King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 382

Jan 2008

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Some Statistical Properties of a Bivariate Chi-Square Probability Density Function

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Keywords and Phrases: Bivariate distribution; Wishart distribution; Bivariate chi-square distribution; correlated chi-square variables; sum, product, and ratio.

AMS Mathematics Subject Classification (2000): 62E15, 60E05, 60E10

Distributions of the sum, difference, product and ratio of two chi-squares variables are well known if the variables are independent. In this paper we derive distributions of some of the above quantities when the variables are correlated through a bivariate chi-square distribution and provided graphs of their density functions. Results match with the independent case when the variables are uncorrelated. An application of the ratio of two correlated chi-squares is referred.

1. Introduction

Fisher (1915) derived the distribution of the bivariate matrix A in order to study the distribution of correlation coefficient for a bivariate normal sample. Wishart (1928) obtained the distribution of p-dimensional matrix A as the joint distribution of sample variances and covariances from multivariate normal population. The bivariate matrix A is said to have a Wishart distribution with parameters $m = N - 1 > 2$ and $\Sigma(2\times 2) > 0$, written as $A \sim W_2(m, \Sigma)$. The joint density function of the elements of A can be written as

$$f(a_{11}, a_{22}, a_{12}) = \frac{(1-\rho^2)^{-m/2}(\sigma_1\sigma_2)^{-m}}{2^m\sqrt{\pi}\Gamma(m/2)\Gamma((m-1)/2)}\left(a_{11}a_{22} - a_{12}^2\right)^{(m-3)/2} \times \exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2} - \frac{a_{22}}{2(1-\rho^2)\sigma_2^2} + \frac{\rho a_{12}}{(1-\rho^2)\sigma_1\sigma_2}\right) \quad (1.1)$$

$a_{11} > 0,\ a_{22} > 0,\ -\sqrt{a_{11}a_{22}} < a_{12} < \sqrt{a_{11}a_{22}}$, $m > 2$ (Anderson, 2003, 123). The quantity ρ $(-1 < \rho < 1)$ is the product moment correlation coefficient between X_1 and X_2.

For the estimation of correlation coefficient by modern techniques, we refer to Ahmed (1992). The joint density function of $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$, called the bivarite chi-square distribution, was derived by Joarder (2007) in the spirit of Krishnaiah, Hagis and Steinberg (1963) who studied the bivariate chi-distribution. The product moment correlation coefficient between U and V can be calculated to be ρ^2. In case the correlation coefficient $\rho = 0$, the density function of U and V becomes that of the product of two independent chi-square variables each with m degrees of freedom. We refer to Kotz, Balakrishnan and Johnson (2000) for other type of bivariate chi-square distribution.

With the help of simple transformations in the density function of the bivariate chi-square distribution (Theorem 3.1), we derive the joint distribution of the sum ($Y = U + V$) and the product ($W = UV$) in Theorem 4.1. Then we exploit simple transformations to derive the distribution of the sum (Y), the product (W), the ratio ($H = U/V$), the ratio ($T = \sqrt{U/V}$) and the ratio ($G = \sigma_1^2 U/(\sigma_2^2 V)$). Wells, Anderson and Cell (1962) derived the distribution of W, the product of two independent chi-square variables with degrees of freedom m_1 and m_2. Note that Springer (1979, 365) derived the same but with some misprints. We derive the distribution of $W = UV$ in Theorem 4.3 when U and V have a bivariate chi-square distribution with common degrees of freedom m. Our contribution is more general than Wells Anderson and Cell (1962) in the sense of accommodating correlated chi-square variables U and V. In case the variables are uncorrelated, Theorem 4.3 matches with Wells, Anderson and Cell for $m_1 = m_2$. Note that some of the results in the paper follows from Krishnaiah, Hagis and Steinberg (1963).

Ratios of two independent chi-squares are widely used in statistical tests of hypotheses. The distribution of $H = U/V$ for $\sigma_1 = \sigma_2$ reported in Kotz, Balakrishnan and Johnson (2000) misses a constant. If $\sigma_1 = \sigma_2$, Finney (1938) derived the sampling distribution of the square root of the ratio of correlated chi-squares variables ($T = \sqrt{U/V}$) directly from the joint distribution of sample variances and correlation coefficient. He compared the variability of the measurements of standing height and stem length for different age groups of schoolboys by his method with the help of Hirschfeld (1937). The distribution of T derived in our paper (section 4) from the bivariate chi-square distribution (Theorem 3.1) matches exactly with Finney (1938). Some works have also been done by Cohen (1986) and Wilcox (1989) for comparing variability of correlated random variables but not resorting to the functional form of the bivariate chi-square distribution. We remark that distribution of linear combinations of correlated chi-squares or some functions of them can also be derived along Provost (1986).

2. Some Preliminaries

Let $f(x, y)$ be the joint density function of X and Y. Then the following lemmas are well known.

Lemma 2.1 Let X and Y be two random variables with common probability density function $f(x)$. Further let $W_1 = X + Y$. Then the density function of W_1 at w is given by

$$g_1(w) = \int_0^\infty f(w - y, y)dy .$$

Lemma 2.3 Let X and Y have the joint density function $f(x, y)$. Further let $W_3 = X/Y$. Then the p.d.f. of W_3 at w is given by

$$g_3(w) = \int_0^\infty f(yw, y)ydy .$$

In what follows we will be using the duplication formula of gamma function

$$\Gamma(2z)\sqrt{\pi} = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{2.1}$$

Multiplying both sides by $2z$ we have

$$(2z)!\sqrt{\pi} = 2^{2z} z!\Gamma\left(z+\tfrac{1}{2}\right). \tag{2.2}$$

The hypergeometric function ${}_pF_q(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots,b_q;z)$ is defined by

$$\begin{aligned}&{}_pF_q(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots,b_q;z)\\&=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+k)}{\Gamma(a_1)}\frac{\Gamma(a_2+k)}{\Gamma(a_2)}\cdots\frac{\Gamma(a_p+k)}{\Gamma(a_p)}\left(\frac{\Gamma(b_1+k)}{\Gamma(b_1)}\frac{\Gamma(b_2+k)}{\Gamma(b_2)}\cdots\frac{\Gamma(b_q+k)}{\Gamma(b_q)}\right)^{-1}\frac{z^k}{k!}.\end{aligned} \tag{2.3}$$

Note that ${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b;c;z)$ can be transformed as

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b;c;z) \tag{2.4}$$

(Gradshtyen and Ryzhik, 1992, 1069)

The error function given by

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{2.5}$$

which enjoys the property: $erf(-x) = -erf(x)$. The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \tag{2.6}$$

which is related to error function (Weisstein, 1999) through

$$E_{1/2}(z) = e^{z^2}[1+erf(z)]. \tag{2.7}$$

The exact distribution of some sampling statistics derived in this paper will involve Macdonald function which admits the following integral representations:

1. $K_\alpha(z) = \dfrac{(\frac{z}{2})^\alpha\sqrt{\pi}}{\Gamma(\alpha+\frac{1}{2})}\displaystyle\int_1^\infty (t^2-1)^{(2\alpha-1)/2}e^{-zt}dt,\ \alpha > -1/2,\ z>0$ (2.8)

(Watson, 1995, 185), (Gradshteyn and Ryzhik, 1994, 969, #8.432(3)).

2. $K_\alpha(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\alpha \displaystyle\int_0^\infty t^{-(\alpha+1)}\exp\left(-t-\frac{z^2}{4t}\right)dt,\ z>0$ (2.9)

(Watson, 1995, 183), (Gradshteyn and Ryzhik, 1994, 969, #8.432(6)).

3. $\int_0^\infty x^{\alpha-1}(x+\beta)^{\alpha-1}e^{-\mu x}dx = \frac{\Gamma(\alpha)}{\sqrt{\pi}}(\beta/\mu)^{\alpha-(1/2)}e^{\beta\mu/2}K_{(1/2)-\alpha}(\beta\mu/2),\ \alpha>0,\ \mu>0$ (2.10)

(Gradshteyn and Ryzhik, 1994, 365, 3.383(8)).

It may be mentioned that the function $K_\alpha(x)$ is variously called as Bessel function of the second kind with imaginary argument, modified Bessel function of the third kind, Macdonald function, Basset function or modified Hankel function. Series representations for different orders are available in Spainer and Oldham (1987, p 502).

3. The Density Function of the Bivariate Chi-square Distribution

Consider a random sample of size N represented by $X_j = (X_{1j}, X_{2j}, \cdots, X_{pj})',\ j = 1,2,\cdots,N$ where $X_j \sim N_p(\mu,\Sigma)$, $\mu = (\mu_1,\mu_2,\cdots,\mu_p)'$ and $\Sigma = (\sigma_{ik})$, $i = 1,2,\cdots,p;\ k = 1,2,\cdots,p$ is a $p\times p$ positive definite matrix. Then $U_i = \sum_{j=1}^{N}(X_{ij}-\bar{X}_i)^2/\sigma^2 \sim \chi_m^2\ (i = 1,2,\cdots,p)$ with $m = N-1$, and the joint distribution of $U_1, U_2, \cdots, U_p$ is called a *p*-variate chi-square distribution (Krishnaiah, Hagis and Steinberg, 1963). In this paper, however, we restrict ourselves to bivariate chi-square distribution $(p = 2)$. The following theorem is due to Joarder (2007) though it follows from Krishnaiah, Hagis and Steinberg (1963).

Theorem 3.1 The random variables U and V are said to have a correlated bivariate chi-square distribution each with m degrees of freedom, if its density function is given by

$$f(u,v) = \frac{(uv)^{(m-2)/2}}{2^m\ \Gamma^2(m/2)(1-\rho^2)^{m/2}}\exp\left(-\frac{u+v}{2-2\rho^2}\right)\ {}_0F_1\left(\frac{m}{2};\frac{\rho^2 uv}{(2-2\rho^2)^2}\right) \qquad (3.1)$$

where ${}_0F_1(;b;z)$ is defined in (2.4).

It is easy to check that the density function in (3.1) has the following representation:

$$f(u,v) = \frac{2^{-(m+1)}(uv)^{(m-2)/2}\ e^{\frac{-(u+v)}{2(1-\rho^2)}}}{\sqrt{\pi}\ \Gamma(\frac{m}{2})(1-\rho^2)^{m/2}}\sum_{k=0}^{\infty}[1+(-1)^k]\left(\frac{\rho\sqrt{uv}}{1-\rho^2}\right)^k\frac{\Gamma\left(\frac{k+1}{2}\right)}{k!\,\Gamma\left(\frac{k+m}{2}\right)}\ . \qquad (3.2)$$

We remark that $f(u,v;\rho) = f(u,v;-\rho)$ and that $f(u,v;\rho) = f(v,u;\rho)$. In case $\rho = 0$, the density function of the joint probability distribution in Theorem 3.1, would be that of the product of two independent chi-square random variables given by

$$f(u,v) = \frac{(uv)^{m/2-1}\ e^{-(u+v)/2}}{2^m\ \Gamma^2(m/2)},\ u>0, v>0.$$

It has been checked with MATHEMATICA 5.0 that the function in (3.1) integrates to 1. Figure 1 in the Appendix shows the bivariate surface of the density function in (3.1) for various values of ρ for m = 5.

4. Some Functions of Correlated Chi-square Variables

Theorem 4.1 Let U and V be two correlated chi-square variables with p.d.f. given by Theorem 3.1. Then the joint density function of $Y = U + V$ and $W = UV$ is given by

$$f_1(y,w) = \frac{w^{(m-2)/2} \exp\left(\dfrac{-y}{2-2\rho^2}\right)(y^2-4w)^{-1/2}}{2^{m-1}\,\Gamma^2(m/2)(1-\rho^2)^{m/2}} \; {}_0F_1\left(\frac{m}{2};\frac{\rho^2 w}{(2-2\rho^2)^2}\right) \tag{4.1}$$

where $\{0<w<y^2/4,\ 0<y<\infty\}$ or, $\{0<w<\infty, 2\sqrt{w}<y<\infty\}$, $m>2, -1<\rho<1$.

Equation (4.1) also has the following alternative computational form:

$$f_1(y,w) = \frac{w^{(m-2)/2} \exp\left(\dfrac{-y}{2-2\rho^2}\right)(y^2-4w)^{-1/2}}{2^{m}\,\sqrt{\pi}\Gamma(m/2)(1-\rho^2)^{m/2}} \times \sum_{k=0}^{\infty}[1+(-1)^k]\left(\frac{\rho\sqrt{w}}{1-\rho^2}\right)^k \frac{\Gamma\left(\frac{k+1}{2}\right)}{k!\,\Gamma\left(\frac{k+m}{2}\right)}$$

Proof. It is well known that $(a+b)^2 > 4ab$ which implies $a+b>2\sqrt{ab}$ and $ab<(a+b)^2/4$. Hence $y=u+v>2\sqrt{uv}=2\sqrt{w}$ and $w=uv<(u+v)^2/4=y^2/4$. With these, the Jacobian of the transformation is $|J(u,v\to y,w)| = \left(y^2-4w\right)^{-1/2}, y>2\sqrt{w}$. Then the joint probability density function of Y and V follows from Theorem 3.1.

Corollary 4.1 If $\rho=0$, then the joint density function of sum and product of two chi-square variables would be

$$f_2^*(y,w) = \frac{w^{(m-2)/2}e^{-y/2}(y^2-4w)^{-1/2}}{2^{m-1}\Gamma^2(m/2)}, \quad 0<w<y^2/4,\ 0<y<\infty,\ m>2. \tag{4.2}$$

By substituting $\rho=0$ into equation (4.2) and integrating out y, we obtain the following density function of W .

$$\frac{w^{(m-2)/2}}{2^{m-1}\Gamma^2(m/2)}K_0(\sqrt{w}).$$

By substituting $\rho=0$ into equation (4.2) and integrating out w, it can be proved that $Y \sim \chi^2_{2m}$.

Theorem 4.2 Let U and V be two correlated chi-square variables with density function given by Theorem 3.1. Then the density function of $Y = U + V$ is given by

$$f_3(y)=\frac{y^{m-1}\exp\left(\frac{-y}{2-2\rho^2}\right)}{2^m\ \Gamma(m)(1-\rho^2)^{m/2}}\ {}_0F_1\left(\frac{m+1}{2};\frac{\rho^2y^2}{(4-4\rho^2)^2}\right),\ y>0 \qquad (4.3)$$

where $m>2, -1<\rho<1$.

The function in (4.3) has the following computational form:

$$f_3(y)=\frac{y^{m-1}\exp\left(\frac{-y}{2-2\rho^2}\right)}{2^{2m}\ (1-\rho^2)^{m/2}\Gamma(m/2)}\sum_{k=0}^{\infty}[1+(-1)^k]\left(\frac{\rho y}{2-2\rho^2}\right)^k\frac{\Gamma\left(\frac{k+1}{2}\right)}{k!\Gamma\left(\frac{k+m+1}{2}\right)},$$

Proof. The density function of y in (4.3) follows from Theorem 4.1 by integrating over w $(0<w<y^2/4)$.

Figure 2 in the Appendix shows the graph of the density function of the sum of two chi-square variables (Theorem 4.2) for various values of ρ for $m=5$. If $\rho=0$, then the density function in Theorem 4.2 becomes that of χ^2_{2m}.

In addition, by substituting $u=4wy^{-2}$ with $dw=(y^2/4)du$ $(0<u<1)$, in (4.2), it can be also be checked that $Y\sim\chi^2_{2m}$.

Alternatively, the density function of $Y=U+V$ can be derived directly from Theorem 3.1 by the use of the convolution formula in Lemma 2.1.

Theorem 4.3 Let U and V be two correlated chi-square variables with density function given by Theorem 3.1. Then the density function of $W=UV$ is given by

$$f_4(w)=\frac{(1-\rho^2)^{-m/2}w^{(m-2)/2}}{2^{m-1}\ \Gamma^2(m/2)}K_0\left(\frac{\sqrt{w}}{1-\rho^2}\right){}_0F_1\left(\frac{m}{2};\frac{\rho^2w}{(2-2\rho^2)^2}\right),\qquad w>0 \qquad (4.4)$$

where $m>2,\ -1<\rho<1$.

The function in (4.4) has the following computational form:

$$f_4(w)=\frac{(1-\rho^2)^{-m/2}w^{(m-2)/2}}{2^m\sqrt{\pi}\ \Gamma(m/2)}K_0\left(\frac{\sqrt{w}}{1-\rho^2}\right)\sum_{k=0}^{\infty}[1+(-1)^k]\left(\frac{\rho\sqrt{w}}{1-\rho^2}\right)^k\frac{\Gamma\left(\frac{k+1}{2}\right)}{k!\Gamma\left(\frac{k+m}{2}\right)}.$$

Proof. Equation (4.4) follows from the joint density function $f_1(y,w)$ in equation (4.1) by integrating out y over the support space $y>2\sqrt{w}$.

Alternatively, the density function of $W=UV$ can be derived directly from Theorem 3.1 by the use of Lemma 2.1.

It has been checked with MATHEMATICA 5.0 that the function $f_4(w)$ in Theorem 4.3 integrates to 1. Figure 3 in the Appendix shows the graph of the density function of the product of two chi-square variables (Theorem 4.3) for various values of ρ for $m=5$. If W is the product of two independent chi-square variables with degrees of freedom m_1 and m_2, then

$$f_5(w)=\frac{w^{\frac{1}{4}(m_1+m_2)-1}}{2^{\frac{1}{2}(m_1+m_2)-1}\Gamma(m_1/2)\Gamma(m_2/2)}K_{\frac{1}{2}(m_1-m_2)}(\sqrt{w}),\ w>0 \quad (4.5)$$

(Wells, Anderson and Cell, 1962) where $K_\alpha(x)$ is the modified Bessel function of the third kind defined in Section 2. Note that Springer (1979, 365) derived the above but there is a misprint in the density function. Note also that if $\rho=0$ and $m_1=m_2$ in $f_5(w)$, it reduces to $f_4(w)$.

Corollary 4.2 In case U and V are independent, the density function of $W=UV$ is given by

$$f_6(w)=\frac{w^{(m-2)/2}}{2^{m-1}\ \Gamma^2(m/2)}K_0(\sqrt{w}),\ \ w>0. \quad (4.6)$$

Similarly substituting $t=y/(2\sqrt{w})$ in (4.2) with $dy=2\sqrt{w}\,dt$ $(0<t<\infty)$, it can be proved that W has the above density function. Note that it matches with Wells, Anderson and Cell (1962) when $m_1=m_2=m$. It has been checked with MATHEMATICA 5.0 that the function $f_6(w)$ in Corollary 4.2 integrates to 1.

Theorem 4.4 The density function of $H=U/V$ is given by

$$f_7(h)=\frac{2^{m-1}(1-\rho^2)^{m/2}}{B\left(\frac{1}{2},\frac{m}{2}\right)}\frac{h^{(m-2)/2}}{(1+h)^m}\left(1-\frac{4\rho^2 h}{(1+h)^2}\right)^{-(m+1)/2},\ h>0. \quad (4.7)$$

Proof. By using Lemma 2.3 and some algebraic manipulation in Theorem 4.1 we have equation (4.7).

Note that Kotz, Balakrishnan and Johnson (2000, 452) misses the constant 2^{m-1} in the above density function. We have checked that the density function $f_7(w)$ in Theorem 4.4 integrates to 1. Figure 4 in the Appendix shows the graph of the density function of the ratio of two chi-square variables (Theorem 4.4) for various values of ρ for $m=5$.

In case $\rho=0$, the density function in the theorem reduces to

$$f_7^*(h)=\frac{2^{m-1}\Gamma((m+1)/2)}{\sqrt{\pi}\ \Gamma(m/2)}\ \frac{h^{(m-2)/2}}{(1+h)^m}$$

which, by virtue of the duplication formula of gamma function (2.4) with $2z=m$, simplifies to

$$f_7^*(h)=\frac{\Gamma(m)}{\Gamma^2(m/2)}\ \frac{h^{(m-2)/2}}{(1+h)^m}$$

which is the density function of the ratio of the two independent chi-square variables each with m degrees of freedom.

It follows from Theorem 4.4 that the density function of $T = \sqrt{H}$ is given by

$$f_8(t) = \frac{2(1-\rho^2)^{m/2}}{B(m/2, m/2)} \frac{t^{m-1}}{(1+t^2)^m} \left(1 - \frac{4\rho^2 t^2}{(1+t^2)^2}\right)^{-(m+1)/2} \tag{4.8}$$

which coincides with Bose (1935) or Finney (1938). The density function of

$$G = \frac{\sigma_1^2 U}{\sigma_2^2 V} = \frac{\sigma_1^2 H}{\sigma_2^2}$$

is given by

$$f_9(g) = \frac{2^{m-1}(\sigma_2^2/\sigma_1^2)^{m/2}(1-\rho^2)^{m/2}}{B(1/2,\ m/2)} \frac{(g\sigma_2^2/\sigma_1^2)^{(m-2)/2}}{(1+g\sigma_2^2/\sigma_1^2)^m} \times \left(1 - \frac{4g\rho^2\sigma_2^2/\sigma_1^2}{(1+g\sigma_2^2/\sigma_1^2)^2}\right)^{-(m+1)/2},\ g > 0. \tag{4.9}$$

<u>5. Marginal and Conditional Distributions</u>

<u>Theorem 5.1</u> Let U and V be two correlated chi-square variables with density function given by Theorem 2.1. Then both U and V will be distributed as independent chi-squares with common degrees of freedom m .

<u>Proof</u>. By integrating out v from the joint density function in (2.1), we have

$$f_1(u) = \frac{u^{(m-2)/2} e^{\frac{-u}{2(1-\rho^2)}}}{2^{m+1}\sqrt{\pi}\ \Gamma(m/2)(1-\rho^2)^{m/2}} \int_0^\infty v^{(m+k-2)/2} e^{\frac{-v}{2(1-\rho^2)}} {}_0F_1\left(\frac{m}{2}; \frac{\rho^2 u}{4(1-\rho^2)^2}\right) dv\ . \tag{5.1}$$

Since

$${}_0F_1\left(\frac{m}{2}; \frac{\rho^2 u}{4(1-\rho^2)^2}\right) = \frac{\Gamma(m/2)}{2\sqrt{\pi}} \sum_{k=0}^\infty [1+(-1)^k] \left(\frac{\rho\sqrt{u}}{1-\rho^2}\right)^k \frac{\Gamma((k+1)/2)}{k!\,\Gamma((k+m)/2)},$$

(see 3.2 for example), evaluating the gamma integral in (5.1), the summand turns into

$$[2(1-\rho^2)]^{m/2} \sum_{k=0}^\infty [1+(-1)^k] \left(\frac{\rho\sqrt{2u}}{\sqrt{1-\rho^2}}\right)^k \frac{\Gamma((k+1)/2)}{k!}\ . \tag{5.2}$$

Using $2z = k+1$ in the duplication formula of gamma function (2.4), we have

$$\sum_{k=0}^{\infty}\left(\frac{\rho\sqrt{2u}}{\sqrt{1-\rho^2}}\right)^k \frac{\Gamma\left((k+1)/2\right)}{k\,!} = \sqrt{\pi}\sum_{k=0}^{\infty}\left(\frac{\rho\sqrt{u}}{\sqrt{2(1-\rho^2)}}\right)^k \frac{1}{\Gamma\left(\frac{k}{2}+1\right)}$$

$$= \sqrt{\pi}E_{1/2}(z)$$

$$= \sqrt{\pi}e^{z^2}[1+erf\,(z)]$$

where $z\sqrt{2(1-\rho^2)} = \rho\sqrt{u}$ and $E_{\alpha}(z)$ is the Mittag-Leffler function given by (2.6). Similarly, it can be proved that

$$\sum_{k=0}^{\infty}\left(\frac{-\rho\sqrt{2u}}{\sqrt{1-\rho^2}}\right)^k \frac{\Gamma\left((k+1)/2\right)}{k\,!} = \sqrt{\pi}E_{1/2}(-z)$$

$$= \sqrt{\pi}e^{z^2}[1+erf\,(-z)]$$

$$= \sqrt{\pi}e^{z^2}[1-erf\,(z)]$$

so that (5.2) turns into

$$[2(1-\rho^2)]^{m/2}\times 2\sqrt{\pi}e^{z^2} = 2^{(m+2)/2}\sqrt{\pi}(1-\rho^2)^{m/2}e^{\frac{u\rho^2}{2(1-\rho^2)}}.$$

If this is plugged into (5.1), $f_1(u)$ simplifies to the density function of univariate chi-square distribution with m degrees of freedom. Krishnaiah, Hagis and Steinberg (1963) reported the a-th moment of a chi variable.

Theorem 5.2 Let U and V be two correlated chi-square variables with density function given by Theorem 2.1. Then the conditional density function of U given $V=v$ will be given by

$$f_2(u\,|v) = \frac{u^{(m-2)/2}}{\Gamma(m/2)(2-2\rho^2)^{m/2}}\exp\left(\frac{-u-v\rho^2)}{2-2\rho^2)}\right){}_0F_1\left(\frac{m}{2};\frac{\rho^2 uv}{(2-2\rho^2)^2}\right) \tag{5.3}$$

Proof. The theorem follows from

$$f_2(u\,|v) = \frac{f\,(u,v)}{f_2(v)}$$

where $f_2(v)$ is the density function of chi-square distribution with m degrees of freedom.

Theorem 5.3 Let U and V be two correlated chi-square variables with density function given by Theorem 2.1**.** Then the regression function of U given $V=v$ is given by

$$E(U\,|V=v) = m(1-\rho^2)+v\rho^2, \quad -1<\rho<1$$

Proof. The regression function of U given $V=v$ follows from the definition

$$E(U|V=v)=\int_0^\infty u f_2(u|v)du.$$

Theorem 5.4 Let U and V be two correlated chi-square variables with density function given by Theorem 2.1. Then the a-th moment of the conditional distribution of U given V is given by

$$E(U^a|V)=\frac{(2-2\rho^2)^a}{\Gamma(m/2)}\exp\left(\frac{-v\rho^2}{2-2\rho^2}\right)\Gamma\left(a+\frac{m}{2}\right){}_1F_1\left(a+\frac{m}{2};\frac{m}{2};\frac{v\rho^2}{2-2\rho^2}\right) \tag{5.4}$$

Proof. By definition, we have $E(U^a|V=v)=\int_0^\infty u^a f_2(u|v)du$

$$E(U^a|V=v)=\frac{v^{(m-2)/2}}{2^{m/2}\Gamma(m/2)(1-\rho^2)^{m/2}}\exp\left(\frac{-v}{2(1-\rho^2)}\right)$$
$$\times\int_0^\infty u^a\exp\left(\frac{-u\rho^2}{2(1-\rho^2)}\right){}_0F_1\left(\frac{m}{2};\frac{\rho^2uv}{4(1-\rho^2)^2}\right)du$$

By expanding the hypergeometric function, and completing the gamma integral we get (5.4). It is easily checked that $E(U^0|V)=1$. With some algebraic manipulations, it can also be checked that $E(U^1|V)=v\rho^2+m(1-\rho^2)$.

Acknowledgements The authors gratefully acknowledge the excellent research facility provided by King Fahd University of Petroleum & Minerals. In particular, the first author gratefully acknowledges the research support provided through the FT 2004-22 project.

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Appendix

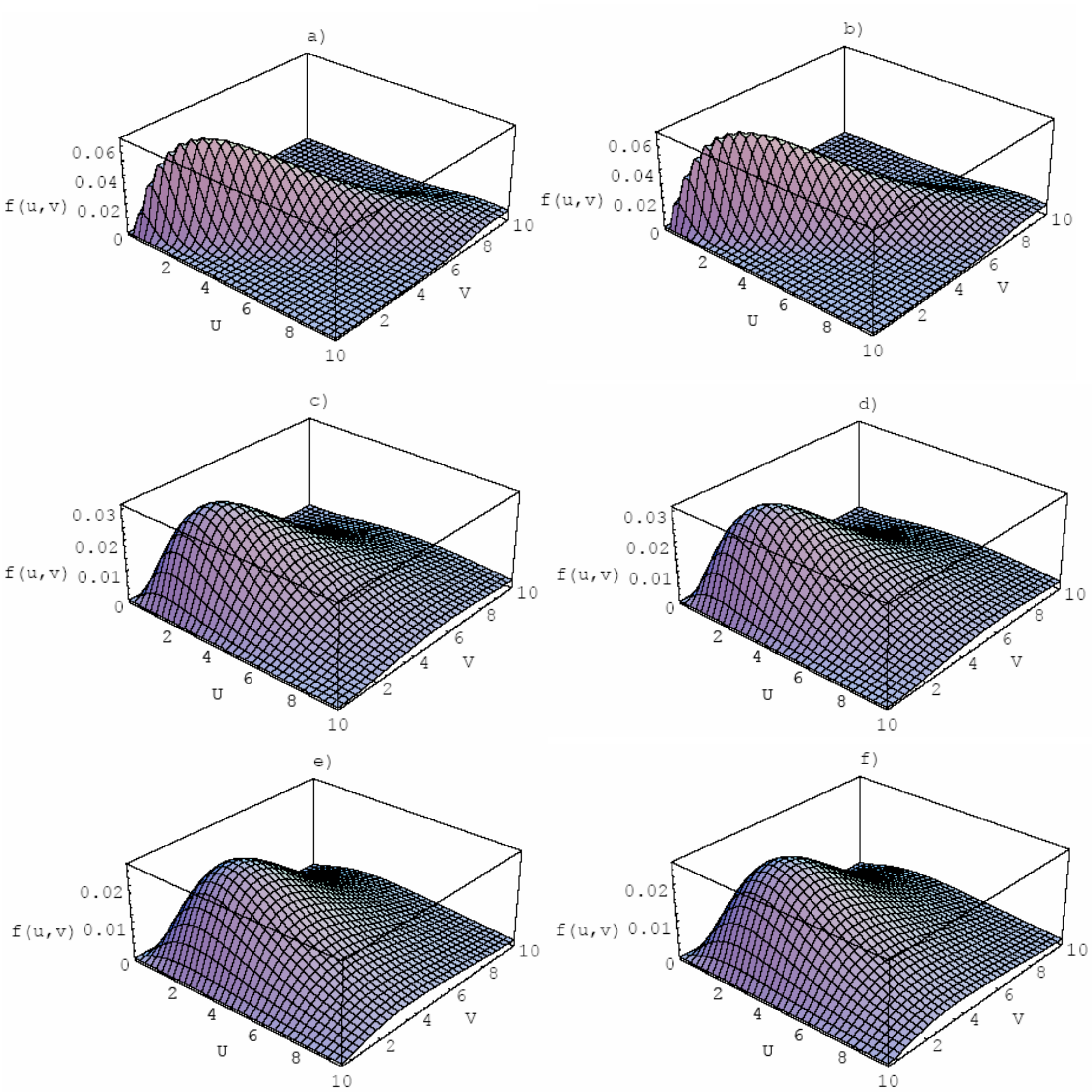
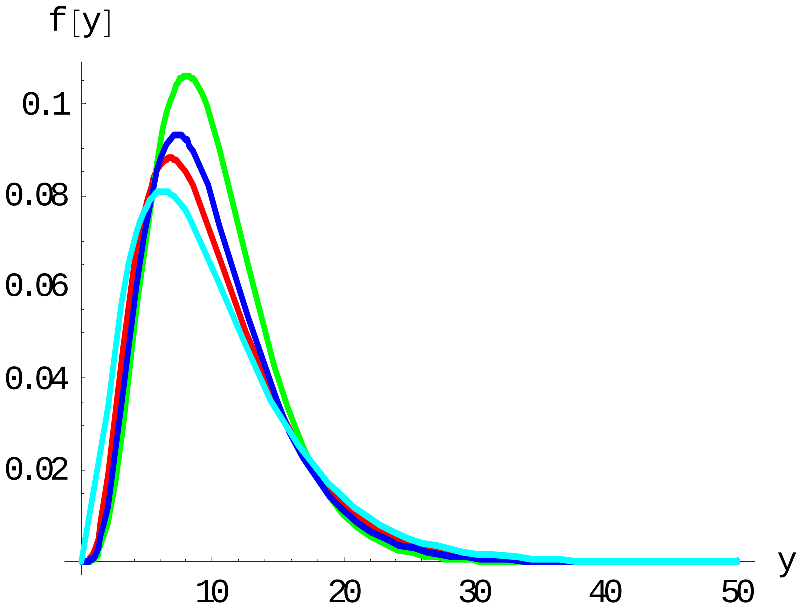
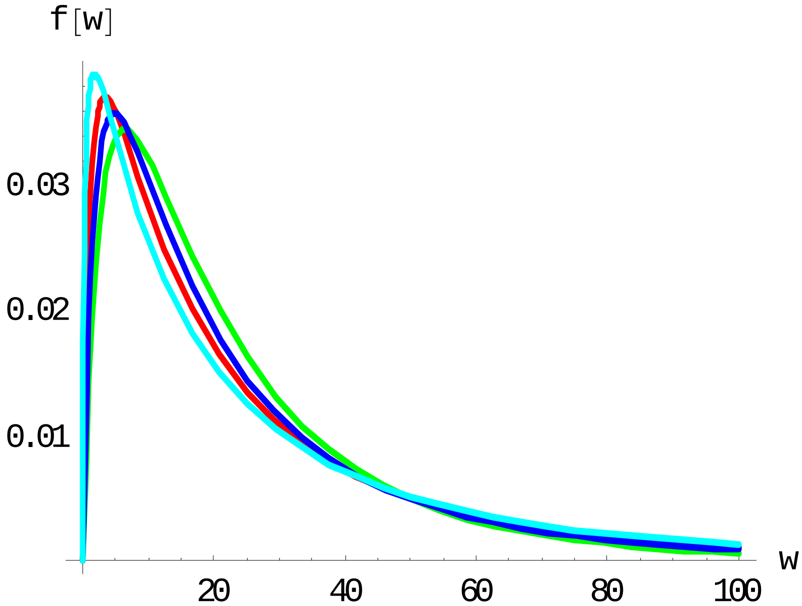


Figure 1. Correlated bivariate Chi-square density *f(x,y)* surface with 5 degrees of freedom at different values of ρ : Graph a (ρ = 0.95), Graph b (ρ = -0.95), Graph c (ρ =0.7), Graph d (ρ = -0.7), Graph e (ρ = 0.5), and Graph f (ρ = -0.5).



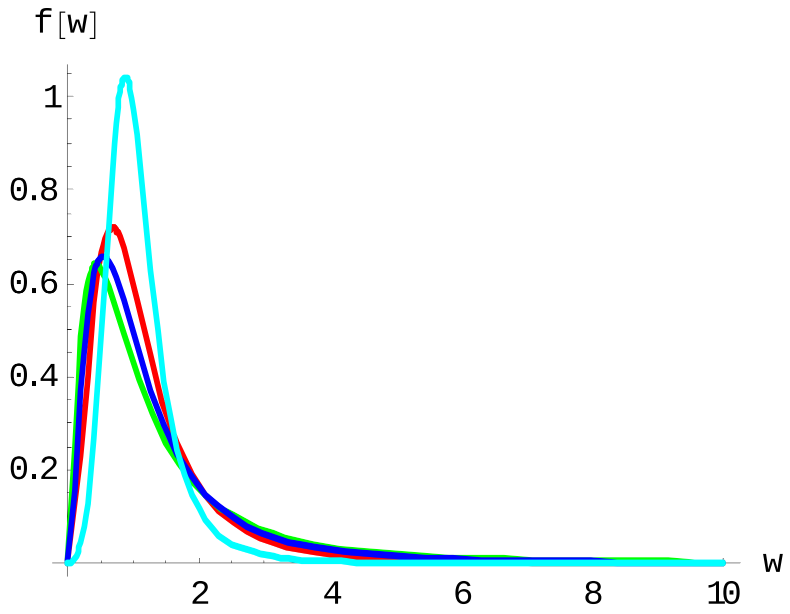
ρ = -0.7 (red), 0(green), 0.5(blue), 0.9 (cyan)

Figure 2. Sum of Chi-square variables for m = 5 and various ρ values



ρ = -0.7 (red), 0(green), 0.5(blue), 0.9 (cyan)

Figure 3. Product of Chi-square variables for m = 5 and various ρ values



ρ = -0.7 (red), 0(green), 0.5(blue), 0.9 (cyan)

Figure 4. Ratio of Chi-square variables for m = 5 and various ρ values