SEMI-CONTINUITY AND SEMI-CONNECTEDNESS IN GENERALIZED SEMI-CLOSURE SPACES

Miguel Caldas, Saeid Jafari, Raja M. Latif and Arafa A. Nasef
SEMI-CONTINUITY AND SEMI-CONNECTEDNESS
IN GENERALIZED SEMI-CLOSURE SPACES *

Miguel Caldas, Saeid Jafari, Raja M. Latif and Arafa A. Nasef

April 6, 2008

Abstract

In this paper, we show that a pointwise symmetric semi-isotonic semi-closure function is uniquely determined by the pairs of sets it separates. We then show that when the semi-closure function of the domain is semi-isotonic and the semi-closure function of the codomain is semi-isotonic and pointwise-semi-symmetric, functions which separate only those pairs of sets which are already separated are semi-continuous.

1 Introduction

The most important legacy of Norman Levine [1] was the introduction of semi-open sets which is one of the well-known notion of generalized open sets.

Throughout the present paper $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) denote topological spaces. Let $A$ be a subset of $X$. We denote the interior and the closure of a set $A$ by $\text{Int}(A)$ and $\text{Cl}(A)$, respectively. $A \subset X$ is called a semi-open set of $X$ [1] if $A \subset \text{Cl}(\text{Int}(A))$. The complement of a semi-open set is called semi-closed [2]. The intersection of all semi-closed sets containing a set $A$ is called the semi-closure of $A$ [2] and is denoted by $s\text{Cl}(A)$.

Definition 1 (1) A generalized semi-closure space is a pair $(X, s\text{Cl})$ consisting of a set $X$ and a semi-closure function $s\text{Cl}$, a function from the power set of $X$ to itself.

(2) The semi-closure of a subset $A$ of $X$, denoted $s\text{Cl}(A)$, is the image of $A$ under $s\text{Cl}$.

---

*2000 Math. Subject Classification: 54C10, 54D10.

Keywords and Phrases: semi-closure-separated, semi-closure function, semi-continuous functions.
The semi-exterior of \(A\) is \(\text{sExt}(A) = X \setminus \text{Cl}(A)\), and the semi-interior of \(A\) is \(\text{sInt}(A) = X \setminus \text{Cl}(X \setminus A)\).

\(A\) is semi-closed if \(A = \text{Cl}(A)\), \(A\) is semi-open if \(A = \text{Int}(A)\) and \(N\) is a semi-neighborhood of \(x\) if \(x \in \text{Int}(N)\).

**Definition 2** A semi-closure function \(\text{sCl}\) defined on \(X\) is:

1. semi-grounded if \(\text{sCl}(\emptyset) = \emptyset\).
2. semi-isotonic if \(\text{sCl}(A) \subseteq \text{sCl}(B)\) whenever \(A \subseteq B\).
3. semi-enlarging if \(A \subseteq \text{sCl}(A)\) for each subset \(A\) of \(X\).
4. semi-idempotent if \(\text{sCl}(A) = \text{sCl}(\text{sCl}(A))\) for each subset \(A\) of \(X\).
5. semi-sub-linear if \(\text{sCl}(A \cup B) \subseteq \text{sCl}(A) \cup \text{sCl}(B)\) for all \(A, B \subseteq X\).

**Definition 3** (1) Subsets \(A\) and \(B\) of \(X\) are said to be semi-closure-separated in a generalized semi-closure space \((X, \text{sCl})\) (or simply, \(\text{sCl}\)-separated) if \(A \cap \text{Cl}(B) = \emptyset\) and \(\text{Cl}(A) \cap B = \emptyset\), or equivalently, if \(A \subseteq \text{Ext}(B)\) and \(B \subseteq \text{Ext}(A)\).

(2) semi-Exterior points are said to be semi-closure-separated in a generalized semi-closure space \((X, \text{sCl})\) if for each \(A \subseteq X\) and for each \(x \in \text{Ext}(A)\), \(\{x\}\) and \(A\) are \(\text{sCl}\)-separated.

**Theorem 1.1** Let \((X, \text{sCl})\) be a generalized semi-closure space in which semi-Exterior points are \(\text{sCl}\)-separated and let \(S\) be the pairs of \(\text{sCl}\)-separated sets in \(X\). Then, for each subset \(A\) of \(X\), the semi-closure of \(A\) is \(\text{sCl}(A) = \{x \in X : \{x\}, A \notin S\}\).

**Proof.** In any generalized semi-closure space \(\text{sCl}(A) \subseteq \{x \in X : \{x\}, A \notin S\}\). Suppose that \(y \notin \{x \in X : \{x\}, A \notin S\}\), that is, \(\{y\}, A \in S\). Then \(\{y\} \cap \text{sCl}(A) = \emptyset\), and so \(y \notin \text{sCl}(A)\).

Now, let \(y \notin \text{sCl}(A)\). By hypothesis, \(\{y\}, A \in S\). Therefore \(y \notin \{x \in X : \{x\}, A \notin S\}\).

### 2 Some basic properties

**Definition 4** A semi-closure function \(\text{sCl}\) defined on a set \(X\) is said to be pointwise semi-symmetric when, for all \(x, y \in X\), if \(x \in \text{sCl}(\{y\})\), then \(y \in \text{sCl}(\{x\})\).

A generalized semi-closure space \((X, \text{sCl})\) is said to be semi-\(R_0\) when, for all \(x, y \in X\), if \(x\) is in each semi-neighborhood of \(y\), then \(y\) is in each semi-neighborhood of \(x\).
Corollary 2.1 Let \((X, s\text{Cl})\) a generalized semi-closure space in which semi-Exterior points are \(s\text{Cl}\)-separated. Then \(s\text{Cl}\) is pointwise semi-symmetric and \((X, s\text{Cl})\) is semi-\(R_0\).

Proof. Let semi-Exterior points be \(s\text{Cl}\)-separated in \((X, s\text{Cl})\). If \(x \in s\text{Cl}(\{y\})\), then \(\{x\}\) and \(\{y\}\) are not \(s\text{Cl}\)-separated. This means that \(y \in s\text{Cl}(\{x\})\). Hence, \(s\text{Cl}\) is pointwise semi-symmetric.

Suppose that \(x\) belongs to every semi-neighborhood of \(y\), that is, \(x \in M\) whenever \(y \in s\text{Int}(M)\). Letting \(A = X \setminus M\) and rewriting contrapositively, \(y \in s\text{Cl}(A)\) whenever \(x \in A\).

Let \(x \in s\text{Int}(N)\). \(x \notin s\text{Cl}(X \setminus N)\), so \(x\) is \(s\text{Cl}\)-separated from \(X \setminus N\). Hence \(s\text{Cl}(\{x\}) \subseteq N\).

\(x \in \{x\}\), so \(y \in s\text{Cl}(\{x\}) \subseteq N\). Hence \((X, s\text{Cl})\) is semi-\(R_0\).

Observe that these three axioms are not equivalent in general, but they are equivalent when the semi-closure function is semi-isotonic:

Theorem 2.2 Let \((X, s\text{Cl})\) be a generalized semi-closure space with \(s\text{Cl}\) semi-isotonic. Then the following are equivalent:

1. sExterior points are \(s\text{Cl}\)-separated.
2. \(s\text{Cl}\) is pointwise semi-symmetric.
3. \((X, s\text{Cl})\) is semi-\(R_0\).

Proof. Suppose that (2) is true. Let \(A \subseteq X\), and let \(x \in s\text{Ext}(A)\). Then, as \(s\text{Cl}\) is semi-isotonic, for each \(y \in A\), \(x \notin s\text{Cl}(\{y\})\), and thus, \(y \notin s\text{Cl}(\{x\})\). Hence \(A \cap s\text{Cl}(\{x\}) = \emptyset\). Therefore (2) implies (1). Moreover, by the previous corollary, (1) implies (2).

Suppose now that (2) is true and let \(x, y \in X\) such that \(x\) is in every semi-neighborhood of \(y\), i.e. \(x \in N\) whenever \(y \in s\text{Int}(N)\). Then \(y \in s\text{Cl}(A)\) whenever \(x \in A\), and in particular, since \(x \in \{x\}\), \(y \in s\text{Cl}(\{x\})\). It follows that \(x \in s\text{Cl}(\{y\})\). Thus if \(y \in B\), then \(x \in s\text{Cl}(\{y\}) \subseteq s\text{Cl}(B)\), as \(s\text{Cl}\) is semi-isotonic. Therefore, if \(x \in s\text{Int}(C)\), then \(y \in C\), that is, \(y\) is in every semi-neighborhood of \(x\). Hence, (2) implies (3).

Now, let \((X, s\text{Cl})\) be semi-\(R_0\) and \(x \in s\text{Cl}(\{y\})\). Since \(s\text{Cl}\) is semi-isotonic, \(x \in s\text{Cl}(B)\) whenever \(y \in B\), or, equivalently, \(y\) is in every semi-neighborhood of \(x\). Since \((X, s\text{Cl})\) is semi-\(R_0\), \(x \in N\) whenever \(y \in s\text{Int}(N)\). Therefore, \(y \in s\text{Cl}(A)\) whenever \(x \in A\), and in particular, since \(x \in \{x\}, y \in s\text{Cl}(\{x\})\). It follows that (3) implies (2).
Theorem 2.3 Let $S$ be a set of unordered pairs of subsets of a set $X$ such that, for all $A, B, C \subseteq X$,

(1) if $A \subseteq B$ and $\{B, C\} \in S$, then $\{A, C\} \in S$ and

(2) if $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, then $\{A, B\} \in S$.

Then there exists a unique pointwise semi-symmetric semi-isotonic semi-closure function $sCl$ on $X$ which semi-closure-separates the elements of $S$.

Proof. Define $sCl$ by $sCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. If $A \subseteq B \subseteq X$ and $x \in sCl(A)$, then $\{\{x\}, A\} \notin S$. Thus, $\{\{x\}, B\} \notin S$, that is, $x \in sCl(B)$. Hence $sCl$ is semi-isotonic. Moreover $x \in sCl(\{y\})$ if and only if $\{\{x\}, \{y\}\} \notin S$ if and only if $y \in sCl(\{x\})$. Thus $sCl$ is pointwise semi-symmetric.

Suppose that $\{A, B\} \in S$. Then $A \cap sCl(B) = A \cap \{x \in X : \{\{x\}, B\} \notin S\} = \{x \in A : \{\{x\}, A\} \notin S\} = \emptyset$. Similarly, $sCl(A) \cap B = \emptyset$. Therefore, if $\{A, B\} \in S$, then $A$ and $B$ are $sCl$-separated.

Now suppose that $A$ and $B$ are $sCl$-separated. Then $\{x \in A : \{\{x\}, B\} \notin S\} = A \cap sCl(B) = \emptyset$ and $\{x \in B : \{\{x\}, A\} \notin S\} = sCl(A) \cap B = \emptyset$. Hence, $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$. Therefore, $\{A, B\} \in S$.

In the following we show that many properties of semi-closure functions can be expressed in terms of the sets they separate:

Theorem 2.4 Let $S$ be the pairs of $sCl$-separated sets of a generalized semi-closure space $(X, sCl)$ in which sExterior points are semi-closure-separates. Then $sCl$ is

(1) semi-grounded if and only if for all $x \in X$ $\{\{x\}, \emptyset\} \in S$.

(2) semi-enlarging if and only if for all $\{A, B\} \in S$, $A$ and $B$ are disjoint.

(3) semi-sub-linear if and only if $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$ and $\{A, C\} \in S$.

Furthermore, if $sCl$ is semi-enlarging and for all $A, B \subseteq X$, $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$, then $sCl$ is semi-idempotent. Now, if $sCl$ is semi-isotonic and semi-idempotent, then $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and
\[\{y\}, A \notin S \text{ for each } y \in B.\]

**Proof.** By Theorem 1.1, \(s\text{Cl}(A) = \{x \in X : \{x\}, A \notin S\}\) for every \(A \subseteq X\). Suppose that for all \(x \in X\), \(\{x\}, \emptyset \in S\). Then \(s\text{Cl}(\emptyset) = \{x \in X : \{x\}, \emptyset \notin S\} = \emptyset\). Hence \(s\text{Cl}\) is semi-grounded. Conversely, if \(\emptyset = s\text{Cl}(\emptyset) = \{x \in X : \{x\}, \emptyset \notin S\}\), then \(\{x\}, \emptyset \in S\), for all \(x \in X\).

Assume that for all \(A, B \in S\), \(A \text{ and } B\) are disjoint. Since \(\{a\}, A \notin S\) if \(a \in A\), \(A \subseteq s\text{Cl}(A)\) for each \(A \subseteq X\). Therefore, \(s\text{Cl}\) is semi-enlarging. Conversely, Let \(s\text{Cl}\) be semi-enlarging and \(\{A, B\} \in S\). Then \(A \cap B \subseteq s\text{Cl}(A) \cap B = \emptyset\). Suppose that \(\{A, B \cup C\} \in S\) whenever \(\{A, B\} \in S\) and \(\{A, C\} \in S\). Let \(x \in X\) and \(B, C \subseteq X\) such that \(\{x\}, B \cup C \notin S\). Then \(\{x\}, B \notin S\) or \(\{x\}, C \notin S\). Hence \(s\text{Cl}(B \cup C) \subseteq s\text{Cl}(B) \cup s\text{Cl}(C)\). Therefore, \(s\text{Cl}\) is semi-sub-linear. Conversely, suppose that \(s\text{Cl}\) is semi-sub-linear and let \(\{A, B\}, \{A, C\} \in S\). Then \(s\text{Cl}(B \cup C) \cap A \subseteq (s\text{Cl}(B) \cup s\text{Cl}(C)) \cap A = (s\text{Cl}(B) \cap A) \cup (s\text{Cl}(C)) \cap A = \emptyset\) and \((B \cup C) \cap s\text{Cl}(A) = (B \cap s\text{Cl}(A)) \cup (C \cap s\text{Cl}(A)) = \emptyset\). Let \(s\text{Cl}\) be semi-enlarging and suppose that \(\{x\}, A \notin S\) whenever \(\{x\}, B \notin S\) and \(\{y\}, A \notin S\) for each \(y \in B\). Then \(s\text{Cl}(s\text{Cl}(A)) \subseteq s\text{Cl}(A)\): If \(x \in s\text{Cl}(s\text{Cl}(A))\), then \(\{x\}, s\text{Cl}(A) \notin S\). \(\{y\}, A \notin S\), for each \(y \in s\text{Cl}(A)\); hence \(\{x\}, A \notin S\). Since \(s\text{Cl}\) is semi-enlarging, \(\text{then } s\text{Cl}(A) \subseteq s\text{Cl}(s\text{Cl}(A))\).

Therefore, \(s\text{Cl}(s\text{Cl}(A)) = s\text{Cl}(A)\), for each \(A \subseteq X\).

**Definition 5** If \((X, (s\text{Cl})_X)\) and \((Y, (s\text{Cl})_Y)\) are generalized semi-closure spaces, then a function \(f : X \to Y\) is said to be

1. semi-closure-preserving if \(f((s\text{Cl})_X(A)) \subseteq (s\text{Cl})_Y(f(A))\) for each \(A \subseteq X\).
2. semi-continuous if \((s\text{Cl})_X(f^{-1}(B)) \subseteq f^{-1}((s\text{Cl})_Y(B))\) for each \(B \subseteq Y\).

Observe that in general, neither condition implies the other. Now, we have the following result:
Theorem 2.5 Let \((X, (sCl)_X)\) and \((Y, (sCl)_Y)\) be generalized semi-closure spaces and let \(f : X \to Y\) be a function.

1. If \(f\) is semi-closure-preserving and \((sCl)_Y\) is semi-isotonic, then \(f\) is semi-continuous.
2. If \(f\) is semi-continuous and \((sCl)_X\) is semi-isotonic, then \(f\) is semi-closure-preserving.

Proof. Let \(f\) be semi-closure-preserving and \((sCl)_Y\) semi-isotonic. Let \(B \subseteq Y\).

\[
f((sCl)_X(f^{-1}(B))) \subseteq (sCl)_Y(f(f^{-1}(B))) \subseteq (sCl)_Y(B)\]

and hence,

\[
(sCl)_X(f^{-1}(B)) \subseteq f^{-1}(f((sCl)_X(f^{-1}(B)))) \subseteq f^{-1}((sCl)_Y(B)).
\]

Suppose that \(f\) is semi-continuous and \((sCl)_X\) is semi-isotonic. Let \(A \subseteq X\). \((sCl)_X(A) \subseteq (sCl)_X(f^{-1}(f(A))) \subseteq f^{-1}((sCl)_Y(f(A))).\) Therefore, \(f((sCl)_X(A)) \subseteq f(f^{-1}((sCl)_Y(f(A)))) \subseteq (sCl)_Y(f(A)).\)

Definition 6 Let \((X, (sCl)_X)\) and \((Y, (sCl)_Y)\) be generalized semi-closure spaces and let \(f : X \to Y\) be a function. If for all \(A, B \subseteq X\), \(f(A)\) and \(f(B)\) are not \((sCl)_Y\)-separated whenever \(A\) and \(B\) are not \((sCl)_X\)-separated, then we say that \(f\) is non-semi-separating.

Observe that \(f\) is non-semi-separating if and only if \(A\) and \(B\) are \((sCl)_X\)-separated whenever \(f(A)\) and \(f(B)\) are \((sCl)_Y\)-separated.

Theorem 2.6 Let \((X, (sCl)_X)\) and \((Y, (sCl)_Y)\) be generalized semi-closure spaces and let \(f : X \to Y\) be a function.

1. If \((sCl)_Y\) is semi-isotonic and \(f\) is non-semi-separating, then \(f^{-1}(C)\) and \(f^{-1}(D)\) are \((sCl)_X\)-separated whenever \(C\) and \(D\) are \((sCl)_Y\)-separated.
2. If \((sCl)_X\) is semi-isotonic and \(f^{-1}(C)\) and \(f^{-1}(D)\) are \((sCl)_X\)-separated whenever \(C\) and \(D\) are \((sCl)_Y\)-separated, then \(f\) is non-semi-separating.

Proof. Suppose that \(C\) and \(D\) are \((sCl)_Y\)-separated subsets, where \((sCl)_Y\) is semi-isotonic. Let \(A = f^{-1}(C)\) and \(B = f^{-1}(D)\). \(f(A) \subseteq C\) and \(f(B) \subseteq D\) and since \((sCl)_Y\) is semi-isotonic, \(f(A)\) and \(f(B)\) are also \((sCl)_Y\)-separated. It follows now that \(A\) and \(B\) are \((sCl)_X\)-separated in \(X\).

Suppose that \((sCl)_X\) is semi-isotonic and let \(A, B \subseteq X\) such that \(C = f(A)\) and \(D = f(B)\)
are $(sCl)_X$-separated. Then $f^{-1}(C)$ and $f^{-1}(D)$ are $(sCl)_X$-separated and since $(sCl)_X$ is semi-isotonic, $A \subseteq f^{-1}(f(A)) = f^{-1}(C)$ and $B \subseteq f^{-1}(f(B)) = f^{-1}(D)$ are $(sCl)_X$-separated as well.

**Theorem 2.7** Let $(X,(sCl)_X)$ and $(Y,(sCl)_Y)$ be generalized semi-closure spaces and let $f : X \to Y$ be a function. If $f$ is semi-closure-preserving, then $f$ is non-semi-separating.

*Proof.* Suppose that $f$ is semi-closure-preserving and $A, B \subseteq X$ are not $(sCl)_X$-separated. Suppose that $(sCl)_X(A) \cap B \neq \emptyset$. Then $\emptyset \neq f((sCl)_X(A) \cap B) \subseteq f((sCl)_X(A)) \cap f(B) \subseteq (sCl)_Y(f(A)) \cap f(B)$. Similarly, if $A \cap (sCl)_X(B) \neq \emptyset$, then $f(A) \cap (sCl)_Y(f(B)) \neq \emptyset$. Hence $f(A)$ and $f(B)$ are not $(sCl)_Y$-separated.

**Corollary 2.8** Let $(X,(sCl)_X)$ and $(Y,(sCl)_Y)$ be generalized semi-closure spaces with $(sCl)_X$ semi-isotonic and let $f : X \to Y$ be a function. If $f$ is semi-continuous, then $f$ is non-semi-separating.

*Proof.* If $f$ is semi-continuous and $(sCl)_X$ semi-isotonic, then by Theorem 2.5 (2) $f$ is semi-closure-preserving. Now, by Theorem 2.7, $f$ is non-semi-separating.

**Theorem 2.9** Let $(X,(sCl)_X)$ and $(Y,(sCl)_Y)$ be generalized semi-closure spaces which semi-Exterior points $(sCl)_Y$-separated in $Y$ and let $f : X \to Y$ be a function. Then $f$ is semi-closure-preserving if and only if $f$ is non-semi-separating.

*Proof.* By Theorem 2.7, if $f$ is semi-closure-preserving, then $f$ is non-semi-separating. Suppose that $f$ is non-semi-separating and let $A \subseteq X$. If $(sCl)_X = \emptyset$, then $f((sCl)_X(A)) = \emptyset \subseteq (sCl)_Y(f(A))$. Suppose $(sCl)_X(A) \neq \emptyset$. Let $S_X$ and $S_Y$ denote the pairs of $(sCl)_X$-separated subsets of $X$ and the pairs of $(sCl)_Y$-separated subsets of $Y$, respectively. Let $y \in f((sCl)_X(A))$ and let $x \in (sCl)_X(A) \cap f^{-1}(\{y\})$. Since $x \in (sCl)_X(A)$, $\{x\}, A \notin S_X$ and since $f$ non-semi-separating, $\{y\}, f(A) \notin S_Y$. Since semi-Exterior points are $(sCl)_Y$-separated, $y \in (sCl)_Y(f(A))$. Thus $f((sCl)_X(A)) \subseteq (sCl)_Y(f(A))$ for each $A \subseteq X$. 

7
Corollary 2.10 Let \((X, (sCl)_X)\) and \((Y, (sCl)_Y)\) be generalized semi-closure spaces with semi-isotonic closure functions and with \((sCl)_Y\)-pointwise-semi-symmetric and let \(f : X \rightarrow Y\) be a function. Then \(f\) is semi-continuous if and only if \(f\) non-semi-separating.

Proof. Since \((sCl)_Y\) is semi-isotonic and pointwise-semi-symmetric, semi-Exterior points are semi-closure separated in \((Y, (sCl)_Y)\) (Theorem 2.2 (1)). Since both semi-closure functions are semi-isotonic, \(f\) is semi-closure-preserving (Theorem 2.5) if and only if \(f\) is semi-continuous. Hence, we can apply the Theorem 2.9.

3 Semi-connected generalized semi-closure spaces

Definition 7 Let \((X, sCl)\) be a generalized semi-closure space. \(X\) is said to be semi-connected if \(X\) is not a union of disjoint nontrivial semi-closure-separated pair of sets.

Theorem 3.1 Let \((X, sCl)\) be a generalized semi-closure space with semi-grounded semi-isotonic semi-enlarging \(sCl\). Then, the following are equivalent:
(1) \((X, sCl)\) is semi-connected,
(2) \(X\) can not be a union of nonempty disjoint semi-open sets.

Proof. (1)\(\Rightarrow\) (2): Let \(X\) be a union of nonempty disjoint semi-open sets \(A\) and \(B\). Then, \(X = A \cup B\) and this implies that \(B = X \setminus A\) and \(A\) is a semi-open set. Thus, \(B\) is semi-closed and hence \(A \cap sCl(B) = A \cap B = \emptyset\). By using similar way, we obtain \(sCl(A) \cap B = \emptyset\). Hence, \(A\) and \(B\) are semi-closure-separated and hence \(X\) is not semi-connected. This is a contradiction.

(2)\(\Rightarrow\) (1): Suppose that \(X\) is not semi-connected. Then \(X = A \cup B\), where \(A\), \(B\) are disjoint semi-closure-separated sets, i.e. \(A \cap sCl(B) = sCl(A) \cap B = \emptyset\). We have \(sCl(B) \subset X \setminus A \subset B\). Since \(sCl\) is semi-enlarging, we obtain \(sCl(B) = B\) and hence, \(B\) is semi-closed. By using \(sCl(A) \cap B = \emptyset\) and similar way, it is obvious that \(A\) is semi-closed. But this is a contradiction.

Definition 8 Let \((X, sCl)\) be a generalized semi-closure space with semi-grounded semi-isotonic \(sCl\). Then, \((X, sCl)\) is called a \(T_1\)-semi-grounded semi-isotonic space if \(sCl(\{x\}) \subset \{x\}\) for all \(x \in X\).
Theorem 3.2 Let \((X, sCl)\) be a generalized semi-closure space with \(\lambda\)-grounded semi-isotonic \(sCl\). Then, the following are equivalent:

1. \((X, sCl)\) is semi-connected,
2. Any semi-continuous function \(f : X \to Y\) is constant for all \(T_1\)-semi-grounded semi-isotonic spaces \(Y = \{0, 1\}\).

Proof. (1)\(\Rightarrow\)(2): Let \(X\) be semi-connected. Suppose that \(f : X \to Y\) is semi-continuous and it is not constant. Then there exists a set \(U \subset X\) such that \(U = f^{-1}(\{0\})\) and \(X \setminus U = f^{-1}(\{1\})\). Since \(f\) is semi-continuous and \(Y\) is \(T_1\)-\(\lambda\)-grounded semi-isotonic space, then we have \(\text{Cl}_A(U) = sCl(f^{-1}(\{0\})) \subset f^{-1}(sCl(\{0\})) \subset f^{-1}(\{0\}) = U\) and hence \(sCl(U) \cap (X \setminus U) = \emptyset\). By using similar way we have \(U \cap sCl(X \setminus U) = \emptyset\). This is a contradiction. Thus, \(f\) is constant.

(2)\(\Rightarrow\)(1): Suppose that \(X\) is not semi-connected. Then there exist semi-closure-separated sets \(U\) and \(V\) such that \(U \cup V = X\). We have \(sCl(U) \subset U\) and \(sCl(V) \subset V\) and \(X \setminus U \subset V\). Since \(sCl\) is semi-isotonic and \(U\) and \(V\) are semi-closure-separated, then \(sCl(X \setminus U) \subset sCl(V) \subset X \setminus U\). If we consider the space \((Y, sCl)\) by \(Y = \{0, 1\}\), \(sCl(\emptyset) = \emptyset\), \(sCl(\{0\}) = \{0\}\), \(sCl(\{1\}) = \{1\}\) and \(sCl(Y) = Y\), then the space \((Y, sCl)\) is a \(T_1\)-semi-grounded semi-isotonic space. We define the function \(f : X \to Y\) as \(f(U) = \{0\}\) and \(f(X \setminus U) = \{1\}\). Let \(A \neq \emptyset\) and \(A \subset Y\). If \(A = Y\), then \(f^{-1}(A) = X\) and hence \(sCl(X) = sCl(f^{-1}(A)) \subset X = f^{-1}(A) = f^{-1}(sCl(A))\). If \(A = \{0\}\), then \(f^{-1}(A) = U\) and hence \(sCl(U) = sCl(f^{-1}(A)) \subset U = f^{-1}(A) = f^{-1}(sCl(A))\). If \(A = \{1\}\), then \(f^{-1}(A) = X \setminus U\) and hence \(sCl(X \setminus U) = sCl(f^{-1}(A)) \subset X \setminus U = f^{-1}(A) = f^{-1}(sCl(A))\). Hence, \(f\) is semi-continuous. Since \(f\) is not constant, this is a contradiction.

Theorem 3.3 Let \(f : (X, sCl) \to (Y, sCl)\) and \(g : (Y, sCl) \to (Z, sCl)\) be semi-continuous functions. Then, \(gof : X \to Z\) is semi-continuous.

Proof. Suppose that \(f\) and \(g\) are semi-continuous. For all \(A \subset Z\) we have \(sCl(gof)^{-1}(A) = sCl(f^{-1}(g^{-1}(A))) \subset f^{-1}(sCl(g^{-1}(A))) \subset f^{-1}(g^{-1}(sCl(A))) = (gof)^{-1}(sCl(A))\). Hence, \(gof : X \to Z\) is semi-continuous.
**Theorem 3.4** Let \((X, sCl)\) and \((Y, sCl)\) be generalized semi-closure spaces with semi-grounded semi-isotonic \(sCl\) and \(f : (X, sCl) \rightarrow (Y, sCl)\) be a semi-continuous function onto \(Y\). If \(X\) is semi-connected, then \(Y\) is semi-connected.

**Proof.** Suppose that \(\{0, 1\}\) is a generalized semi-closure space with semi-grounded semi-isotonic \(sCl\) and \(g : Y \rightarrow \{0, 1\}\) is a semi-continuous function. Since \(f\) is semi-continuous, by Theorem 3.3, \(gof : X \rightarrow \{0, 1\}\) is semi-continuous. Since \(X\) is semi-connected, \(gof\) is constant and hence \(g\) is constant. By Theorem 3.2, \(Y\) is semi-connected.

**Definition 9** Let \((Y, sCl)\) be a generalized semi-closure space with semi-grounded semi-isotonic \(sCl\) and more than one element. A generalized semi-closure space \((X, sCl)\) with semi-grounded semi-isotonic \(sCl\) is called \(Y\)-semi-connected if any semi-continuous function \(f : X \rightarrow Y\) is constant.

**Theorem 3.5** Let \((Y, sCl)\) be a generalized semi-closure space with semi-grounded semi-isotonic semi-enlarging \(sCl\) and more than one element. Then every \(Y\)-semi-connected generalized semi-closure space with semi-grounded semi-isotonic is semi-connected.

**Proof.** Let \((X, sCl)\) be a \(Y\)-semi-connected generalized semi-closure space with semi-grounded semi-isotonic \(sCl\). Suppose that \(f : X \rightarrow \{0, 1\}\) is a semi-continuous function, where \(\{0, 1\}\) is a \(T_1\)-semi-grounded semi-isotonic space. Since \(Y\) is a generalized semi-closure space with semi-grounded semi-isotonic semi-enlarging \(sCl\) and more than one element, then there exists a semi-continuous injection \(g : \{0, 1\} \rightarrow Y\). By Theorem 3.3, \(gof : X \rightarrow Y\) is semi-continuous. Since \(X\) is \(Y\)-semi-connected, then \(gof\) is constant. Thus, \(f\) is constant and hence, by Theorem 3.2, \(X\) is semi-connected.

**Theorem 3.6** Let \((X, sCl)\) and \((Y, sCl)\) be generalized semi-closure spaces with semi-grounded semi-isotonic \(sCl\) and \(f : (X, sCl) \rightarrow (Y, sCl)\) be a semi-continuous function onto \(Y\). If \(X\) is \(Z\)-semi-connected, then \(Y\) is \(Z\)-semi-connected.

**Proof.** Suppose that \(g : Y \rightarrow Z\) is a semi-continuous function. Then \(gof : X \rightarrow Z\) is semi-continuous. Since \(X\) is \(Z\)-semi-connected, then \(gof\) is constant. This implies that \(g\) is constant. Thus, \(Y\) is \(Z\)-semi-connected.
Definition 10 A generalized semi-closure space \((X, sCl)\) is strongly semi-connected if there is no countable collection of pairwise semi-closure-separated sets \(\{A_n\}\) such that \(X = \bigcup A_n\).

Theorem 3.7 Every strongly semi-connected generalized semi-closure space with semi-grounded semi-isotonic \(sCl\) is semi-connected.

Theorem 3.8 Let \((X, sCl)\) and \((Y, sCl)\) be generalized semi-closure spaces with semi-grounded semi-isotonic \(sCl\) and \(f : (X, sCl) \rightarrow (Y, sCl)\) be a semi-continuous function onto \(Y\). If \(X\) is strongly semi-connected, then \(Y\) is strongly semi-connected.

Proof. Suppose that \(Y\) is not strongly semi-connected. Then, there exists a countable collection of pairwise semi-closure-separated sets \(\{A_n\}\) such that \(Y = \bigcup A_n\). Since \(f^{-1}(A_n) \cap sCl(f^{-1}(A_m)) \subset f^{-1}(A_n) \cap f^{-1}(sCl(A_m)) = \emptyset\) for all \(n \neq m\), then the collection \(\{f^{-1}(A_n)\}\) is pairwise semi-closure-separated. This is a contradiction. Hence, \(Y\) is strongly semi-connected.

Theorem 3.9 Let \((X, (sCl)_X)\) and \((Y, (sCl)_Y)\) be generalized semi-closure spaces. Then the following are equivalent for a function \(f : X \rightarrow Y\):

1. \(f\) is semi-continuous,
2. \(f^{-1}(sInt(B)) \subseteq sInt(f^{-1}(B))\) for each \(B \subseteq Y\).

Theorem 3.10 Let \((X, sCl)\) be a generalized semi-closure space with semi-grounded semi-isotonic \(sCl\). Then \((X, sCl)\) is strongly semi-connected if and only if \((X, sCl)\) is \(Y\)-semi-connected for any countable \(T_1\)-semi-grounded semi-isotonic space \((Y, sCl)\).

Proof. \((\Rightarrow)\): Let \((X, sCl)\) be strongly semi-connected. Suppose that \((X, sCl)\) is not \(Y\)-semi-connected for some countable \(T_1\)-semi-grounded semi-isotonic space \((Y, sCl)\). There exists a semi-continuous function \(f : X \rightarrow Y\) which is not constant and hence \(K = f(X)\) is a countable set with more than one element. For each \(y_n \in K\), there exists \(U_n \subset X\) such that \(U_n = f^{-1}\{y_n\}\) and hence \(Y = \cup U_n\). Since \(f\) is semi-continuous and \(Y\) is semi-grounded, then for each \(n \neq m\), \(U_n \cap sCl(U_m) = f^{-1}\{y_n\} \cap sCl(f^{-1}\{y_m\}) \subset f^{-1}\{y_n\} \cap \)
\[ f^{-1}(sCl(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap f^{-1}(\{y_m\}) = \emptyset. \] This contradicts with the strong semi-connectedness of \( X \). Thus, \( X \) is \( Y \)-semi-connected.

\( (\Leftarrow) \): Let \( X \) be \( Y \)-semi-connected for any countable \( T_1 \)-semi-grounded semi-isotonic space \((Y, sCl)\). Suppose that \( X \) is not strongly semi-connected. There exists a countable collection of pairwise semi-closure-separated sets \( \{U_n\} \) such that \( X = \cup U_n \). We take the space \((Z, sCl)\), where \( Z \) is the set of integers and \( sCl : P(Z) \to P(Z) \) is defined by \( sCl(K) = K \) for each \( K \subset Z \). Clearly \((Z, sCl)\) is a countable \( T_1 \)-semi-grounded semi-isotonic space. Put \( U_k \in \{U_n\} \). We define a function \( f : X \to Z \) by \( f(U_k) = \{x\} \) and \( f(X \setminus U_k) = \{y\} \) where \( x, y \in Z \) and \( x \neq y \). Since \( sCl(U_k) \cap U_n = \emptyset \) for all \( n \neq k \), then \( sCl(U_k) \cap \cup_{n \neq k} U_n = \emptyset \) and hence \( sCl(U_k) \subset U_k \). Let \( \emptyset \neq K \subset Z \). If \( x, y \in K \) then \( f^{-1}(K) = X \) and \( sCl(f^{-1}(K)) = sCl(X) \subset X = f^{-1}(K) = f^{-1}(sCl(K)) \). If \( x \in K \) and \( y \notin K \), then \( f^{-1}(K) = U_k \) and \( sCl(f^{-1}(K)) = sCl(U_k) \subset U_k = f^{-1}(K) = f^{-1}(sCl(K)) \). If \( y \in K \) and \( x \notin K \) then \( f^{-1}(K) = X \setminus U_k \). Since \( sCl(K) = K \) for each \( K \subset Z \), then \( sInt(K) = K \) for each \( K \subset Z \). Also, \( X \setminus U_k \subset \cup_{n \neq k} U_n \subset X \setminus sCl(U_k) \) = \( sInt(X \setminus U_k) \). Therefore, \( f^{-1}(sInt(K)) = X \setminus U_k = f^{-1}(K) \subset sInt(X \setminus U_k) \) = \( sInt(f^{-1}(K)) \). Hence we obtain that \( f \) is semi-continuous. Since \( f \) is not constant, this is a contradiction with the \( Z \)-semi-connectedness of \( X \). Hence, \( X \) is strongly semi-connected.

**Acknowledgement.** The third author is highly and greatly indebted to the King Fahd University of Petroleum and Minerals, for providing necessary research facilities during the preparation of this paper.

**References**


Addresses:

Miguel Caldas
Departamento de Matemática Aplicada
Universidade Federal Fluminense
Rua Mário Santos Braga, s/n
24020-140, Niterói, RJ BRASIL.
E-mail: gmamccs@vm.uff.br

Saeid Jafari
College of Vestsjaelland South
Herrestraede 11
4200 Slagelse, DENMARK.
E-mail: jafari@stofanet.dk

Raja M. Latif
Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals
Dhahran 31261 SAUDI ARABIA.
E-mail: raja@kfupm.edu.sa

Arafa A. Nasef
Department of Physics and Engineering Mathematics
Faculty of Engineering
Kafr El-Sheikh University
Kafr El-Sheikh, EGYPT.
E-mail: nasefa50@yahoo.com