abelian groups* $\{\beta_n : \mathrm{Ext}_R^{n-1}(A,N) \dashrightarrow \mathrm{Ext}_R^n(A,L)\}_{n\geq 1}$ *such that the following* long sequence *of Abelian groups is exact:*

$$\begin{array}{llllll}
0 & \to & \mathrm{Hom}_R(A,L) & \to & \mathrm{Hom}_R(A,M) & \to & \mathrm{Hom}_R(A,N) \\
& \overset{\beta_1}{\dashrightarrow} & \mathrm{Ext}_R^1(A,L) & \to & \mathrm{Ext}_R^1(A,M) & \to & \mathrm{Ext}_R^1(A,N) \\
& \overset{\beta_2}{\dashrightarrow} & \mathrm{Ext}_R^2(A,L) & \to & \mathrm{Ext}_R^2(A,M) & \to & \mathrm{Ext}_R^2(A,N) \\
& \dashrightarrow & \ldots & \to & \ldots & \to & \ldots \\
& \overset{\beta_{n-1}}{\dashrightarrow} & \mathrm{Ext}_R^{n-1}(A,L) & \to & \mathrm{Ext}_R^{n-1}(A,M) & \to & \mathrm{Ext}_R^{n-1}(A,N) \\
& \overset{\beta_n}{\dashrightarrow} & \mathrm{Ext}_R^n(A,L) & \to & \mathrm{Ext}_R^n(A,M) & \to & \mathrm{Ext}_R^n(A,N) \\
& \overset{\beta_{n+1}}{\dashrightarrow} & \mathrm{Ext}_R^{n+1}(A,L) & \to & \mathrm{Ext}_R^{n+1}(A,M) & \to & \mathrm{Ext}_R^{n+1}(A,N) \\
& \dashrightarrow & \ldots & \to & \ldots & \to & \ldots
\end{array}$$

2. *For every R-module B, we have a sequence of* connecting morphisms $\{\gamma^n : \mathrm{Ext}_R^{n-1}(L,B) \to \mathrm{Ext}_R^n(N,B)\}_{n\geq 1}$ *such that the following* long sequence *of Abelian groups is exact:*

$$\begin{array}{llllll}
0 & \to & \mathrm{Hom}_R(N,B) & \to & \mathrm{Hom}_R(M,B) & \to & \mathrm{Hom}_R(L,B) \\
& \overset{\gamma^1}{\dashrightarrow} & \mathrm{Ext}_R^1(N,B) & \to & \mathrm{Ext}_R^1(M,B) & \to & \mathrm{Ext}_R^1(L,B) \\
& \overset{\gamma^2}{\dashrightarrow} & \mathrm{Ext}_R^2(N,B) & \to & \mathrm{Ext}_R^2(M,B) & \to & \mathrm{Ext}_R^2(L,B) \\
& \dashrightarrow & \ldots & \to & \ldots & \to & \ldots \\
& \overset{\gamma^{n-1}}{\dashrightarrow} & \mathrm{Ext}_R^{n-1}(N,B) & \to & \mathrm{Ext}_R^{n-1}(M,B) & \to & \mathrm{Ext}_R^{n-1}(L,B) \\
& \overset{\gamma^n}{\dashrightarrow} & \mathrm{Ext}_R^n(N,B) & \to & \mathrm{Ext}_R^n(M,B) & \to & \mathrm{Ext}_R^n(L,B) \\
& \overset{\gamma^{n+1}}{\dashrightarrow} & \mathrm{Ext}_R^{n+1}(N,B) & \to & \mathrm{Ext}_R^{n+1}(M,B) & \to & \mathrm{Ext}_R^{n+1}(L,B) \\
& \dashrightarrow & \ldots & \to & \ldots & \to & \ldots
\end{array}$$

The Projective (Injective) Dimension:

Definition 448. The **projective dimension** of an R-module A is defined as

$$\text{proj.dim.}({}_RA) := \inf\{n \mid \text{Ext}_R^{n+1}(A, \bullet) \equiv 0\}.$$

Theorem 449. ([Wei-2003, Lemma 4.1.6.]) *The following are equivalent for an R-module A and $n \geq 0$:*

1. $\text{proj.dim.}(A) \leq n$;
2. $\text{Ext}_R^{n+l}(A, M) = 0$ *for every R-module M;*
3. $\text{Ext}_R^{n+l}(A, M) = 0$ *for all $l \geq 1$ and all R-modules M;*
4. *There exist projective R-modules $P_0, ..., P_n$ fitting in an exact sequences of R-modules*
$$0 \rightarrow P_n \xrightarrow{f_n} ... \rightarrow P_0 \xrightarrow{f_0} A \rightarrow 0;$$
5. *In any projective resolution*
$$... \rightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \rightarrow ... \rightarrow P_0 \xrightarrow{\partial_0} A \rightarrow 0,$$
*of A, the n**th syzygy module** $K_n := \text{Ker}(\partial_n)$ is projective.*

Definition 450. The **injective dimension** of an R-module B is defined as

$$\text{inj.dim.}({}_RB) := \inf\{n \mid \text{Ext}_R^{n+1}(\bullet, B) \equiv 0\}.$$

Theorem 451. ([Wei-2003, Lemma 4.1.6.]) *The following are equivalent for an R-module B :*

1. $\text{inj.}\dim(B) \leq n$;
2. $\text{Ext}_R^{n+l}(M, B) = 0$ *for all $l \geq 1$ and all R-modules M;*
3. $\text{Ext}_R^{n+1}(M, B) = 0$ *for every R-module M;*
4. $\text{Ext}_R^{n+1}(R/I, B) = 0$ *for every left R-ideal I;*

5. *There exist injective R-modules $E^0, ..., E^n$ fitting in an exact sequences of R-modules*

$$0 \to B \xrightarrow{\iota} E^0 \xrightarrow{g^1} E^1 \to ... \to E^{n-1} \xrightarrow{g^n} E^n \to 0;$$

6. *In any injective coresolution*

$$0 \to B \xrightarrow{\iota} E^0 \xrightarrow{\delta^1} E^1 \to ... \to E^{n-1} \xrightarrow{\delta^n} E^n \xrightarrow{\delta^{n+1}} E^{n+1} \to$$

*of B, the n**th cozyzygy module** $C^n := \mathrm{Coker}(\delta^n)$ is injective.*

The Functor $\mathrm{tor}^R(A, \bullet)$:

452. Let A be a right R-module and pick a *flat presentation*

$$0 \xrightarrow{\iota} K \longrightarrow F \xrightarrow{\pi} A \to 0.$$

of A_R. For any left R-module ${}_RB$, applying the *right exact functor* $- \otimes_R B$, yields the exact sequence of Abelian groups

$$K \otimes_R B \xrightarrow{\iota \otimes id_B} F \otimes_R B \xrightarrow{\pi \otimes id_B} A \otimes_R B \to 0.$$

We define

$$\overline{\mathrm{tor}}^R(A, B) := \mathrm{Ker}(\iota \otimes id_B), \tag{6.26}$$

i.e. $\mathrm{tor}^R(A, B)$ to be the *Abelian group* that yields an exact sequence of Abelian groups

$$0 \longrightarrow \mathrm{tor}^R(A, B) \longrightarrow K \otimes_R B \xrightarrow{\iota \otimes id_B} F \otimes_R B \xrightarrow{\pi \otimes id_B} A \otimes_R B \longrightarrow 0. \tag{6.27}$$

In fact, for each left R-module B, the Abelian groups $\mathrm{tor}^R(A, B)$ are (up to isomorphism) independent of the flat presentations of A_R used to compute them; and, moreover, we have an additive covariant functor

$$\mathrm{tor}^R(A, \bullet) : \ {}_R\mathbb{M} \to \mathbf{Ab}.$$

Theorem 453. *The following are equivalent for a right R-module A :*

1. *A_R is flat;*
2. *$A \otimes_R - : {}_R\mathbb{M} \to \mathbf{Ab}$ is (left) exact;*
3. *$\mathrm{tor}^R(A, M) = 0$ for every left R-module M.*

The Functor $\overline{\mathrm{tor}}^R(\bullet, B)$:

454. Let B be a left R-module and pick a flat presentation of B

$$0 \to S \xrightarrow{\bar{\iota}} U \xrightarrow{\bar{\pi}} B \to 0.$$

For any right R-module A, applying the *right exact functor* $A \otimes_R -$ yields the exact sequence of Abelian groups

$$A \otimes_R S \overset{id_A \otimes \bar{\iota}}{\longrightarrow} A \otimes_R U \overset{id_A \otimes \bar{\pi}}{\longrightarrow} A \otimes_R B \longrightarrow 0.$$

We define

$$\overline{\mathrm{tor}}^R(A, B) := \mathrm{Ker}(id_A \otimes \bar{\iota}), \tag{6.28}$$

i.e. $\overline{\mathrm{tor}}^R(A, B)$ is the *Abelian group* that yields an exact sequence of Abelian groups

$$0 \longrightarrow \overline{\mathrm{tor}}^R(A, B) \longrightarrow A \otimes_R S \overset{id_A \otimes \bar{\iota}}{\longrightarrow} A \otimes_R U \overset{id_A \otimes \bar{\pi}}{\longrightarrow} A \otimes_R B \longrightarrow 0. \tag{6.29}$$

It can be shown that for any right R-module A_R, the Abelian groups $\overline{\mathrm{tor}}^R(A, B)$ are (up to isomorphism) independent of the flat presentation of ${}_RB$ used to evaluate them; and that, moreover, we have an additive covariant functor

$$\overline{\mathrm{tor}}^R(\bullet, B) : \mathbb{M}_R \to \mathbf{Ab}.$$

Theorem 455. *The following are equivalent for a left R-module B :*

1. *${}_RB$ is flat;*
2. *$- \otimes_R B : \mathbb{M}_R \to \mathbf{Ab}$ is (left) exact;*
3. *$\overline{\mathrm{tor}}^R(M, B) = 0$ for every right R-module M.*

The functor $\mathrm{tor}_n^R(A, \bullet)$:

456. Let A be a right R-module and pick a *flat resolution* of A_R :

$$... \to F_{n+1} \overset{d_{n+1}}{\to} F_n \overset{d_n}{\to} F_{n-1} \to ... \to F_1 \overset{d_1}{\to} F_0 \overset{\pi}{\to} A \to 0. \qquad (6.30)$$

For any left R-module B, applying the *right exact functor* $- \otimes_R B$ to (6.30) yields a complex chain of Abelian groups

$$\mathbf{F}_{\widetilde{A}} \otimes_R B : \ ... \to F_{n+1} \otimes_R B \to F_n \otimes_R B \to F_{n-1} \otimes_R B \to ... \to F_0 \otimes_R B \to 0.$$

We define

$$\mathrm{tor}_n^R(A, B) := H_n(\mathbf{F}_{\widetilde{A}} \otimes_R B) = \mathrm{Ker}(d_n \otimes_R id_B)/\mathrm{Im}(d_{n+1} \otimes_R id_B). \qquad (6.31)$$

It can be shown that the Abelian groups $\mathrm{tor}_n^R(A, B)$ are (up to isomorphism) independent of the flat resolution of A_R used to compute them; and that, moreover, we have an additive covariant functor

$$\mathrm{tor}_n^R(A, \bullet) := H_n(\mathbf{F}_{\widetilde{A}} \otimes_R \bullet) : \ {}_R\mathbb{M} \to \mathbf{Ab}.$$

The Functors $\overline{\mathrm{tor}}_n^R(\bullet, B)$:

457. Let B be a left R-module and pick a *flat resolution* of ${}_RB$:

$$\mathbf{F}_B : \ ... \to F_{n+1} \overset{d_{n+1}}{\to} F_n \overset{d_n}{\to} F_{n-1} \to ... \to F_1 \overset{d_1}{\to} F_0 \overset{\pi}{\to} B \to 0. \qquad (6.32)$$

For any right R-module A_R, applying the *right exact functor* $A \otimes_R -$ to the chain complex of R-modules $\mathbf{F}_{\widetilde{B}}$ yields a complex chain of Abelian groups

$$A \otimes_R \mathbf{F}_{\widetilde{B}} : \ ... \to A \otimes_R F_{n+1} \to A \otimes_R F_n \to A \otimes_R F_{n-1} \to ... \to A \otimes_R F_0 \to 0.$$

We define

$$\overline{\mathrm{tor}}_n^R(A, B) := H_n(A \otimes_R \mathbf{F}_{\widetilde{B}}) = \mathrm{Ker}(id_A \otimes_R d_n)/\mathrm{Im}(id_A \otimes_R d_{n+1}). \qquad (6.33)$$

It can be shown that for each ${}_RB$ the Abelian groups $\overline{\mathrm{tor}}_n^R(A, B)$ are (up to isomorphism) independent of the flat resolution of ${}_RB$ used to compute them; and that, moreover, we have covariant functors

$$\overline{\mathrm{tor}}_n^R(\bullet, B) := H_n(\bullet \otimes_R \mathbf{F}_{\widetilde{B}}) : \mathbb{M}_R \to \mathbf{Ab}, \ \text{for } n \geq 0.$$

The Functors $\mathrm{Tor}_R^n(\bullet, B)$:

458. Let B be a left R-module and consider the *additive right exact covariant functor* $- \otimes_R B : \mathbb{M}_R \to \mathbf{Ab}$. Associated to ${}_RB$ is a set of functors

$$\mathrm{Tor}_n^R(\bullet, B) := \mathcal{L}^n(- \otimes_R B)(\bullet), \ n \geq 0. \tag{6.34}$$

As shown above, to compute $\mathrm{Tor}_n^R(A, B) := \mathcal{L}^n(- \otimes_R B)(A)$, for some right R-module A, pick some projective resolution $\mathbf{P}_A$ of A and find the homology groups

$$\mathrm{Tor}_n^R(A, B) = H_n(\mathbf{P}_{\widetilde{A}} \otimes_R B), \ \text{for } n \geq 0.$$

As a consequence of Remark 419 we get

Proposition 459. *Let B be a left R-module. For every right R-module A we have $\mathrm{Tor}_0^R(A, B) = A \otimes_R B$.*

The Functors $\overline{\mathrm{Tor}}_R^n(A, \bullet)$:

460. Let A be a right R-module and consider the *right exact covariant* functor $A \otimes_R - : {}_R\mathbb{M} \to \mathbf{Ab}$. Associated to A is a set of functors

$$\overline{\mathrm{Tor}}_n^R(A, \bullet) := \mathcal{L}^n(A \otimes_R -)(\bullet), \ \text{for } n \geq 0. \tag{6.35}$$

As shown above, to compute $\overline{\mathrm{Tor}}_n^R(A, B)$ for any left R-module B, pick a *projective resolution* $\mathbf{P}_B$ of B and find the homology groups

$$\overline{\mathrm{Tor}}_n^R(A, B) := H_n(A \otimes_R \mathbf{P}_{\widetilde{B}}), \ \text{for } n \geq 0.$$

As a consequence of Remark 419 we get

Proposition 461. *Let A be a right R-module. For every left R-module B we have $\overline{\mathrm{Tor}}_0^R(A, B) \simeq A \otimes_R B$.*

Proposition 462. *For any $n \geq 0$ and any $A \in \mathbb{M}_R$ and $B \in {}_R\mathbb{M}$ we have natural isomorphisms of Abelian groups*

$$\mathrm{Tor}_n^R(A, B) \simeq \overline{\mathrm{tor}}_n^R(A, B) \simeq \mathrm{tor}_n^R(A, B) \simeq \overline{\mathrm{Tor}}_n^R(A, B), \ \textit{for } n \geq 0.$$

Lemma 463. *For any $A \in \mathbb{M}_R$ and $B \in {}_R\mathbb{M}$ we have a natural isomorphism of Abelian groups*

$$\begin{array}{rclcl} \mathrm{Tor}_1^R(A,B) & \simeq & \overline{\mathrm{Tor}}_1^R(A,B) & \simeq & \\ & \simeq & \mathrm{tor}_1^R(A,B) & \simeq & \overline{\mathrm{tor}}_1^R(A,B) \\ & & \mathrm{tor}^R(A,B) & \simeq & \overline{\mathrm{tor}}^R(A,B). \end{array}$$

Lemma 464. *Let A be a right R-module and B a left R-module.*

1. *Let $\{M_\lambda\}_\Lambda$ be a class of left R-modules. For every $n \geq 0$, we have isomorphisms of Abelian groups*

$$\mathrm{Tor}_n^R(A, \bigoplus_{\lambda\in\Lambda} M_\lambda) = \bigoplus_{\lambda\in\Lambda} \mathrm{Tor}_n^R(A, M_\lambda). \tag{6.36}$$

2. *Let $\{N_\lambda\}_\Lambda$ be a class of right R-modules. For every $n \geq 0$, we have isomorphisms of Abelian groups*

$$\mathrm{Tor}_n^R(\bigoplus_{\lambda\in\Lambda} N_\lambda, B) = \bigoplus_{\lambda\in\Lambda} \mathrm{Tor}_n^R(N_\lambda, B).$$

Theorem 465. *1. Consider a short exact sequence of left R-modules*

$$0 \to L \to M \to N \to 0.$$

For every right R-module A_R, we have a sequence of connecting morphisms $\{\kappa_n : \mathrm{Tor}_n^R(A,N) \to \mathrm{Tor}_{n-1}^R(A,L)\}_{n\geq 1}$ *such that the following* long sequence *of Abelian groups is exact:*

$$\begin{array}{lclclcl}
... & \to & ... & \to & ... & \dashrightarrow & \\
\mathrm{Tor}_{n+1}^R(A,L) & \to & \mathrm{Tor}_{n+1}^R(A,M) & \to & \mathrm{Tor}_{n+1}^R(A,N) & \overset{\kappa_{n+1}}{\dashrightarrow} & \\
\mathrm{Tor}_{n}^R(A,L) & \to & \mathrm{Tor}_{n}^R(A,M) & \to & \mathrm{Tor}_{n}^R(A,N) & \overset{\kappa_{n}}{\dashrightarrow} & \\
\mathrm{Tor}_{n-1}^R(A,L) & \to & \mathrm{Tor}_{n-1}^R(A,M) & \to & \mathrm{Tor}_{n-1}^R(A,N) & \overset{\kappa_{n-1}}{\dashrightarrow} & \\
... & \to & ... & \to & ... & \dashrightarrow & \\
\mathrm{Tor}_{1}^R(A,L) & \to & \mathrm{Tor}_{1}^R(A,M) & \to & \mathrm{Tor}_{1}^R(A,N) & \overset{\kappa_{1}}{\dashrightarrow} & \\
A \otimes_R L & \to & A \otimes_R M & \to & A \otimes_R N & \to & 0
\end{array}$$

2. Consider a short exact sequence of right R-modules

$$0 \to L \to M \to N \to 0.$$

For every left R-module ${}_RB$, we have a sequence of connecting morphisms $\{\kappa_n : \mathrm{Tor}_n^R(N,B) \to \mathrm{Tor}_{n-1}^R(L,B)\}_{n\geq 1}$ *such that the following* long sequence *of Abelian groups is exact:*

$$\begin{array}{lclclcl}
... & \to & ... & \to & ... & \dashrightarrow & \\
\mathrm{Tor}_{n+1}^R(L,B) & \to & \mathrm{Tor}_{n+1}^R(M,B) & \to & \mathrm{Tor}_{n+1}^R(N,B) & \overset{\lambda_{n+1}}{\dashrightarrow} & \\
\mathrm{Tor}_{n}^R(L,B) & \to & \mathrm{Tor}_{n}^R(M,B) & \to & \mathrm{Tor}_{n}^R(N,B) & \overset{\lambda_{n}}{\dashrightarrow} & \\
\mathrm{Tor}_{n-1}^R(L,B) & \to & \mathrm{Tor}_{n-1}^R(M,B) & \to & \mathrm{Tor}_{n-1}^R(N,B) & \overset{\lambda_{n-1}}{\dashrightarrow} & \\
... & \to & ... & \to & ... & \dashrightarrow & \\
\mathrm{Tor}_{1}^R(L,B) & \to & \mathrm{Tor}_{1}^R(M,B) & \to & \mathrm{Tor}_{1}^R(N,B) & \overset{\lambda_{1}}{\dashrightarrow} & \\
L \otimes_R B & \to & M \otimes_R B & \to & N \otimes_R B & \to & 0
\end{array}$$

Proposition 466. *Let $n \geq 0$. For any right R-module A_R and any left R-module ${}_RB$, we have a natural isomorphism of Abelian groups*

$$\operatorname{Tor}_n^R(A, B) \simeq \operatorname{Tor}_n^{R^{op}}(B, A).$$

Proposition 467. *If R is a commutative ring, then we have for any $n \geq 0$ and any R-modules A, B a natural isomorphism of R*-modules

$$\operatorname{Tor}_n^R(A, B) \simeq \operatorname{Tor}_n^R(B, A).$$

The Flat Dimension:

Definition 468. We define the

flat dimension of a right R-module A_R as

$$\text{flat.dim.}(A_R) := \inf\{n \mid \mathrm{Tor}_R^{n+1}(A, \bullet) \equiv 0\};$$

the **flat dimension** of a left R-module ${}_RB$ as

$$\text{flat.dim.}({}_RB) := \inf\{n \mid \mathrm{Tor}_R^{n+1}(\bullet, B) \equiv 0\};$$

Theorem 469. ([Wei-2003, Lemma 4.1.6.]) *The following are equivalent for a left R-module ${}_RB$ and $n \geq 0$:*

1. $\text{flat.dim.}({}_RB) \leq n$;
2. $\mathrm{Tor}^R_{n+l}(M, B) = 0$ *for all* $l \geq 1$ *and all right R-modules* M;
3. $\mathrm{Tor}^R_{n+1}(M, B) = 0$ *for every right R-module* M;
4. $\mathrm{Tor}^R_{n+1}(R/I, B) = 0$ *for every finitely generated left R-ideal* I;
5. *There exist flat R-modules $F_0, ..., F_n$ fitting in an exact sequences of R-modules*
$$0 \to F_n \overset{f_n}{\to} ... \to F_0 \overset{f_0}{\to} B \to 0;$$
6. *In any flat resolution*
$$... \to F_{n+1} \overset{\partial_{n+1}}{\to} F_n \overset{\partial_n}{\to} F_{n-1} \to ... \to F_0 \overset{\partial_0}{\to} B \to 0,$$
*of B, the n**th syzygy module** $K_n := \mathrm{Ker}(\partial_n)$ is flat.*

Global and Weak Dimensions

Definition 470. For the ring R, we define the
left global dimension of R as

$$\text{LG.dim.}(R) := \sup\{\text{proj.dim.}({}_RM) \mid M \text{ is a left } R\text{-module}\};$$

right global dimension of R as

$$\text{RG.dim.}(R) := \sup\{\text{proj.dim.}(M_R) \mid M \text{ is a right } R\text{-module}\};$$

left weak dimension of R as

$$\text{LW.dim.}(R) := \sup\{\text{flat.dim.}({}_RM) \mid M \text{ is a left } R\text{-module}\};$$

right weak dimension of R as

$$\text{RW.dim.}(R) := \sup\{\text{flat.dim.}(M_R) \mid M \text{ is a right } R\text{-module}\}.$$

Remarks 471.

1. There exist rings R for which $\text{LG.dim.}(R) \neq \text{RG.dim.}(R)$.
2. For any ring, $\text{LW.dim.}(R) = \text{RW.dim.}(R)$ (called the **weak dimension** of R and denoted by $\text{W.dim.}(R)$).

Theorem 472. *1. We have*

$$\begin{array}{lcl} \text{LG.dim.}(R) & := & \sup\{\text{proj.dim.}({}_RM) \mid M \textit{ is a left } R\textit{-module}\}; \\ & = & \inf\{n \geq 0 \mid \text{Ext}_R^{n+1}(\bullet, \bullet) \equiv 0\}; \\ & = & \sup\{\text{inj.dim.}({}_RM) \mid M \textit{ is a left } R\textit{-module}\}; \\ & = & \sup\{\text{proj.dim.}(R/I) \mid I \textit{ is a left ideal of } R\}; \end{array}$$

2. If $\text{LG.dim.}(R) > 0$, *then*

$$\text{LG.dim.}(R) = 1 + \sup\{\text{proj.dim.}({}_RI) \mid I \textit{ is a left ideal of } R\};$$

3. *We have*

$$\begin{aligned}
\mathrm{W.dim.}(R) &:= \sup\{\mathrm{flat.dim.}({}_RM) \mid M \textit{ is a left } R\textit{-module}\};\\
&= \sup\{\mathrm{flat.dim.}(R/I) \mid I \textit{ is a left ideal of } R\};\\
&= \inf\{n \geq 0 \mid \mathrm{Tor}^R_{n+1}(\bullet, \bullet) \equiv 0\};\\
&= \sup\{\mathrm{flat.dim.}(N_R) \mid N \textit{ is a right } R\textit{-module}\};\\
&= \sup\{\mathrm{flat.dim.}(R/I) \mid I \textit{ is a right ideal of } R\};
\end{aligned}$$

4. *If* $\mathrm{W.dim.}(R) > 0$, *then*

$$\begin{aligned}
\mathrm{W.dim.}(R) &= 1 + \sup\{\mathrm{flat.dim.}({}_RI) \mid I \textit{ is a f.g. left ideal of } R\};\\
&= 1 + \sup\{\mathrm{flat.dim.}({}_RI) \mid I \textit{ is a f.g. right ideal of } R\}.
\end{aligned}$$

Corollary 473. *For the ring R we have*

1. $\mathrm{LG.dim.}(R) \leq 1$ *if and only if every left R-ideal is projective;*

2. $\mathrm{W.dim.}(R) \leq 1$ *if and only if every finitely generated left (right) R-ideal is flat.*

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