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Abstract

In this paper, we introduce and study a new class of function by using the notions of b-θ-open sets and b-θ-closure operator called weakly BR-open functions. The connections between this function and other existing topological functions are studied.

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1 Introduction and Preliminaries

In 2006, Park [14] has introduced a new class of sets called b-θ-open sets. He showed that the b-θ-cluster points can be characterized by b-regular sets and the class of b-θ-open sets include the class of b-regular sets and introduced the notion of strongly b-θ-continuous function. Recently in 2008, Ekici [8] continued the work of Park and also introduced a new class called weakly BR-continuity. In this paper we define the notion of weakly BR-openness as a natural dual to the weakly BR continuity by using the notion of b-θ-open and b-θ-closed sets. We obtain several characterizations and properties of these functions. Moreover, we also study these functions comparing with other types of already known functions. It turns out that b-θ-openness implies weak b-θ-openness but not conversely. We show that under a certain condition the converse is also true.

Throughout the present paper, (X, τ) and (Y, σ) (or X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, the closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$. A subset A is said to be preopen [11] (resp. b-open [1], α-open [12]) if $A \subset Int(Cl(A))$ (resp. $A \subset Int(Cl(A)) \cup Cl(Int(A))$, $A \subset Int(Cl(Int(A)))$). A point $x \in X$ is called a θ-cluster point of X [17] if $A \cap Cl(U) \neq \emptyset$ for each open set U containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $Cl_\theta(A)$. A subset A is called θ-closed [17] if $Cl_\theta(A) = A$. The complement of a θ-closed set is called θ-open set. The complement of a b-open set is said to be b-closed. The intersection all b-closed sets of X containing A is called the b-closure of A and is denoted by $Cl_b(A)$. The union of all b-open sets of X contained in a subset A is called b-interior [1] and is denoted by $Int_b(A)$. The family of all b-open (resp. b-regular) sets is denoted by $BO(X)$ $(resp.BR(X))$. A point $x \in X$ is said to be a b-θ-cluster point of X [14] if $A \cap Cl_b(U) \neq \emptyset$ for each b-open set U of X containing x. The set of all b-θ-cluster points of A is called the b-θ-closure of A and is denoted by b-θ-$Cl(A)$. A subset A is said to be b-θ-closed if b-θ-$Cl(A) = A$. The complement of a b-θ-closed set is called b-θ-open set. The family of all b-θ-open (resp. b-θ-closed) sets is denoted by $B\theta O(X)$ (resp. $B\theta C(X)$). A point x of X is said to be a b-θ-Interior point of a subset A, denoted by b-θ-$Int(A)$, if there exists a b-regular set U of X containing x such that $U \subset A$.

It should be mentioned that the following diagram holds for a subset A of a space X:

b-regular $\Rightarrow$ b-θ-open $\Rightarrow$ b-open.

A function $f : (X, \tau) \to (Y, \sigma)$ is called:
(i) weakly open [15, 16] if $f(U) \subset Int(f(Cl(U)))$ for each open set $U \subset X$.
(ii) contra b-θ-open (resp. contra b-θ-closed) if $f(U)$ is b-θ-closed (resp. b-θ-open) in Y for each open (resp. closed) set U of X.
(iii) contra b-θ-closed if $f(U)$ is b-θ-open in Y for each closed set U of X.
(iv) strongly continuous [10, 2] if for every subset A of X, $f(Cl(A)) \subset f(A)$.

2 Weakly BR-open Functions

In this section, We define the concept of weak BR-openness as a natural dual to the weak BR-continuity due to Ekici [8].

Definition 1 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly BR-open if $f(U) \subset b$-θ-$Int(f(Cl(U)))$ for each open set U of X.*

Definition 2 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be b-θ-open if $f(U)$ is b-θ-open in Y for each open set U of X.*

Observe that every b-θ-open function is weakly BR-open, but the converse is not true in general.

Example 2.1 *(i) A weakly BR-open function need not be b-θ-open.*
Let $X = \{a, b\}$, $\tau = \{\emptyset, \{b\}, X\}$, $Y = \{x, y\}$ and $\sigma = \{\emptyset, \{x\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is weakly BR-open, but it is not b-θ-open since $f(b)$ is not a b-θ-open set in Y.

(ii) A weakly-open function need not be weakly BR-open (hence not open).
Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}, \sigma = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$ such that $f : (X, \tau) \to (X, \sigma)$ is the identity function. Then f is weakly-open but it is not weakly BR-open since $f(\{a\}) \not\subset b$-θ-$Int(f(Cl(\{a\})))$.

Lemma 2.2 *[14]. Let A be a subset of a space X. Then:*
(1) b-θ-$Cl(A) = \cap\{V : A \subset V and V \subset BR(X)\}$.
(2) $x \in$ b-θ-$Cl(A)$ if and only if $A \cap U \neq \emptyset$ for each b-regular set U of X containing x.
(3) b-θ-$Cl(A)$ is b-θ-closed.
(4) Any intersections of b-θ-closed sets is b-θ-closed and any union of b-θ-open sets is b-θ-open.
(5) A is b-θ-open in X if and only if for each $x \in A$ there exists a b-regular set U containing x such that $x \in U \subset A$.

Theorem 2.3 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:*
(i) f is weakly BR-open.
(ii) $f(Int_\theta(A)) \subset b\text{-}\theta\text{-}Int(f(A))$ for every subset A of X.
(iii) $Int_\theta(f^{-1}(B)) \subset f^{-1}(b\text{-}\theta\text{-}Int(B))$ for every subset B of Y.
(iv) $f^{-1}(b\text{-}\theta\text{-}Cl(B)) \subset Cl_\theta(f^{-1}(B))$ for every subset B of Y.
(v) $f(Int(F)) \subset b\text{-}\theta\text{-}Int(f(F))$ for each closed subset F of X.
(vi) $f(Int(Cl(U))) \subset b\text{-}\theta\text{-}Int(f(Cl(U)))$ for each open subset U of X.
(vii) $f(U) \subset b\text{-}\theta\text{-}Int(f(Cl(U)))$ for every regular open subset U of X.
(viii) $f(U) \subset b\text{-}\theta\text{-}Int_\theta(f(Cl(U)))$ for every α-open subset U of X.
(ix) For each $x \in X$ and each open set U of X containing x, there exists a b-θ-open set V of Y containing $f(x)$ such that $V \subset f(Cl(U))$.

Proof. $(i) \Rightarrow (ii)$: *Let A be any subset of X and $x \in Int_\theta(A)$. Then there exists an open set U such that $x \in U \subset Cl(U) \subset A$. Then, $f(x) \in f(U) \subset f(Cl(U)) \subset f(A)$. Since f is weakly BR-open, $f(U) \subset b\text{-}\theta\text{-}Int(f(Cl(U))) \subset b\text{-}\theta\text{-}Int(f(A))$. It implies that $f(x) \in b\text{-}\theta\text{-}Int(f(A))$. This shows that $x \in f^{-1}(b\text{-}\theta\text{-}Int(f(A)))$. Thus $Int_\theta(A) \subset f^{-1}(b\text{-}\theta\text{-}Int(f(A)))$. Thus $f(Int_\theta(A)) \subset b\text{-}\theta\text{-}Int(f(A))$.*
$(ii) \Rightarrow (i)$: *Let U be an open set in X. As $U \subset Int_\theta(Cl(U))$ implies, $f(U) \subset f(Int_\theta(Cl(U))) \subset b\text{-}\theta\text{-}Int(f(Cl(U)))$. Hence f is weakly BR-open.*
$(ii) \Rightarrow (iii)$: *Let B be any subset of Y. Then by (ii), $f(Int_\theta(f^{-1}(B))) \subset b\text{-}\theta\text{-}Int(B)$. Therefore $Int_\theta(f^{-1}(B)) \subset f^{-1}(b\text{-}\theta\text{-}Int(B))$.*
$(iii) \Rightarrow (ii)$: *This is obvious.*
$(iii) \Rightarrow (iv)$: *Let B be any subset of Y. Using (iii), we have $X - Cl_\theta(f^{-1}(B)) = Int_\theta(X - f^{-1}(B)) = Int_\theta(f^{-1}(Y - B)) \subset f^{-1}(b\text{-}\theta\text{-}Int(Y - B)) = f^{-1}(Y - b\text{-}\theta\text{-}Cl(B)) = X - f^{-1}(b\text{-}\theta\text{-}Cl(B))$. Therefore, we obtain $f^{-1}(b\text{-}\theta\text{-}Cl(B)) \subset Cl_\theta(f^{-1}(B))$.*
$(iv) \Rightarrow (iii)$: *Similarly we obtain, $X - f^{-1}(b\text{-}\theta\text{-}Int(B)) \subset X - Int_\theta(f^{-1}(B))$ for every subset B of Y, i.e., $Int_\theta(f^{-1}(B)) \subset f^{-1}(b\text{-}\theta\text{-}Int(B))$.*
Proofs of $(i) \Rightarrow (v) \Rightarrow (vi) \to (vii) \Rightarrow (viii) \Rightarrow (i)$ are straightforward and are omitted.
$(i) \Rightarrow (ix)$: *Let $x \in X$ and U be an open set in X with $x \in U$. Since f is weakly BR-open, $f(x) \in f(U) \subset b\text{-}\theta\text{-}Int(f(Cl(U)))$. Let $V = b\text{-}\theta\text{-}Int(f(Cl(U)))$. Hence $V \subset f(Cl(U))$ with*

$f(x) \in V$.

$(ix) \Rightarrow (i)$: Let U be an open set in X and $y \in f(U)$. It follows from (ix) that $V \subset f(Cl(U))$ for some b-θ-open set V in Y containing y. Hence we have, $y \in V \subset$ b-θ-$Int(f(Cl(U)))$. This shows that $f(U) \subset$ b-θ-$Int(f(Cl(U)))$, i.e., f is a weakly b-θ-open function.

Theorem 2.4 *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective function. Then the following statements are equivalent.*

(i) f is weakly BR-open,

(ii) For each $x \in X$ and each open set U of X containing x, there exists a b-regular set V containing $f(x)$ such that $V \subset f(Cl(U))$,

(iii) b-θ-$Cl(f(Int(Cl(U)))) \subset f(Cl(U))$ for each subset U of X,

(iv) b-θ-$Cl(f(Int(F)) \subset f(F)$ for each regular closed subset F of X,

(v) b-θ-$Cl(f(U)) \subset f(Cl(U))$ for each open subset U of X,

(vi) b-θ-$Cl(f(U)) \subset f(Cl(U))$ for each preopen subset U of X,

(vii) $f(U) \subset$ b-θ-$Int(f(Cl(U)))$ for each preopen subset U of X,

(viii) $f^{-1}($b-θ-$Cl(B)) \subset Cl_\theta(f^{-1}(B))$ for each subset B of Y,

(ix) b-θ-$Cl(f(U)) \subset f(Cl_\theta(U))$ for each subset U of X,

(x) b-θ-$Cl(f(Int(Cl_\theta(U)))) \subset f(Cl_\theta(U))$ for each subset U of X.

Proof. *$(i) \Rightarrow (ii)$: Let $x \in X$ and U be any open subset of X containing x. Since f is weakly BR-open, $f(x) \in f(U) \subset$ b-θ-$Int(f(Cl(U)))$. Then , there exists a b-regular set V of Y containing $f(x)$ such that $V \subset f(Cl(U))$.*

$(ii) \Rightarrow (iii)$: Let $x \in X$, $U \subset X$ and $f(x) \in Y - f(Cl(U))$. We have $x \in X - Cl(U)$. This implies that there exists an open set G containing x such that $G \cap U = \emptyset$. We have $Cl(G) \cap Int(Cl(U)) = \emptyset$. By (ii) there exists a b-regular set V of Y containing $f(x)$ such that $V \subset f(Cl(G))$. We have $V \cap f(Int(Cl(U)) = \emptyset$ and hence $f(x) \in X -$b-θ-$Cl(f(Int(Cl(U))))$. Thus b-θ-$Cl(f(Int(Cl(U)))) \subset f(Cl(U))$.

$(iii) \Rightarrow (iv)$: Let F be any regular closed set of X. Then b-θ-$Cl(f(Int(F)) =$ b-θ-$Cl(f(Int(Cl(Int(F))))) \subset f(Cl(Int(F))) = f(F)$.

$(iv) \Rightarrow (v)$: Let U be an open subset of X. Since $Cl(U)$ is regular closed in X, b-θ-$Cl(f(U)) \subset$ b-θ-$Cl(f(Int(Cl(U)))) \subset f(Cl(U))$.

(v) ⇒ (i): Let U be any open subset of X. Since $X - Cl(U)$ is open in X, then $Y - b$ -θ-$Int(f(Cl(U)))$= b-θ-$Cl(f(X - (Cl(U))) \subset f(Cl(X - Cl(U))) \subset Y - f(U)$. Thus $f(U) \subset$ b-θ-$Int(f(Cl(U)))$. Hence f is weakly BR-open.

(ii) ⇒ (vi): Let U be any preopen subset of X and $y \in Y - f(Cl(U))$. Then there exists an open set W of X containing $f^{-1}(y)$ such that $W \cap U = \emptyset$. Also, we have $Cl(W \cap U) = \emptyset$. Since U is preopen, then $U \cap Cl(W) \subset Int(Cl(U)) \cap Cl(W) \subset Cl(Int(Cl(U)) \cap W) \subset Cl(Int(Cl(U) \cap W)) \subset Cl(Int(Cl(U \cap W))) \subset Cl(U \cap W) = \emptyset$. By (ii) and W is an open set in X containing $f^{-1}(y)$, there exists a b-regular set S of Y containing y such that $S \subset f(Cl(W))$. Thus $f^{-1}(S) \cap U = \emptyset$ and hence $S \cap f(U) = \emptyset$. Furthermore, $y \in Y-$ b-θ-$Cl(f(U)))$ and hence b-θ-$Cl(f(U)) \subset f(Cl(U))$.

(vi) ⇒ (vii): Let U be any preopen subset of X. Since $X - Cl(U)$ is open in X, then $Y -$ b-θ-$Int(f(Cl(U)))$= b-θ-$Cl(f(X - Cl(U))) \subset f(Cl(X - Cl(U))) \subset Y - f(U)$. Hence $f(U) \subset$ b-θ-$Int(f(Cl(U)))$.

(vii) ⇒ (ii): Let $x \in X$, U be any open subset of X containing x. Then $f(x) \in f(U) \subset b - \theta - Int(f(Cl(U)))$. This implies that there exists a b-regular set V of Y containing $f(x)$ such that $V \subset f(Cl(U))$.

(ii) ⇒ (viii): Let $B \subset Y$ and $x \in f^{-1}($b-θ-$Cl(B))$. Then $f(x) \in$ b-θ-$Cl(B)$. Let W be any open set of X containing x. There exists a b-regular set S containing $f(x)$ such that $S \subset f(Cl(W))$. Since $f(x) \in$ b-θ-$Cl(B)$, then $S \cap B \neq \emptyset$. We have $\emptyset \neq f^{-1}(S) \cap f^{-1}(B) \subset Cl(W) \cap f^{-1}(B)$ and then $x \in Cl_\theta(f^{-1}(B))$. Thus $f^{-1}($b-θ-$Cl(B)) \subset Cl_\theta(f^{-1}(B))$

(viii) ⇒ (ix): Let $U \subset X$. Then $f^{-1}($b-θ-$Cl(f(U))) \subset Cl_\theta(f^{-1}(f(U))) \subset Cl_\theta(U)$ and hence b-θ-$Cl(f(U)) \subset f(Cl_\theta(U))$.

(ix) ⇒ (x): Let $U \subset X$. Since $Cl_\theta(U)$ is closed in X, then b-θ-$Cl(f(Int(Cl_\theta(U)))) \subset f(Cl_\theta(Int(Cl_\theta(U)))) = f(Cl(Int(Cl_\theta(U)))) \subset f(Cl_\theta(U))$.

(x) ⇒ (i): Let U be any open subset of X. We have $U \subset Int(Cl(U)) = Int(Cl_\theta(U)$. Moreover b-θ-$Cl(f(U)) \subset$ b-θ-$Cl(f(Int(Cl_\theta(U)))) \subset f(Cl_\theta(U)) = f(Cl(U))$. It follows by (v) ⇔ (i).

Theorem 2.5 *If X is a regular space, then for a bijective function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:*

(i) f is weakly BR-open.

(ii) For each θ-open set A in X, $f(A)$ is b-θ-open in Y.

(iii) For any set B of Y and any θ-closed set A in X containing $f^{-1}(B)$, there exists a b-θ-closed set F in Y containing B such that $f^{-1}(F) \subset A$.

Proof. *(i) $\Rightarrow$ (ii): Let A be a θ-open set in X. Then $Y - f(A)$ is a set in Y such that by (i) and Theorem 2.4(iv), $f^{-1}(b\text{-}\theta\text{-}Cl(Y - f(A))) \subset Cl_\theta(f^{-1}(Y - f(A)))$. Therefore, $X - f^{-1}(b\text{-}\theta\text{-}Int(f(A))) \subset Cl_\theta(X - A) = X - A$. Then, we have $A \subset f^{-1}(b\text{-}\theta\text{-}Int(f(A)))$ which implies $f(A) \subset b\text{-}\theta\text{-}Int(f(A))$. Hence $f(A)$ is a b-θ-open subset of Y.*

(ii) $\Rightarrow$ (iii): Let B be any set in Y and A be a θ-closed set in X such that $f^{-1}(B) \subset A$. Since $X - A$ is θ-open in X, by (ii), $f(X - A)$ is b-θ-open in Y. Let $F = Y - f(X - A)$. Then F is b-θ-closed and $B \subset F$. Now, $f^{-1}(F) = f^{-1}(Y - f(X - A)) = X - f^{-1}(f(A)) \subset A$.

(iii) $\Rightarrow$ (i) : Let B be any set in Y. Let $A = Cl_\theta(f^{-1}(B))$. Since A is θ-closed set in X and $f^{-1}(B) \subset A$. Then there exists a b-θ-closed set F in Y containing B such that $f^{-1}(F) \subset A$. Since F is b-θ-closed $f^{-1}(\, b\text{-}\theta\text{-}Cl_\theta(B)) \subset f^{-1}(F) \subset Cl_\theta(f^{-1}(B))$. Therefore by Theorem 2.4, f is a weakly BR-open function.

Theorem 2.6 *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective function. If X is regular, the following are equivalent:*

(i) f is weakly BR-open.

(ii) f is b-θ-open.

(iii) For each $x \in X$ and each open set U of X containing x, there exists a b-open set V of Y containing $f(x)$ such that $C_b(V) \subset f(U)$.

Proof. *(i) $\Rightarrow$ (ii): Let W be a nonempty open subset of X. For each x in W, let U_x be an open set such that $x \in U_x \subset Cl(U_x) \subset W$. Hence we have $W = \cup\{U_x : x \in W\} = \cup\{Cl(U_x) : x \in W\}$ and , $f(W) = \cup\{f(U_x) : x \in W\} \subset \cup\{b\text{-}\theta\text{-}Int(f(Cl(U_x))) : x \in W\} \subset b\text{-}\theta\text{-}Int(f(\cup\{Cl(U_x) : x \in W\}) = b\text{-}\theta\text{-}Int(f(W))$. Thus f is b-θ-open.*

(ii) $\Rightarrow$ (i): Obvious.

(i) $\Rightarrow$ (iii): Let $x \in X$ and U be an open set of X containing x. Since X is regular, there

exists an open set W of X containing x such that $W \subset Cl(W) \subset U$. Since f is weakly BR-open, there exists a b-regular (hence, b-open and b-closed) set V of Y containing $f(x)$ such that $Cl_b(V) = V \subset f(Cl(W)) \subset f(U)$.

(iii) $\Rightarrow$ (i): It follows from ([14], Theorem 3.1). It is $A \in BO(X)$ if and only if $Cl_b(A) \in BR(X)$.

Theorem 2.7 *If $f : (X, \tau) \to (Y, \sigma)$ is weakly BR-open and strongly continuous, then f is b-θ-open.*

Proof. *Let U be an open subset of X. Since f is weakly BR-open, $f(U) \subset$ b-θ-$Int(f(Cl(U)))$. However, because f is strongly continuous, $f(U) \subset$ b-θ-$Int(f(U))$ and therefore $f(U)$ is b-θ-open.*

The following example shows that strong continuity does not yield a decomposition of b-θ-openness.

Example 2.8 *A b-θ-open function need not be strongly continuous.*

Let $X = \{a, b, c\}$ and τ be the indiscrete topology for X. Then the identity function $f : (X, \tau) \to (X, \tau)$ is a b-θ-open (hence weakly BR-open) function which is not strongly continuous.

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be relatively weakly open [4] provide that $f(U)$ is open in $f(Cl(U))$ for every open subset U of X.

Theorem 2.9 *A function $f : (X, \tau) \to (Y, \sigma)$ is b-θ-open if f is weakly BR-open and relatively weakly open.*

Proof. *Assume f is weakly BR-open and relatively weakly open. Let U be an open subset of X and let $y \in f(U)$. Since f is relatively weakly open , there is an open subset V of Y for which $f(U) = f(Cl(U)) \cap V$. Because f is weakly BR-open, it follows that $f(U) \subset$ b-θ-$Int(f(Cl(U)))$. Then $y \in$ b-θ-$Int(f(Cl(U))) \cap V \subset f(Cl(U)) \cap V = f(U)$ and therefore $f(U)$ is b-θ-open.*

Theorem 2.10 *If $f : (X,\tau) \to (Y,\sigma)$ is contra b-θ-closed, then f is a weakly BR-open function.*

Proof. *Let U be an open subset of X. Then, we have $f(U) \subset f(Cl(U)) =$ b-θ-$Int(f(Cl(U)))$.*

Theorem 2.11 *If $f : (X,\tau) \to (Y,\sigma)$ is bijective contra b-θ-open, then f is a weakly BR-open function.*

Proof. *Let U be an open subset of X. Since $f(U)$ is b-θ-closed, then b-θ-$Cl(f(U)) = f(U) \subset f(Cl(U))$. By Theorem 24(v), f is weakly BR-open.*

Theorem 2.12 *Let $f : (X,\tau) \to (Y,\sigma)$ be a bijective function. If $f(Cl_\theta(U)$ is b-θ-closed in Y for every subset U of X, then f is weakly BR-open.*

Proof. *Let U be an open subset of X. Since $f(Cl_\theta(U))$ is b-θ-closed, then b-θ-$Cl(f(U)) \subset$ b-θ-$Cl(f(Cl_\theta(U))) = f(Cl_\theta(U))$. By Theorem 2.4(ix), f is weakly BR-open.*

Next, we define a dual form called complementary weakly BR-open function as follows:

Definition 3 *A function $f : (X,\tau) \to (Y,\sigma)$ is called complementary weakly BR-open (written as c.w.br-o) if for each open set U of X, $f(Fr(U))$ is b-θ-closed in Y, where $Fr(U)$ denotes the frontier of U.*

Example 2.13 *A weakly BR-open function need not be c.w.br-o.*
Let (X,τ) be as in Example 2.1, i.e., let $X = \{a,b\}$, $\tau = \{\emptyset,\{b\},X\}$, $Y = \{x,y\}$ and $\sigma = \{\emptyset,\{x\},Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is weakly BR-open. But it is not c.w.br-o., since $F_r(\{b\}) = Cl(\{b\}) - \{b\} = \{a\}$ and $f(F_r(\{b\})) = \{x\}$ is not a b-θ-closed set in Y.

Example 2.14 *c.w.br-o. does not imply weakly b-θ-open.*
Let $X = \{a,b\}$, $\tau = \{\emptyset,\{a\},\{b\},X\}$, $Y = \{x,y\}$ and $\sigma = \{\emptyset,\{y\},Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is not weakly BR-open, but f is c.w.br-o., since $F_r(\{a\}) = Cl(\{a\}) - \{a\} = \emptyset$, $F_r(\{b\}) = Cl(\{b\}) - \{b\} = \emptyset$ and $f(\emptyset) = \emptyset$ is b-θ-closed.

Examples 2.13 and 2.14 demonstrate the independence of complementary weakly b-θ-openness and weakly BR-openness.

Theorem 2.15 *Let $B\theta O(Y)$ be closed under intersections. If $f : (X, \tau) \to (Y, \sigma)$ is bijective weakly BR-open and c.w.br-o, then f is b-θ-open.*

Proof. *Let U be an open subset of X with $x \in U$. Since f is weakly BR-open, by Theorem 2.3(ix) there exists a b-θ-open set V containing $f(x) = y$ such that $V \subset f(Cl(U))$. Now $Fr(U) = Cl(U) - U$ and thus $x \notin Fr(U)$. Hence $y \notin f(Fr(U))$ and therefore $y \in V - f(Fr(U))$. Put $V_y = V - f(Fr(U))$. Then V_y a b-θ-open set since f is c.w.br-o. Since $y \in V_y$, then $y \in f(Cl(U))$. But $y \notin f(Fr(U))$ and thus $y \notin f(Fr(U)) = f(Cl(U)) - f(U)$ which implies that $y \in f(U)$. Therefore $f(U) = \cup\{V_y : V_y \in B\theta O(Y), y \in f(U)\}$. Hence f is b-θ-open.*

The following theorem is a variation of a result of C.Baker [3] in which contra-closedness is replaced with weakly BR-open and closed by contra-M-b-θ-closed. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra M-b-θ-closed, provided that $f(F)$ is b-θ-open for each b-θ-closed subset F of X.

Theorem 2.16 *Let $B\theta C(X)$ be closed under unions. If $f : (X, \tau) \to (Y, \sigma)$ is weakly BR-open and if for each b-θ-closed subset F of X and each fiber $f^{-1}(y) \subset X - F$ there exists an open subset U of X for which $F \subset U$ and $f^{-1}(y) \cap Cl(U) = \phi$, then f is contra-M-b-θ-closed.*

Proof. *Assume that F is a b-θ-closed subset of X and let $y \in Y - f(F)$. Thus $f^{-1}(y) \subset X - F$ and hence there exists an open subset U of X for which $F \subset U$ and $f^{-1}(y) \cap Cl(U) = \phi$. Therefore $y \in Y - f(Cl(U)) \subset Y - f(F)$. Since f is weakly BR-open, then $f(U) \subset$ b-θ-$Int(f(Cl(U)))$. By complement, we obtain $y \in$ b-θ-$Cl(Y - f(Cl(U))) \subset Y - f(F)$. Let $B_y =$ b-θ-$Cl(Y - f(Cl(U)))$. Then B_y is a b-θ-closed subset of Y containing y. Hence $Y - f(F) = \cup\{B_y : y \in Y - f(F)\}$ is b-θ-closed and therefore $f(F)$ is b-θ-open.*

Recall that a space X is said to be b-connected [9] if X can not be expressed as the union of two nonempty disjoint b-open sets.

Theorem 2.17 *If $f : (X, \tau) \to (Y, \sigma)$ is a bijective weakly BR-open of a space X onto a b-connected space Y, then X is connected.*

Proof. *Suppose that X is not connected. There exist nonempty open sets U_1 and U_2 such that $U_1 \cap U_2 = \phi$ and $U_1 \cup U_2 = X$. Then U_1 and U_2 are clopen in X. Since f is weakly BR-open, we have $f(U_i) \subset$ b-θ-$Int(f(Cl(U_i)))$ =b-θ-$Int(f(U_i))$ for i = 1, 2. This implies that $f(U_i)$ is b-open in Y for i=1,2. We have $f(U_1) \cap f(U_2) = \phi$ and $f(U_1) \cup f(U_2) = Y$. Moreover, $f(U_1)$ and $f(U_2)$ are nonempty and Y is not b-connected.*

Definition 4 *A space X is said to be hyperconnected [13] if every nonempty open subset of X is dense in X.*

Theorem 2.18 *If X is a hyperconnected space, then a function $f : (X, \tau) \to (Y, \sigma)$ is weakly BR-open if and only if $f(X)$ is b-θ-open in Y.*

Proof. *The sufficiency is clear. For the necessity observe that for any open subset U of X, $f(U) \subset f(X) =$ b-θ-$Int(f(X)) =$ b-θ-$Int(f(Cl(U)))$.*

Theorem 2.19 *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective weakly BR-open function. The following hold:*
(i) $f(B)$ is b-θ-closed in Y for every θ-closed set B of X.
(ii) $f(A)$ is b-θ-open in Y for every θ-open set A of X.

Proof. *It follows from Theorem 2.4.*

3 Description

Generalized open and closed sets proved to be useful not only in fuzzy topology, digital topology and multifunctions with application in economy but also in relation to notions and results which may directly or indirectly be applicable to topology and its applications as a whole. In this paper we are using b-θ-open sets. In a 0-dimensional submaximal and extremally disconnected space b-θ-open set is equivalent with preopen sets [11] which, among others, *proved to be very important with respect to the mathematical model of the computer screen (see [5] and [6] and [7]). We believe that the class of functions introduced in this paper and the related notions may have different range of applications not only in fuzzy theory, digital topology, multifunctions and state transforms in theoretical computer science but also frame theory and lattice theory which have applications in different respects.*

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References

[1] *D. Andrijevic, On b-open sets, Mat. Vesnik,* **48** *(1996), 59-64.*

[2] *S.P. Arya and R. Gupta, Strongly continuous mappings, Kyungpook Math.J.,* **14** *(1974), 131-143.*

[3] *C.W. Baker, Contra-open functions and contra-closed functions, Math. Today* **15** *(1997), 19-24.*

[4] *C.W. Baker, Decomposition of openness, Internat. J. Math. & Math. Sci.,* **17**, *(1994), 413-415.*

[5] *R. Devi, K. Bhuvananeswari and H. Maki, Weak form on $g\rho$-closed sets, where $\rho \in \{\alpha, \alpha^*, \alpha^{**}\}$, and digital planes, Mem. Fac. Kochi Univ. Ser. A Math., 14(1993), 41-54.*

[6] *R. Devi, S. N. rajappriya, K. Muthukumaraswamy and H. Maki, ξ-closed sets in topological spaces and digital planes, Sci. Math. Jap., 64(3)(2006), 533-549.*

[7] *M. Fujimoto, H. Maki, T. Noiri and S. Takigawa, The digital plane is quasi-submaximal, Q and A in General Topology, 22(2004), 163-168.*

[8] *E. Ekici, Generalization of weakly clopen and strongly θ-b-continuous functions, Chaos Solitons and Fractals,* **38**, *(2008), 79-88.*

[9] *E. Ekici, On separated sets and connected spaces, Demonstrat. Math.* **XL***(1), (2007), 209-217.*

[10] *N. Levine, Strong continuity in topological spaces, Amer. Math. Monthly,* **67** *(1960), 269.*

[11] *A. Mashhour, M. Abd El-Monsef and S. El-Deeb, On pre-continuous and weak pre-continuous mappings, Proc. Math. Phys. Soc. Egypt* **53** *(1982), 47-53.*

[12] *O. Njastad, On some classes of nearly open sets, Pacific J. Math.,***15** *(1965), 961-970.*

[13] T. Noiri, *A generalization of closed mappings*, Atti. Accad. Naz. Lince Rend. Cl. Sci. Fis. Mat. Natur., **8** (1973), 210-214.

[14] J.H. Park, *Strongly θ-b-continuous functions*, Acta Math. Hung., **110**(4) (2006), 347-359.

[15] D.A. Rose, *On weak openness and almost openness*, Internat.J. Math.&Math. Sci., **7** (1984), 35-40.

[16] D.A. Rose and D.S. Jankovic,. *Weakly closed functions and Hausdorff spaces*, Math. Nachr., **130** (1987),105-110.

[17] N.V. Velicko, *H-closed topological spaces*, Amer. Math. Soc. Transl., **78** (1968), 103-118.

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