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Abstract

In this paper, we show that a pointwise symmetric semi-isotonic semi-closure function is uniquely determined by the pairs of sets it separates. We then show that when the semi-closure function of the domain is semi-isotonic and the semi-closure function of the codomain is semi-isotonic and pointwise-semi-symmetric, functions which separate only those pairs of sets which are already separated are semi-continuous.

1 Introduction

The most important legacy of Norman Levine [1] was the introduction of semi-open sets which is one of the well-known notion of generalized open sets.

Throughout the present paper (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$, respectively. $A \subset X$ is called a semi-open set of X [1] if $A \subset Cl(Int(A))$. The complement of a semi-open set is called semi-closed [2]. The intersection of all semi-closed sets containing a set A is called the semi-closure of A [2] and is denoted by $sCl(A)$.

Definition 1 *(1) A generalized semi-closure space is a pair (X, sCl) consisting of a set X and a semi-closure function sCl, a function from the power set of X to itself.*
(2) The semi-closure of a subset A of X, denoted $sCl(A)$, is the image of A under sCl.

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(3) The semi-exterior of A is $sExt(A) = X \backslash sCl(A)$, and the semi-Interior of A is $sInt(A) = X \backslash sCl(X \backslash A)$.
(4) A is semi-closed if $A = sCl(A)$, A is semi-open if $A = sInt(A)$ and N is a semi-neighborhood of x if $x \in sInt(N)$.

Definition 2 *A semi-closure function sCl defined on X is:*
(1) semi-grounded if $sCl(\emptyset) = \emptyset$.
(2) semi-isotonic if $sCl(A) \subseteq sCl(B)$ whenever $A \subseteq B$.
(3) semi-enlarging if $A \subseteq sCl(A)$ for each subset A of X.
(4) semi-idempotent if $sCl(A) = sCl(sCl(A))$ for each subset A of X.
(5) semi-sub-linear if $sCl(A \cup B) \subseteq sCl(A) \cup sCl(B)$ for all $A, B \subseteq X$.

Definition 3 *(1) Subsets A and B of X are said to be semi-closure-separated in a generalized semi-closure space (X, sCl) (or simply, sCl-separated) if $A \cap sCl(B) = \emptyset$ and $sCl(A) \cap B = \emptyset$, or equivalently, if $A \subseteq sExt(B)$ and $B \subseteq sExt(A)$.*
(2) semi-Exterior points are said to be semi-closure-separated in a generalized semi-closure space (X, sCl) if for each $A \subseteq X$ and for each $x \in sExt(A)$, $\{x\}$ and A are sCl-separated.

Theorem 1.1 *Let (X, sCl) be a generalized semi-closure space in which semi-Exterior points are sCl-separated and let S be the pairs of sCl-separated sets in X. Then, for each subset A of X, the semi-closure of A is $sCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$.*

Proof. In any generalized semi-closure space $sCl(A)) \subseteq \{x \in X : \{\{x\}, A\} \notin S\}$. Suppose that $y \notin \{x \in X : \{\{x\}, A\} \notin S\}$, that is, $\{\{y\}, A\} \in S$. Then $\{y\} \cap sCl(A) = \emptyset$, and so $y \notin sCl(A)$.
Now, let $y \notin sCl(A)$. By hypothesis, $\{\{y\}, A\} \in S$. Therefore $y \notin \{x \in X : \{\{x\}, A\} \notin S\}$.

2 Some basic properties

Definition 4 *A semi-closure function sCl defined on a set X is said to be pointwise semi-symmetric when, for all $x, y \in X$, if $x \in sCl(\{y\})$, then $y \in sCl(\{x\})$.*

A generalized semi-closure space (X, sCl) is said to be semi-R_0 when, for all $x, y \in X$, if x is in each semi -neighborhood of y, then y is in each semi-neighborhood of x.

Corollary 2.1 *Let (X, sCl) a generalized semi-closure space in which semi-Exterior points are sCl -separated. Then sCl is pointwise semi-symmetric and (X, sCl) is semi-R_0.*

Proof. Let semi-Exterior points be sCl-separated in (X, sCl). If $x \in sCl(\{y\})$, then $\{x\}$ and $\{y\}$ are not sCl-separated. This means that $y \in sCl(\{x\})$. Hence, sCl is pointwise semi -symmetric.
Suppose that x belongs to every semi-neighborhood of y, that is, $x \in M$ whenever $y \in sInt(M)$. Letting $A = X \backslash M$ and rewriting contrapositively, $y \in sCl(A)$ whenever $x \in A$. Let $x \in sInt(N)$. $x \notin sCl(X \backslash N)$, so x is sCl-separated from $X \backslash N$. Hence $sCl(\{x\}) \subseteq N$. $x \in \{x\}$, so $y \in sCl(\{x\}) \subseteq N$. Hence (X, sCl) is semi-R_0.

Observe that these three axioms are not equivalent in general, but they are equivalent when the semi-closure function is semi-isotonic:

Theorem 2.2 *Let (X, sCl) be a generalized semi-closure space with sCl semi-isotonic. Then the following are equivalent:*
(1) sExterior points are sCl-separated.
(2) sCl is pointwise semi-symmetric.
(3) (X, sCl) is semi-R_0.

Proof. Suppose that (2) is true. Let $A \subseteq X$, and let $x \in sExt(A)$. Then, as sCl is semi-isotonic, for each $y \in A$, $x \notin sCl(\{y\})$, and thus, $y \notin sCl(\{x\})$. Hence $A \cap sCl(\{x\}) = \emptyset$. Therefore (2) implies (1). Moreover, by the previous corollary, (1) implies (2).
Suppose now that (2) is true and let $x, y \in X$ such that x is in every semi-neighborhood of y, i.e. $x \in N$ whenever $y \in sInt(N)$. Then $y \in sCl(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}$, $y \in sCl(\{x\})$. It follows that $x \in sCl(\{y\})$. Thus if $y \in B$, then $x \in sCl(\{y\}) \subseteq sCl(B)$, as sCl is semi-isotonic. Therefore, if $x \in sInt(C)$, then $y \in C$, that is, y is in every semi-neighborhood of x. Hence, (2) implies (3).
Now, let (X, sCl) be semi-R_0 and $x \in sCl(\{y\})$. Since sCl is semi-isotonic, $x \in sCl(B)$ whenever $y \in B$, or, equivalently, y is in every semi-neighborhood of x. Since (X, sCl) is semi-R_0, $x \in N$ whenever $y \in sInt(N)$. Therefore, $y \in sCl(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}$, $y \in sCl(\{x\})$. It follows that (3) implies (2).

Theorem 2.3 *Let S be a set of unordered pairs of subsets of a set X such that, for all $A, B, C \subseteq X$,*
(1) if $A \subseteq B$ and $\{B, C\} \in S$, then $\{A, C\} \in S$ and
(2) if $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, then $\{A, B\} \in S$.
Then there exists a unique pointwise semi-symmetric semi-isotonic semi-closure function sCl on X which semi -closure-separates the elements of S.

Proof. Define sCl by $sCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. If $A \subseteq B \subseteq X$ and $x \in sCl(A)$, then $\{\{x\}, A\} \notin S$. Thus, $\{\{x\}, B\} \notin S$, that is, $x \in sCl(B)$. Hence sCl is semi-isotonic. Moreover $x \in sCl(\{y\})$ if and only if $\{\{x\}, \{y\}\} \notin S$ if and only if $y \in sCl(\{x\})$. Thus sCl is pointwise semi-symmetric.
Suppose that $\{A, B\} \in S$. Then $A \cap sCl(B) = A \cap \{x \in X : \{\{x\}, B\} \notin S\} = \{x \in A : \{\{x\}, A\} \notin S\} = \emptyset$. Similarly, $sCl(A) \cap B = \emptyset$. Therefore, if $\{A, B\} \in S$, then A and B are sCl-separated.
Now suppose that A and B are sCl-separated. Then $\{x \in A : \{\{x\}, B\} \notin S\} = A \cap sCl(B) = \emptyset$ and $\{x \in B : \{\{x\}, A\} \notin S\} = sCl(A) \cap B = \emptyset$. Hence, $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$. Therefore, $\{A, B\} \in S$.

In the following we show that many properties of semi-closure functions can be expressed in terms of the sets they separate:

Theorem 2.4 *Let S be the pairs of sCl-separated sets of a generalized semi-closure space (X, sCl) in which sExterior points are semi-closure-separates. Then sCl is*
(1) semi-grounded if and only if for all $x \in X$ $\{\{x\}, \emptyset\} \in S$.
(2) semi-enlarging if and only if for all $\{A, B\} \in S$, A and B are disjoint.
(3) semi-sub-linear if and only if $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$ and $\{A, C\} \in S$.

Furthermore, if sCl is semi-enlarging and for all $A, B \subseteq X$, $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$, then sCl is semi -idempotent. Now, if sCl is semi-isotonic and semi -idempotent, then $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and

$\{\{y\}, A\} \notin S$ for each $y \in B$.

Proof. By Theorem 1.1, $sCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. Suppose that for all $x \in X$, $\{\{x\}, \emptyset\} \in S$. Then $sCl(\emptyset) = \{x \in X : \{\{x\}, \emptyset\} \notin S\} = \emptyset$. Hence sCl is semi-grounded. Conversely, if $\emptyset = sCl(\emptyset) = \{x \in X : \{\{x\}, \emptyset\} \notin S\}$, then $\{\{x\}, \emptyset\} \in S$, for all $x \in X$.

Assume that for all $\{A, B\} \in S$, A and B are disjoint. Since $\{\{a\}, A\} \notin S$ if $a \in A$, $A \subseteq sCl(A)$ for each $A \subseteq X$. Therefore, sCl is semi-enlarging. Conversely, Let sCl be semi-enlarging and $\{A, B\} \in S$. Then $A \cap B \subseteq sCl(A) \cap B = \emptyset$. Suppose that $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$ and $\{A, C\} \in S$. Let $x \in X$ and $B, C \subseteq X$ such that $\{\{x\}, B \cup C\} \notin S$. Then $\{\{x\}, B\} \notin S$ or $\{\{x\}, C\} \notin S$. Hence $sCl(B \cup C) \subseteq sCl(B) \cup sCl(C)$. Therefore, sCl is semi-sub-linear. Conversely, suppose that sCl is semi-sub-linear and let $\{A, B\}, \{A, C\} \in S$. Then $sCl(B \cup C) \cap A \subseteq (sCl(B) \cup sCl(C)) \cap A = (sCl(B) \cap A) \cup (sCl(C)) \cap A) = \emptyset$ and $(B \cup C) \cap sCl(A) = (B \cap sCl(A)) \cup (C \cap sCl(A)) = \emptyset$. Let sCl be semi -enlarging and suppose that $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$. Then $sCl(sCl(A)) \subseteq sCl(A)$: If $x \in sCl(sCl(A))$, then $\{\{x\}, sCl(A)\} \notin S$. $\{\{y\}, A\} \notin S$, for each $y \in sCl(A)$; hence $\{\{x\}, A\} \notin S$. Since sCl is semi-enlarging, $then sCl(A) \subseteq sCl(sCl(A))$. Therefore, $sCl(sCl(A)) = sCl(A)$, for each $A \subseteq X$.
suppose that sCl is semi-isotonic and semi -idempotent. Let $x \in X$ and $A, B \subseteq X$ such that $\{\{x\}, B\} \notin S$ and, for each $y \in B$, $\{\{y\}, A\} \notin S$. Then $x \in sCl(B)$ and for each $y \in B$, $y \in sCl(A)$, i.e. $B \subseteq sCl(A)$. Therefore, $x \in sCl(B) \subseteq sCl(sCl(A)) = sCl(A)$.

Definition 5 *If $(X, (sCl)_X)$ and $(Y, (sCl)_Y)$ are generalized semi-closure spaces, then a function $f : X \to Y$ is said to be*
(1) semi-closure-preserving if $f((sCl)_X(A)) \subseteq (sCl)_Y(f(A))$ for each $A \subseteq X$.
(2) semi-continuous if $(sCl)_X(f^{-1}(B)) \subseteq f^{-1}((sCl)_Y(B))$ for each $B \subseteq Y$.

Observe that in general, neither condition implies the other. Now, we have the following result:

Theorem 2.5 *Let* $(X, (sCl)_X)$ *and* $(Y, (sCl)_Y)$ *be generalized semi-closure spaces and let* $f : X \to Y$ *be a function.*
(1) If f *is semi-closure-preserving and* $(sCl)_Y$ *is semi-isotonic, then* f *is semi-continuous.*
(2) If f *is semi-continuous and* $(sCl)_X$ *is semi -isotonic, then* f *is semi-closure-preserving.*

Proof. Let f be semi-closure-preserving and $(sCl)_Y$ semi-isotonic. Let $B \subseteq Y$.

$f(sCl)_X(f^{-1}(B))) \subseteq (sCl)_Y(f(f^{-1}(B))) \subseteq (sCl)_Y(B)$ and hence,
$(sCl)_X(f^{-1}(B)) \subseteq f^{-1}(f((sCl)_X(f^{-1}(B)))) \subseteq f^{-1}((sCl)_Y(B))$.
Suppose that f is semi-continuous and $(sCl)_X$ is semi-isotonic. Let $A \subseteq X$. $(sCl)_X(A) \subseteq (sCl)_X(f^{-1}(f(A))) \subseteq f^{-1}((sCl)_Y(f(A)))$. Therefore, $f((sCl)_X(A)) \subseteq f(f^{-1}((sCl)_Y(f(A)))) \subseteq (sCl)_Y(f(A))$.

Definition 6 *Let* $(X, (sCl)_X)$ *and* $(Y, (sCl)_Y)$ *be generalized semi-closure spaces and let* $f : X \to Y$ *be a function. If for all* $A, B \subseteq X$, $f(A)$ *and* $f(B)$ *are not* $(sCl)_Y$*-separated whenever* A *and* B *are not* $(sCl)_X$*-separated, then we say that* f *is non-semi-separating.*

Observe that f is non-semi-separating if and only if A and B are $(sCl)_X$-separated whenever $f(A)$ and $f(B)$ are $(sCl)_Y$-separated.

Theorem 2.6 *Let* $(X, (sCl)_X)$ *and* $(Y, (sCl)_Y)$ *be generalized semi-closure spaces and let* $f : X \to Y$ *be a function.*
(1) If $(sCl)_Y$ *is semi-isotonic and* f *is non-semi -separating, then* $f^{-1}(C)$ *and* $f^{-1}(D)$ *are* $(sCl)_X$*-separated whenever* C *and* D *are* $(sCl)_Y$*-separated.*
(2) If $(sCl)_X$ *is semi-isotonic and* $f^{-1}(C)$ *and* $f^{-1}(D)$ *are* $(sCl)_X$*-separated whenever* C *and* D *are* $(sCl)_Y$*-separated, then* f *is non-semi-separating.*

Proof. Suppose that C and D are $(sCl)_Y$-separated subsets, where $(sCl)_Y$ is semi-isotonic. Let $A = f^{-1}(C)$ and $B = f^{-1}(D)$. $f(A) \subseteq C$ and $f(B) \subseteq D$ and since $(sCl)_Y$ is semi-isotonic, $f(A)$ and $f(B)$ are also $(sCl)_Y$-separated. It follows now that A and B are $(sCl)_X$-separated in X.
Suppose that $(sCl)_X$ is semi-isotonic and let $A, B \subseteq X$ such that $C = f(A)$ and $D = f(B)$

are $(sCl)_X$-separated. Then $f^{-1}(C)$ and $f^{-1}(D)$ are $(sCl)_X$-separated and since $(sCl)_X$ is semi-isotonic, $A \subseteq f^{-1}(f(A)) = f^{-1}(C)$ and $B \subseteq f^{-1}(f(B)) = f^{-1}(D)$ are $(sCl)_X$-separated as well.

Theorem 2.7 *Let $(X, (sCl)_X)$ and $(Y, (sCl)_Y)$ be generalized semi-closure spaces and let $f : X \to Y$ be a function. If f is semi-closure-preserving, then f is non-semi-separating.*

Proof. Suppose that f is semi-closure-preserving and $A, B \subseteq X$ are not $(sCl)_X$-separated. Suppose that $(sCl)_X(A) \cap B \neq \emptyset$. Then $\emptyset \neq f((sCl)_X(A) \cap B) \subseteq f((sCl)_X(A)) \cap f(B) \subseteq (sCl)_Y(f(A)) \cap f(B)$. Similarly, if $A \cap (sCl)_X(B) \neq \emptyset$, then $f(A) \cap (sCl)_Y(f(B)) \neq \emptyset$. Hence $f(A)$ and $f(B)$ are not $(sCl)_Y$-separated.

Corollary 2.8 *Let $(X, (sCl)_X)$ and $(Y, (sCl)_Y)$ be generalized semi-closure spaces with $(sCl)_X$ semi-isotonic and let $f : X \to Y$ be a function. If f is semi-continuous, then f is non-semi-separating.*

Proof. If f is semi-continuous and $(sCl)_X$ semi-isotonic, then by Theorem 2.5 (2) f is semi -closure-preserving. Now, by Theorem 2.7, f is non-semi-separating.

Theorem 2.9 *Let $(X, (sCl)_X)$ and $(Y, (sCl)_Y)$ be generalized semi-closure spaces which semi-Exterior points $(sCl)_Y$-separated in Y and let $f : X \to Y$ be a function. Then f is semi-closure-preserving if and only if f is non-semi -separating.*

Proof. By Theorem 2.7, if f is semi-closure-preserving, then f is non-semi-separating. Suppose that f is non-semi -separating and let $A \subseteq X$. If $(sCl)_X = \emptyset$, then $f((sCl)_X(A)) = \emptyset \subseteq (sCl)_Y(f(A))$.

Suppose $(sCl)_X(A) \neq \emptyset$. Let S_X and S_Y denote the pairs of $(sCl)_X$-separated subsets of X and the pairs of $(sCl)_Y$-separated subsets of Y, respectively. Let $y \in f((sCl)_X(A))$ and let $x \in (sCl)_X(A) \cap f^{-1}(\{y\})$. Since $x \in (sCl)_X(A)$, $\{\{x\}, A\} \notin S_X$ and since f non-semi-separating, $\{\{y\}, f(A)\} \notin S_Y$. Since semi-Exterior points are $(sCl)_Y$-separated, $y \in (sCl)_Y(f(A))$. Thus $f((sCl)_X(A)) \subseteq (sCl)_Y(f(A))$ for each $A \subseteq X$.

Corollary 2.10 *Let $(X,(sCl)_X)$ and $(Y,(sCl)_Y)$ be generalized semi-closure spaces with semi-isotonic closure functions and with $(sCl)_Y$-pointwise-semi-symmetric and let $f : X \to Y$ be a function. Then f is semi-continuous if and only if f non-semi -separating.*

Proof. Since $(sCl)_Y$ is semi-isotonic and pointwise-semi-symmetric, semi-Exterior points are semi -closure separated in $(Y,(sCl)_Y)$ (Theorem 2.2 (1)). Since both semi-closure functions are semi-isotonic, f is semi -closure-preserving (Theorem 2.5) if and only if f is semi -continuous. Hence, we can apply the Theorem 2.9.

3 Semi-connected generalized semi-closure spaces

Definition 7 *Let (X, sCl) be a generalized semi-closure space. X is said to be semi-connected if X is not a union of disjoint nontrivial semi-closure-separated pair of sets.*

Theorem 3.1 *Let (X, sCl) be a generalized semi-closure space with semi-grounded semi-isotonic semi-enlarging sCl. Then, the following are equivalent:*
(1) (X, sCl) is semi-connected,
(2) X can not be a union of nonempty disjoint semi-open sets.

Proof. (1)⇒(2): Let X be a union of nonempty disjoint semi-open sets A and B. Then, $X = A \cup B$ and this implies that $B = X \backslash A$ and A is a semi-open set. Thus, B is semi -closed and hence $A \cap sCl(B) = A \cap B = \varnothing$. By using similar way, we obtain $sCl(A) \cap B = \varnothing$. Hence, A and B are semi-closure-separated and hence X is not semi-connected. This is a contradiction.

(2)⇒(1): Suppose that X is not semi-connected. Then $X = A \cup B$, where A, B are disjoint semi-closure-separated sets, i.e. $A \cap sCl(B) = sCl(A) \cap B = \varnothing$. We have $sCl(B) \subset X \backslash A \subset B$. Since sCl is semi-enlarging, we obtain $sCl(B) = B$ and hence, B is semi-closed. By using $sCl(A) \cap B = \varnothing$ and similar way, it is obvious that A is semi-closed. But this is a contradiction.

Definition 8 *Let (X, sCl) be a generalized semi-closure space with semi-grounded semi-isotonic sCl. Then, (X, sCl) is called a T_1-semi-grounded semi-isotonic space if $sCl(\{x\}) \subset \{x\}$ for all $x \in X$.*

Theorem 3.2 *Let (X, sCl) be a generalized semi-closure space with λ-grounded semi-isotonic sCl. Then, the following are equivalent:*
(1) (X, sCl) is semi-connected,
(2) Any semi-continuous function $f : X \to Y$ is constant for all T_1-semi-grounded semi-isotonic spaces $Y = \{0, 1\}$.

Proof. (1)⇒(2): Let X be semi-connected. Suppose that $f : X \to Y$ is semi-continuous and it is not constant. Then there exists a set $U \subset X$ such that $U = f^{-1}(\{0\})$ and $X \backslash U = f^{-1}(\{1\})$. Since f is semi-continuous and Y is T_1-λ-grounded semi-isotonic space, then we have $Cl_\lambda(U) = sCl(f^{-1}(\{0\})) \subset f^{-1}(sCl\{0\}) \subset f^{-1}(\{0\}) = U$ and hence $sCl(U) \cap (X \backslash U) = \varnothing$. By using similar way we have $U \cap sCl(X \backslash U) = \varnothing$. This is a contradiction. Thus, f is constant.

(2)⇒(1): Suppose that X is not semi-connected. Then there exist semi-closure-separated sets U and V such that $U \cup V = X$. We have $sCl(U) \subset U$ and $sCl(V) \subset V$ and $X \backslash U \subset V$. Since sCl is semi-isotonic and U and V are semi-closure-separated, then $sCl(X \backslash U) \subset sCl(V) \subset X \backslash U$. If we consider the space (Y, sCl) by $Y = \{0, 1\}$, $sCl(\varnothing) = \varnothing$, $sCl(\{0\}) = \{0\}$, $sCl(\{1\}) = \{1\}$ and $sCl(Y) = Y$, then the space (Y, sCl) is a T_1-semi-grounded semi-isotonic space. We define the function $f : X \to Y$ as $f(U) = \{0\}$ and $f(X \backslash U) = \{1\}$. Let $A \neq \varnothing$ and $A \subset Y$. If $A = Y$, then $f^{-1}(A) = X$ and hence $sCl(X) = sCl(f^{-1}(A)) \subset X = f^{-1}(A) = f^{-1}(sCl(A))$. If $A = \{0\}$, then $f^{-1}(A) = U$ and hence $sCl(U) = sCl(f^{-1}(A)) \subset U = f^{-1}(A) = f^{-1}(sCl(A))$. If $A = \{1\}$, then $f^{-1}(A) = X \backslash U$ and hence $sCl(X \backslash U) = sCl(f^{-1}(A)) \subset X \backslash U = f^{-1}(A) = f^{-1}(sCl(A))$. Hence, f is semi-continuous. Since f is not constant, this is a contradiction.

Theorem 3.3 *Let $f : (X, sCl) \to (Y, sCl)$ and $g : (Y, sCl) \to (Z, sCl)$ be semi-continuous functions. Then, $gof : X \to Z$ is semi-continuous.*

Proof. Suppose that f and g are semi-continuous. For all $A \subset Z$ we have $sCl(gof)^{-1}(A) = sCl(f^{-1}(g^{-1}(A))) \subset f^{-1}(sCl(g^{-1}(A))) \subset f^{-1}(g^{-1}(sCl(A))) = (gof)^{-1}(sCl(A))$. Hence, $gof : X \to Z$ is semi-continuous.

Theorem 3.4 *Let (X, sCl) and (Y, sCl) be generalized semi -closure spaces with semi-grounded semi-isotonic sCl and $f : (X, sCl) \to (Y, sCl)$ be a semi -continuous function onto Y. If X is semi-connected, then Y is semi-connected.*

Proof. Suppose that $\{0, 1\}$ is a generalized semi-closure space with semi-grounded semi-isotonic sCl and $g : Y \to \{0, 1\}$ is a semi-continuous function. Since f is semi-continuous, by Theorem 3.3, $gof : X \to \{0, 1\}$ is semi-continuous. Since X is semi-connected, gof is constant and hence g is constant. By Theorem 3.2, Y is semi-connected.

Definition 9 *Let (Y, sCl) be a generalized semi-closure space with semi-grounded semi-isotonic sCl and more than one element. A generalized semi-closure space (X, sCl) with semi-grounded semi-isotonic sCl is called Y-semi-connected if any semi-continuous function $f : X \to Y$ is constant.*

Theorem 3.5 *Let (Y, sCl) be a generalized semi-closure space with semi-grounded semi-isotonic semi-enlarging sCl and more than one element. Then every Y-semi-connected generalized semi-closure space with semi-grounded semi-isotonic is semi-connected.*

Proof. Let (X, sCl) be a Y-semi-connected generalized semi-closure space with semi-grounded semi-isotonic sCl. Suppose that $f : X \to \{0, 1\}$ is a semi-continuous function, where $\{0, 1\}$ is a T_1-semi-grounded semi-isotonic space. Since Y is a generalized semi-closure space with semi-grounded semi-isotonic semi-enlarging sCl and more than one element, then there exists a semi-continuous injection $g : \{0, 1\} \to Y$. By Theorem 3.3, $gof : X \to Y$ is semi-continuous. Since X is Y-semi-connected, then gof is constant. Thus, f is constant and hence, by Theorem 3.2, X is semi-connected.

Theorem 3.6 *Let (X, sCl) and (Y, sCl) be generalized semi-closure spaces with semi-grounded semi-isotonic sCl and $f : (X, sCl) \to (Y, sCl)$ be a semi-continuous function onto Y. If X is Z-semi-connected, then Y is Z-semi-connected.*

Proof. Suppose that $g : Y \to Z$ is a semi-continuous function. Then $gof : X \to Z$ is semi-continuous. Since X is Z-semi-connected, then gof is constant. This implies that g is constant. Thus, Y is Z-semi-connected.

Definition 10 *A generalized semi-closure space* (X, sCl) *is strongly semi-connected if there is no countable collection of pairwise semi-closure-separated sets* $\{A_n\}$ *such that* $X = \cup A_n$.

Theorem 3.7 *Every strongly semi-connected generalized semi-closure space with semi-grounded semi-isotonic* sCl *is semi-connected.*

Theorem 3.8 *Let* (X, sCl) *and* (Y, sCl) *be generalized semi-closure spaces with semi-grounded semi-isotonic* sCl *and* $f : (X, sCl) \to (Y, sCl)$ *be a semi-continuous function onto* Y. *If* X *is strongly semi-connected, then* Y *is strongly semi-connected.*

Proof. Suppose that Y is not strongly semi-connected. Then, there exists a countable collection of pairwise semi-closure-separated sets $\{A_n\}$ such that $Y = \cup A_n$. Since $f^{-1}(A_n) \cap sCl(f^{-1}(A_m)) \subset f^{-1}(A_n) \cap f^{-1}(sCl(A_m)) = \varnothing$ for all $n \neq m$, then the collection $\{f^{-1}(A_n)\}$ is pairwise semi-closure-separated. This is a contradiction. Hence, Y is strongly semi-connected.

Theorem 3.9 *Let* $(X, (sCl)_X)$ *and* $(Y, (sCl)_Y)$ *be generalized semi-closure spaces. Then the following are equivalent for a function* $f : X \to Y$:
(1) f *is semi-continuous,*
(2) $f^{-1}(sInt(B)) \subseteq sInt(f^{-1}(B))$ *for each* $B \subseteq Y$.

Theorem 3.10 *Let* (X, sCl) *be a generalized semi-closure space with semi-grounded semi-isotonic* sCl. *Then* (X, sCl) *is strongly semi-connected if and only if* (X, sCl) *is* Y*-semi-connected for any countable* T_1*-semi-grounded semi-isotonic space* (Y, sCl).

Proof. ($\Rightarrow$): Let (X, sCl) be strongly semi-connected. Suppose that (X, sCl) is not Y-semi-connected for some countable T_1-semi-grounded semi-isotonic space (Y, sCl). There exists a semi-continuous function $f : X \to Y$ which is not constant and hence $K = f(X)$ is a countable set with more than one element. For each $y_n \in K$, there exists $U_n \subset X$ such that $U_n = f^{-1}(\{y_n\})$ and hence $Y = \cup U_n$. Since f is semi-continuous and Y is semi-grounded, then for each $n \neq m$, $U_n \cap sCl(U_m) = f^{-1}(\{y_n\}) \cap sCl(f^{-1}(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap$

$f^{-1}(sCl(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap f^{-1}(\{y_m\}) = \varnothing$. This contradicts with the strong semi-connectedness of X. Thus, X is Y-semi-connected.

($\Leftarrow$): Let X be Y-semi-connected for any countable T_1-semi-grounded semi-isotonic space (Y, sCl). Suppose that X is not strongly semi-connected. There exists a countable collection of pairwise semi-closure-separated sets $\{U_n\}$ such that $X = \cup U_n$. We take the space (Z, sCl), where Z is the set of integers and $sCl : P(Z) \to P(Z)$ is defined by $sCl(K) = K$ for each $K \subset Z$. Clearly (Z, sCl) is a countable T_1-semi-grounded semi-isotonic space. Put $U_k \in \{U_n\}$. We define a function $f : X \to Z$ by $f(U_k) = \{x\}$ and $f(X \backslash U_k) = \{y\}$ where x, $y \in Z$ and $x \neq y$. Since $sCl(U_k) \cap U_n = \varnothing$ for all $n \neq k$, then $sCl(U_k) \cap \cup_{n \neq k} U_n = \varnothing$ and hence $sCl(U_k) \subset U_k$. Let $\varnothing \neq K \subset Z$. If x, $y \in K$ then $f^{-1}(K) = X$ and $sCl(f^{-1}(K)) = sCl(X) \subset X = f^{-1}(K) = f^{-1}(sCl(K))$. If $x \in K$ and $y \notin K$, then $f^{-1}(K) = U_k$ and $sCl(f^{-1}(K)) = sCl(U_k) \subset U_k = f^{-1}(K) = f^{-1}(sCl(K))$. If $y \in K$ and $x \notin K$ then $f^{-1}(K) = X \backslash U_k$. Since $sCl(K) = K$ for each $K \subset Z$, then $sInt(K) = K$ for each $K \subset Z$. Also, $X \backslash U_k \subset \cup_{n \neq k} U_n \subset X \backslash sCl(U_k) = sInt(X \backslash U_k)$. Therefore, $f^{-1}(sInt(K)) = X \backslash U_k = f^{-1}(K) \subset sInt(X \backslash U_k) = sInt(f^{-1}(K))$. Hence we obtain that f is semi-continuous. Since f is not constant, this is a contradiction with the Z-semi-connectedness of X. Hence, X is strongly semi-connected.

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