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Abstract

In this paper we consider a one-dimensional linear thermoelastic system of Timoshenko type, where the heat conduction is given by Green and Naghdi theories. We prove the exponential stability by using the energy method.
Keywords : exponential decay, Timoshenko, thermoelasticity type III.
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1. Introduction

In 1921, Timoshenko [28] gave the following system of coupled hyperbolic equations

$$\begin{aligned} \rho u_{tt} &= (K(u_x-\varphi))_x, \quad \text{in } (0,L)\times(0,+\infty) \\ I_\rho \varphi_{tt} &= (EI\varphi_x)_x + K(u_t-\varphi), \ \text{ in } \ (0,L)\times(0,+\infty), \end{aligned} \tag{1.1}$$

together with boundary conditions of the form

$$EI\varphi_x|_{x=0}^{x=L} = 0, \quad (u_x-\varphi)|_{x=0}^{x=L} = 0,$$

as a simple model describing the transverse vibrations of a beam. Here t denotes the time variable and x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

Kim and Renardy [11] considered (1.1) together with two boundary controls of the form

$$\begin{aligned} K\varphi(L,t) - K\frac{\partial u}{\partial x}(L,t) &= \alpha\frac{\partial u}{\partial t}(L,t) \quad \forall t \geq 0 \\ EI\frac{\partial \varphi}{\partial x}(L,t) &= -\beta\frac{\partial \varphi}{\partial t}(L,t) \quad \forall t \geq 0 \end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (1.1). They also provided numerical estimates to the eigenvalues of the operator associated with the system. An analogous result was also established by Feng *et al.* [6], where the stabilization of vibrations in a Timoshenko system was studied. Raposo *et al.* [23] studied (1.1) with homogeneous Dirichlet boundary conditions and two linear frictional dampings. Precisely, they looked into the following system

$$\begin{array}{l} \rho_1 u_{tt} - K(u_x - \varphi) + u_t = 0, \ \text{ in } (0,L)\times(0,+\infty) \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + K(u_x - \varphi) + \varphi_t = 0, \ \text{ in } (0,L)\times(0,+\infty) \\ u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad t > 0 \end{array} \tag{1.2}$$

and proved that the energy associated with (1.2) decays exponentially. Soufyane and Wehbe [26] showed that it is possible to stabilize uniformly (1.1) by using a unique locally distributed feedback. They considered

$$\begin{array}{l} \rho u_{tt} = (K(u_x - \varphi))_x, \ \text{ in } (0,L)\times(0,+\infty) \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi) - b\varphi_t, \ \text{ in } (0,L)\times(0,+\infty) \\ u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad t > 0, \end{array} \tag{1.3}$$

where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall\, x \in [a_0, a_1] \subset [0, L].$$

In fact, they proved that the uniform stability of (1.3) holds if and only if the wave speeds are equal $\left(\frac{K}{\rho} = \frac{EI}{I_\rho}\right)$; otherwise only the asymptotic stability has been proved. This result improves earlier ones by Soufyane [27], where an exponential decay of the solution energy of (1.1), together with two locally distributed feedbacks, had been proved. Rivera and Racke [15] obtained a similar result in a work, where the damping function $b = b(x)$ is allowed to change sign. Also, Rivera and Racke [14] treated a nonlinear Timoshenko-type system of the form

$$\begin{array}{l} \rho_1 \varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0 \\ \rho_2 \psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0 \end{array}$$

in a one-dimensional bounded domain. The dissipation here is through frictional damping which is only in the equation for the rotation angle. The authors gave an alternative proof for a sufficient and necessary condition for exponential stability in the linear case and then proved a polynomial stability in general. Moreover, they investigated the global existence of small smooth solutions and exponential stability in the nonlinear case. Xu and Yung [29] studied a system of Timoshenko beams with pointwise feedback controls, sought information about the eigenvalues and eigenfunctions of the system, and used this information to examine the stability of the system.

The nonuniform Timoshenko beam has also been studied by Ammar-Khodja *et al.* [2] and a similar result to that in [26] has been established. Also, Ammar-Khodja *et al.* [1] considered a linear Timoshenko-type system with memory of the form

$$\begin{array}{l} \rho_1\varphi_{tt} - K(\varphi_x+\psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x+\psi) = 0 \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x) \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x) \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \end{array} \tag{1.4}$$

in $(0,L)\times(0,+\infty)$, and proved, using the multiplier techniques, that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their results. This result has been later improved by Guesmia *et al.* [10], where the technical conditions on g'' have been removed and those on $g^{'}$ have been weakened.

The feedback of memory type has also been used by Santos [25]. He considered a Timoshenko system and showed that the presence of two feedbacks of memory type at a portion of the boundary stabilizes the system uniformly. He also obtained the rate of decay of the energy, which is exactly the rate of decay of the relaxation functions. This last result has been improved and generalized by Messaoudi and Soufyane [12].

For Timenshinko systems in classical thermoelasticity, Rivera and Racke [13] considered

$$\begin{array}{ll} \rho_1\varphi_{tt} - \sigma(\varphi_x,\psi)_x = 0 & \text{in } (0,\infty)\times(0,L) \\ \rho_2\psi_{tt} - b\psi_{xx} + k\,(\varphi_x+\psi) + \gamma\theta_x = 0 & \text{in } (0,\infty)\times(0,L) \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0 & \text{in } (0,\infty)\times(0,L)\,, \end{array} \tag{1.5}$$

where φ,ψ, and θ are functions of (x,t) which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions of σ,ρ_i,b,k,γ, they proved several exponential decay results for the linearized system and non exponential stability result for the case of different wave speeds.

In the above system, the heat flux is given by Fourier's law. As a result, this theory predicts an infinite speed of heat propagation. That is any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in a finite speed. This phenomenon in dielectric crystals is called second sound. To overcome this physical paradox, many theories have merged such as thermoelastcity by second sound or thermoelasticity type III.

By the end of the last century, Green and Naghdi [7-9] introduced three types of thermoelastic theories based on an entropy equality instead of the usual entropy inequality. In each of these theories, the heat flux is given by a different constitutive assumption. As a results, three theories are obtained and were called thermoelasticity type I, type II, and type III respectively. This theory is developed in a rational way in order to obtain a fully consistent explanation, which will incorporate thermal pulse transmission in a very logical manner and elevate the unphysical infinite speed of heat propagation induced by the classical theory of heat conduction. When the theory of type I is linearized the parabolic equation of the heat conduction arises. Whereas the theory of type II does not admit dissipation of energy and it is known as thermoelasticity without dissipation. It is a limiting case of thermoelasticity type III. See in this regard [3-5], [18] , and [20] for more details.

To understand these new theories and their application, several mathematical and physical contributions have been made; see for example [3-5], [16-22] and [24]. In particular, we must mention the survey paper of Chandrasekharaiah [5], in which the author has focussed attention on the work done during the last 10 or 12 years. He reviewed the theory of thermoelasticity with thermal relaxation and the temperature-rate dependent thermoelasticity. He also described the thermoelasticity without dissipation and clarified its properties. By the end of his paper, he made a brief discussion to the new theories, including what is called dual-phase-lag effects.

Zhang and Zuazua [30] analyzed the long time behavior of the solution of the system

$$\begin{array}{ll} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \nabla\theta = 0 & \text{in } (0,\infty)\times\Omega \\ \theta_{tt} - \Delta\theta - \Delta\theta_t + \text{div}\, u_{tt} = 0 & \text{in } (0,\infty)\times\Omega \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad \theta_t(.,0) = \theta_1, & x\in\Omega \\ u = \theta = 0, & \text{on } (0,\infty)\times\Gamma \end{array}$$

and concluded the following: For most domains, the energy of the system does not decay uniformly. But under suitable conditions on the domain, which might be described in terms of geometric optics, the energy of the system decays exponentially. For most domains in two space dimension, the energy of smooth solutions decays in a polynomial rate.

In [21] , Quintanilla and Racke considered a system similar to (1.1) and used the spectral analysis method and the energy method to obtain the exponential stability in one dimension for different boundary conditions; (Dirichlet- Dirichlet or Dirichlet-Neuman). They also proved a decay of energy result for the radially symmetric situations in multi- dimensional case $(n = 2, 3)$.

We also recall the contribution of Quintanilla [20] , in which he proved that solutions of thermoelasticity of type III converge to solutions of the classical thermoelasticity as well as to the solution of thermoelasticity without energy dissipation and Quintanilla [18], in which he established a structural stability result on the coupling coefficients and continuous dependence on the external data in thermoelasticity type III.

In the present work we consider the following system

$$
\begin{array}{ll}
\rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x = 0 & \text{in } (0,\infty)\times(0,1)\,, \\
\rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \beta\theta_x = 0 & \text{in } (0,\infty)\times(0,1)\,, \\
\rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0 & \text{in } (0,\infty)\times(0,1) \\
\varphi\left(.,0\right) = \varphi_0,\ \varphi_t\left(.,0\right) = \varphi_1,\ \psi\left(t.,0\right) = \psi_0,\ \psi_1\left(.,0\right) = \psi_1, & \\
\theta\left(.,0\right) = \theta_0,\ \theta_t\left(.,0\right) = \theta_1 & \\
\varphi\left(0,t\right) = \varphi\left(1,t\right) = \psi\left(0,t\right) = \psi\left(1,t\right) = \theta\left(0,t\right) = \theta\left(1,t\right). &
\end{array}
\tag{1.6}
$$

and prove an exponential decay similar to the one in [13]. This system models the transverse vibration of a thick beam, taking in account the heat conduction given by Green and Naghdi' s theory

2. Main result

In this section, we state and prove our main decay result. In order to exhibit the dissipative nature of system (1.6) , we introduce the new variables $\phi = \varphi_t$ and $\Psi = \psi_t$, So, problem (1.6) takes the form

$$
\begin{array}{ll}
\rho_1\phi_{tt} - K\left(\phi_x + \Psi\right)_x = 0 & \text{in } (0,\infty)\times(0,1) \\
\rho_2\Psi_{tt} - b\Psi_{xx} + K\left(\phi_x + \Psi\right) + \beta\theta_{tx} = 0 & \text{in } (0,\infty)\times(0,1) \\
\rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\Psi_{tx} - k\theta_{txx} = 0 & \text{in } (0,\infty)\times(0,1) \\
\phi\left(.,0\right) = \phi_0,\ \phi_t\left(.,0\right) = \phi_1,\ \Psi\left(.,0\right) = \Psi_0,\ \Psi_t\left(.,0\right) = \Psi_1 & \\
\theta\left(.,0\right) = \theta_0,\ \theta_t\left(.,0\right) = \theta_1 & \\
\phi\left(0,t\right) = \phi\left(1,t\right) = \Psi\left(0,t\right) = \Psi\left(1,t\right) = \theta_x\left(0,t\right) = \theta_x\left(1,t\right) = 0. &
\end{array}
\tag{2.1}
$$

In order to be able to use Poincaré's inequallity for θ, let

$$
\overline{\theta}\left(x,t\right) = \theta\left(x,t\right) - t\int_0^1 \theta_1\left(x\right)dx - \int_0^1 \theta_0\left(x\right)dx
$$

Then by $(2.1)_3$ we have

$$
\int_0^1 \overline{\theta}\left(x,t\right)dx = 0, \quad \forall t \geq 0. \tag{2.2}
$$

In this case, Poincaré's inequality is applicable for $\overline{\theta}$ and on the other hand it is easy to check that $\left(\phi,\Psi,\overline{\theta}\right)$ satisfies the same equations and boundary conditions in (2.1).
Remark 2.1. We can also do this by the same way as in [30] by putting : $\phi = \varphi, \Psi = \psi$, and

$$
\Theta(x,t) = \int_0^t \theta\left(x,s\right)ds + \chi\left(x\right),
$$

where $\chi\left(x\right) \in H_0^1\left(0,1\right)$ solves

$$
\left\{
\begin{array}{c}
\chi_{xx} = \rho_3\theta_1 - k\theta_{0xx} + \gamma\psi_{1x},\ \text{in } (0,1) \\
\chi = 0,\ x = 0,1
\end{array}
\right. .
$$

In the sequel we will work with $\overline{\theta}$ but for convenience, we write θ instead of $\overline{\theta}$. Therefore, the associated energy is given by

$$E(t) = \frac{\gamma}{2}\int_0^1 \left(\rho_1\phi_t^2 + \rho_2\Psi_t^2 + K\left|\phi_x + \Psi\right|^2 + b\Psi_x^2\right)dx + \frac{\beta}{2}\int_0^1 \left(\rho_3\theta_t^2 + \delta\theta_x^2\right)dx. \quad (2.3)$$

Theorem 2.1 *Suppose that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b} \quad (2.4)$$

and

$$\phi_0, \Psi_0,\ \theta_0 \in H_0^1(0,1),\ \phi_1, \Psi_1,\ \theta_1 \in L^2(0,1).$$

Then the energy $E(t)$ decays exponentially as time tends to infinity; that is, there exist two positive constants C and ξ independent of the initial data, such that

$$E(t) \leq CE(0)\, e^{-\xi t},\ \forall t > 0. \quad (2.5)$$

The proof of our result will be established through several lemmas.
Lemma 2.1 *Let (ϕ, Ψ, θ) be a solution of* (2.1). *Then, we have*

$$E'(t) = -\beta k\int_0^1 \theta_{tx}^2 dx. \quad (2.6)$$

Proof.
Multiplying equation $(2.1)_1$ by $\gamma\phi_t$, $(2.1)_2$ by $\gamma\Psi_t$ and $(2.1)_3$ by $\beta\theta_t$, integrating over $(0,1)$ and summing up to obtain (2.6) .□

As in [13], let

$$I_1 := \int_0^1 \left(\rho_2\Psi_t\Psi + \rho_1\phi_t\omega\right)dx. \quad (2.7)$$

where ω is the solution of

$$-\omega_{xx} = \Psi_x, \qquad \omega(0) = \omega(1) = 0, \quad (2.8)$$

Lemma 2.2 *Let (ϕ, Ψ, θ) be a solution of* (2.1). *Then we have, $\forall\varepsilon_1 > 0$,*

$$I_1' \leq -\frac{b}{2}\int_0^1 \Psi_x^2 dx + \varepsilon_1\rho_1\int_0^1 \phi_t^2 dx + \left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right)\int_0^1 \Psi_t^2 dx + \frac{\beta^2}{2b}\int_0^1 \theta_{tx}^2 dx. \quad (2.9)$$

Proof.

By taking a derivative of (2.7) and using equations (2.1) we conclude

$$\begin{aligned} I_1' &= -b\int_0^1 \Psi_x^2 dx + \rho_2\int_0^1 \Psi_t^2 dx - K\int_0^1 \Psi^2 dx - \beta\int_0^1 \Psi\theta_{tx} dx \\ &\quad + K\int_0^1 \omega_x^2 dx + \rho_1\int_0^1 \phi_t\omega_t dx. \end{aligned}$$

By using the inequalities

$$\begin{aligned}\int_0^1 \omega_x^2 dx &\leq \int_0^1 \Psi^2 dx \leq \int_0^1 \Psi_x^2 dx \\ \int_0^1 \omega_t^2 dx &\leq \int_0^1 \omega_{tx}^2 dx \leq \int_0^1 \Psi_t^2 dx,\end{aligned}$$

and Youmg's inequality, we find that

$$\begin{aligned}I_1' \leq\ & -b\int_0^1 \Psi_x^2 dx + \varepsilon_1 \rho_1 \int_0^1 \phi_t^2 dx + \left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right)\int_0^1 \Psi_t^2 dx \\ & + \frac{\beta^2}{2b}\int_0^1 \theta_{tx}^2 dx + \frac{b}{2}\int_0^1 \Psi_x^2 dx.\end{aligned}$$

□

Next, we set

$$I_2 := \rho_2 \rho_3 \int_0^1 \int_0^x \theta_t (t, y)\, dy \Psi_t (t, x)\, dx - \delta \int_0^1 \theta_x \Psi dx. \tag{2.10}$$

Lemma 2.3 *Let (ϕ, Ψ, θ) be a solution of (2.1). Then we have, $\forall \varepsilon_2 > 0$,*

$$I_2' \leq -\frac{\gamma\rho_2}{2}\int_0^1 \Psi_t^2 dx + \varepsilon_2 \int_0^1 \Psi_x^2 dx + \varepsilon_2 \int_0^1 \phi_x^2 dx + C(\varepsilon_2)\int_0^1 \theta_{tx}^2 dx. \tag{2.11}$$

Proof.

Using equations (2.1) and (2.10) we get

$$\begin{aligned}& \rho_2 \rho_3 \frac{d}{dt}\int_0^1 \int_0^x \theta_t (t, y)\, dy \Psi_t (t, x)\, dx \\ =\ & \int_0^1 \int_0^x (\delta\theta_{xx} - \gamma\Psi_{tx} + k\theta_{txx})\, dy \rho_2 \Psi_t dx \\ & + \int_0^1 \int_0^x \rho_3 \theta_t (t, y)\, dy\, (b\Psi_{xx} - K(\phi_x + \Psi) - \beta\theta_{tx})\, dx \\ =\ & \int_0^1 (\delta\theta_x - \gamma\Psi_t + k\theta_{tx})\, \rho_2 \Psi_t dx - K\int_0^1 \int_0^x \theta_t (t, y)\, dy \Psi dx \\ & -\rho_3 b \int_0^1 \theta_t \Psi_x dx - \rho_3 K \int_0^1 \theta_t \phi dx + \beta\rho_3 \int_0^1 \theta_t^2 dx \\ & + \left[\rho_3 \left(\int_0^x \theta_t (t, y)\, dy\right)(b\Psi_x - K\phi - \beta\theta_t)\right]_{x=0}^{x=1}.\end{aligned}$$

By using $(2.1)_1$, $(2.1)_6$ and keeping in mind that θ stands for $\overline{\theta}$, we easily see that

$$\int_0^1 \theta_t (y, t)\, dy = \frac{d}{dt}\int_0^1 \theta (y, t)\, dy = 0.$$

Consequently, we get

$$\left[\rho_3\left(\int_0^x \theta_t(t,y)\,dy\right)(b\Psi_x - K\phi - \beta\theta_t)\right]_{x=0}^{x=1} = 0.$$

Thus,

$$\begin{aligned} I_2' &= -\gamma\rho_2\int_0^1 \Psi_t^2 dx - \delta\rho_2\int_0^1 \theta\Psi_{tx}dx + k\rho_2\int_0^1 \theta_{tx}\Psi_t dx \\ &\quad -K\rho_2\int_0^1\int_0^x \theta_t(t,y)\,dy\Psi dx - \rho_3 b\int_0^1 \theta_t\Psi_x dx \\ &\quad -\rho_2 K\int_0^1 \theta_t\phi dx + \beta\rho_3\int_0^1 \theta_t^2 dx. \end{aligned}$$

The assertion of the lemma then fallows, using Young's and Poincaré's inequalities. □

Next we introduce the functional

$$J(t) := \rho_2\int_0^1 \Psi_t(\phi_x + \Psi)\,dx + \rho_2\int_0^1 \Psi_x\phi_t dx. \tag{2.12}$$

Lemma 2.4 *Let* (ϕ, Ψ, θ) *be a solution of* (2.1). *Assume that* (2.4) *holds. Then we have*

$$J'(t) \leq [b\phi_x\Psi_x]_{x=0}^{x=1} - \frac{K}{2}\int_0^1 (\phi_x + \Psi)^2 dx + \rho_2\int_0^1 \Psi_t^2 dx + \frac{\beta^2}{2K}\int_0^1 \theta_{tx}^2 dx. \tag{2.13}$$

Proof.

A differentiation of (2.12) gives

$$J'(t) = \int_0^1 \rho_2\Psi_{tt}(\phi_x+\Psi)\,dx + \int_0^1 \rho_2\Psi_t(\phi_x+\Psi)_t\,dx + \rho_2\int_0^1 \Psi_x\phi_{tt}dx + \rho_2\int_0^1 \Psi_{tx}\phi_t dx,$$

Then use of equations (2.1) yields

$$J'(t) = [b\phi_x\Psi_x]_{x=0}^{x=1} - K\int_0^1 (\phi_x+\Psi)^2 dx - \beta\int_0^1 (\phi_x+\Psi)\,\theta_{tx}dx + \rho_2\int_0^1 \Psi_t^2 dx$$

Consequently, (2.13) follows by Young's inequality. □

Next, in order to absorbe the boundary terms, appearing in (2.13), we exploit, as in [13], the function

$$q(x) = 2 - 4x, \qquad x \in (0,1).$$

Lemma 2.5 *Let* (ϕ, Ψ, θ) *be a solution of* (2.1). *Then we have,* $\forall\varepsilon_3 > 0$,

$$\begin{aligned} [b\phi_x\Psi_x]_{x=0}^{x=1} &\leq -\frac{\varepsilon_3}{K}\frac{d}{dt}\int_0^1 q\phi_t\phi_x dx - \frac{b\rho_2}{4\varepsilon_3}\frac{d}{dt}\int_0^1 q\Psi_t\Psi_x dx \\ &\quad +3\varepsilon_3\int_0^1 \phi_x^2 dx + \left(\varepsilon_3 + \frac{3b^2}{4\varepsilon_3} + \frac{b^2}{4\varepsilon_3^3}\right)\int_0^1 \Psi_x^2 dx + \frac{2\rho_1\varepsilon_3}{K}\int_0^1 \phi_t^2 dx \\ &\quad +\frac{\rho_2 b}{2\varepsilon_3}\int_0^1 \Psi_t^2 dx + \frac{K^2}{4}\varepsilon_3\int_0^1 (\phi_x+\Psi)^2 dx + \frac{\beta^2}{4\varepsilon_3}\int_0^1 \theta_{tx}^2 dx. \end{aligned} \tag{2.14}$$

Proof.

By using Young's inequality, we easily see that, $\forall\ \varepsilon_3 > 0$,

$$[b\phi_x \Psi_x]_{x=0}^{x=1} \leq \varepsilon_3 \left[\phi_x^2(1) + \phi_x^2(0)\right] + \frac{b^2}{4\varepsilon_3}\left[\Psi_x^2(1) + \Psi_x^2(0)\right]. \tag{2.15}$$

Also,

$$\begin{aligned} \frac{d}{dt}\int_0^1 b\rho_2 q\Psi_t\Psi_x dx &= \frac{b^2}{2}\left[q\Psi_x^2\right]_{x=0}^{x=1} - \frac{b^2}{2}\int_0^1 q_x\Psi_x^2 dx - \frac{\rho_2 b}{2}\int_0^1 q_x\Psi_t^2 dx \\ & -Kb\int_0^1 q\Psi_x(\phi_x+\Psi)\,dx - \beta b\int_0^1 q\Psi_x\theta_{tx}dx, \end{aligned}$$

then

$$\begin{aligned} \frac{d}{dt}\int_0^1 b\rho_2 q\Psi_t\Psi_x dx &\leq -b^2\left[\Psi_x^2(1) + \Psi_x^2(0)\right] + 3b^2\int_0^1 \Psi_x^2 dx \\ & +2\rho_2 b\int_0^1 \Psi_t^2 dx + \varepsilon_3^2 K^2\int_0^1 (\phi_x+\Psi)^2 dx \qquad (2.16) \\ & +\frac{b^2}{\varepsilon_3^2}\int_0^1 \Psi_x^2 dx + \beta^2\int_0^1 \theta_{tx}^2 dx. \end{aligned}$$

Similarly, we arrive at

$$\begin{aligned} \frac{d}{dt}\int_0^1 \rho_1 q\phi_t\phi_x dx &\leq -K\left[\phi_x^2(1) + \phi_x^2(0)\right] \qquad (2.17) \\ & +3K\int_0^1 \phi_x^2 dx + K\int_0^1 \Psi_x^2 dx + 2\rho_1\int_0^1 \phi_t^2 dx. \end{aligned}$$

Hence the assertion of the lemma follows, combining $(2.15) - (2.17)$. □

Let's introduce the functional

$$\mathcal{K}(t) := -\rho_1\int_0^1 \phi_t\phi dx - \rho_2\int_0^1 \Psi_t\Psi dx.$$

It easily follows, by using $\int_0^1 \Psi^2 dx \leq \int_0^1 \Psi_x^2 dx$ and equations (2.1) that

$$\begin{aligned} \mathcal{K}'(t) &\leq -\rho_1\int_0^1 \phi_t^2 dx - \rho_2\int_0^1 \Psi_t^2 dx + \left(b+\frac{1}{2}\right)\int_0^1 \Psi_x^2 dx \\ & +K\int_0^1 (\phi_x+\Psi)^2 dx + \frac{\beta^2}{2}\int_0^1 \theta_{tx}^2 dx. \qquad (2.18) \end{aligned}$$

Finally let

$$\Theta(t) := \int_0^1 \left(\rho_3\theta_t\theta + \frac{k}{2}\theta_x^2 + \gamma\Psi_x\theta\right)dx,$$

Lemma 2.6 *Let (ϕ, Ψ, θ) be a solution of (2.1). Then we have, $\forall \varepsilon_2 > 0$*

$$\Theta'(t) \leq -\delta\int_0^1 \theta_x^2 dx + \left(\rho_3 + \frac{\gamma^2}{4\varepsilon_2}\right)\int_0^1 \theta_t^2 dx + \varepsilon_2\int_0^1 \Psi_x^2 dx. \tag{2.19}$$

Proof.

A simple differentiation leads to

$$\begin{aligned}\Theta'(t) &= \rho_3\int_0^1 \theta_t^2 dx + \rho_3\int_0^1 \theta_{tt}\theta dx + k\int_0^1 \theta_x\theta_{tx}dx \\ &+\gamma\int_0^1 \Psi_{tx}\theta dx + \gamma\int_0^1 \Psi_x\theta_t dx.\end{aligned}$$

By using equation $(2.1)_3$, we arrive at

$$\Theta'(t) = \rho_3\int_0^1 \theta_t^2 dx - \delta\int_0^1 \theta_x^2 dx + \gamma\int_0^1 \Psi_x\theta_t dx$$

Finally, by Young's inequality, (2.19) is proved. □

To finalize the proof of Theorem 2.1, we define the Lyapunov functional $\mathcal{L}$ as follows

$$\begin{aligned}\mathcal{L}(t) &: = NE(t) + N_1I_1 + N_2I_2 + J(t) + \frac{\varepsilon_3}{K}\int_0^1 \rho_1 q\phi_t\phi_x dx \\ &+\frac{\rho_2 b}{4\varepsilon_3}\int_0^1 q\Psi_t\Psi_x dx + \mu\mathcal{K}(t) + \Theta(t). \end{aligned} \tag{2.20}$$

A combination of (2.6), (2.9), (2.11), (2.13), (2.14), (2.18), (2.19), and use of

$$\int_0^1 \theta_t^2 dx \le \int_0^1 \theta_{tx}^2 dx$$

and

$$\int_0^1 \phi_x^2 dx \le 2\int_0^1 (\phi_x+\Psi)^2 dx + 2\int_0^1 \Psi_x^2 dx,$$

give

$$\begin{aligned}\mathcal{L}'(t) &\le \left[-\beta kN + N_1\frac{\beta^2}{2b} + N_2C(\varepsilon_2) + \frac{\beta^2}{2K} + \frac{\beta^2}{4\varepsilon_3} + \frac{\mu\beta^2}{2} + \rho_3 + \frac{\gamma^2}{4\varepsilon_2}\right]\int_0^1 \theta_{tx}^2 dx \\ &+\left[-\frac{N_1 b}{2} + 3\varepsilon_2 N_2 + 7\varepsilon_3 + \frac{3b^2}{4\varepsilon_3} + \frac{b^2}{4\varepsilon_3^3} + \mu\left(\frac{1}{2}+b\right) + \varepsilon_2\right]\int_0^1 \Psi_x^2 dx \\ &+\rho_1\left[N_1\varepsilon_1 + \frac{2\varepsilon_3}{K} - \mu\right]\int_0^1 \phi_t^2 dx - \delta\int_0^1 \theta_x^2 dx \\ &+\left[N_1\left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right) - \rho_2\left(\frac{N_2\gamma}{2} - 1 - \frac{1}{2\varepsilon_3} + \mu\right)\right]\int_0^1 \Psi_t^2 dx \\ &+\left(-\frac{K}{2} + \left(\frac{K^2}{4}+6\right)\varepsilon_3 + \varepsilon_2 N_2 + \mu K\right)\int_0^1 (\phi_x+\Psi)^2 dx.\end{aligned} \tag{2.21}$$

At this point, we have to choose our constants very carefully. First, let's take $\mu = \frac{1}{16}$ and choose $\varepsilon_3 = \min\left(\frac{\mu K}{4}, \frac{1}{2}\frac{7K}{16\left(\frac{K^2}{4}+6\right)}\right)$. Now select N_1 large enough such that

$$\frac{N_1 b}{4} - \left[7\varepsilon_3 + \frac{3b^2}{4\varepsilon_3} + \frac{b^2}{4\varepsilon_3^3} + \mu\left(\frac{1}{2}+b\right)\right] > 0$$

then pick ε_1 so small that

$$N_1\varepsilon_1 + \frac{2\varepsilon_3}{K} - \mu \leq N_1\varepsilon_1 - \frac{\mu}{2} \leq -\frac{\mu}{4}$$

That is $\varepsilon_1 < \frac{\mu}{4N_1}$. We then choose N_2 large enough so that

$$\rho_2\left(\frac{N_2\gamma}{2} - 1 - \frac{b}{2\varepsilon_3} + \mu\right) - N_1\left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right) \geq \frac{\rho_2 N_2\gamma}{4}.$$

That is

$$N_2 \geq \frac{4}{\rho_2\gamma}\left[N_1\left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right) - \mu + 1 + \frac{b}{2\varepsilon_3}\right]$$

Next, we choose ε_2 so small that

$$\varepsilon_2 < \min\left(\frac{7K}{64N_2}, \frac{N_1 b}{4(3N_2+1)}\right).$$

Finally, we choose N large enough so that (2.21) becomes

$$\mathcal{L}'(t) \leq -\eta\int_0^1 \left(\theta_t^2 + \theta_{xt}^2 + \Psi_x^2 + \Psi_t^2 + \phi_t^2 + (\phi_x+\Psi)^2\right)dx \leq -CE(t) \qquad (2.22)$$

for some positive constants η, C. Moreover, we may choose N even larger (if needed) so that

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \quad \forall t \geq 0, \qquad (2.23)$$

for some positive constants $\beta_1, \beta_2 > 0$. Combining (2.22), (2.23), we conclude

$$\mathcal{L}'(t) \leq -\xi\mathcal{L}(t), \quad \forall t \geq 0. \qquad (2.24)$$

A simple integration of (2.24) leads to

$$\mathcal{L}(t) \leq \mathcal{L}(0)\, e^{-\xi t}, \quad \forall t \geq 0. \qquad (2.25)$$

Again, the use of (2.23) and (2.25) yields the desired result (2.5).
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