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θ -SOBRIETY VIA NETS

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Abstract

P. Sünnerhauf [11] studied the important notion of sobriety in terms of nets. In this paper, by the same token, we present and study the notion of θ-sobriety by utilizing the notion of θ-open sets.

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1 Introduction

In 1943, Fomin [5] (see, also [6]) introduced the notion of θ-continuity. The notions of θ-open subsets, θ-closed subsets and θ-closure were introduced by Veličko [12] for the purpose of studying the important class of H-closed spaces in terms of arbitrary filterbases. Dickman and Porter [2], [3], Joseph [8] continued the work of Veličko. Recently Noiri and Jafari [10] have also obtained several new and interesting results related to these sets.

In what follows (X, τ) and (Y, σ) (or X and Y) denote topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by Int(A) and Cl(A), respectively. A point x ∈ X is called a θ-cluster point of A if A ∩ Cl(U) ≠ ∅ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure of A. A subset A is called θ-closed if A and its θ-closure coincide. The complement of a θ-closed set is called θ-open. It is shown in [9] that the collection of all θ-open sets in a space X forms a topology denoted by τθ. A topological space (X, τ) is called θ-compact [7] if every cover of the space by θ-open sets has a finite subcover. We denote the filter of θ-open neighbourhoods [1] of some point x in X by Ωθ(x).
Definition 1 A function \( f : (X, \tau) \to (Y, \sigma) \) is called \( \theta \)-continuous if for each \( x \in X \) and each open set \( V \) in \( Y \) containing \( f(x) \), there exists an open set \( U \) in \( X \) containing \( x \) such that \( f(Cl(U)) \subseteq Cl(V) \).

Definition 2 Two topological spaces \( (X, \tau) \) and \( (Y, \sigma) \) are \( \theta \)-homeomorphic [12] if there exists a one-to-one and onto function \( f : (X, \tau) \to (Y, \sigma) \) such that \( f \) and \( f^{-1} \) are both \( \theta \)-continuous.

2 \( \theta \)-Sobriety

Definition 3 A space \( (X, \tau) \) is said to be \( \theta \)-sober if it is \( \theta \)-homeomorphic with the space of points of its frame of \( \theta \)-open sets.

Recall that the \( \theta \)-saturated set and the \( \theta \)-kernel of a set \( A \) [1] are the same, i.e. \( \cap \{ O \in \tau_{\theta} \mid A \subseteq O \} \).

\( \theta \)-sobriety implies the existence of LUB (= least upper bound) of subsets which are directed with respect to the order of \( \theta \)-specialization, i.e. if \( (X, \tau) \) is a topological space, then the order of \( \theta \)-specialization of \( X \) is defined by \( x \leq_{\tau} y \iff x \in Cl_{\theta}(\{y\}) \).

Theorem 2.1 Let \( X \) be a \( \theta \)-sober space for which finite intersection of \( \theta \)-compact \( \theta \)-saturated subsets are \( \theta \)-compact.

1. Every cover of a \( \theta \)-compact set by \( \theta \)-open sets contains a finite subcover.
2. If the intersection of \( \theta \)-compact \( \theta \)-saturated sets is contained in a \( \theta \)-open set, then the same is true for an intersection of finitely many of them.

Now we offer a new notion called \( \theta \)-observative net by which we characterize \( \theta \)-sobriety.

Definition 4 A net \( (x_i)_i \in I \) in a space \( X \) is \( \theta \)-observative if for all \( i \in I \) and for all \( U \in \tau_{\theta} \), we have that \( x_i \in U \) implies that the net is eventually in the set \( U \).

Recall that a filter base \( \mathcal{F} \) is called \( \theta \)-convergent [13] to a point \( x \) in \( X \) if for any open set \( U \) containing \( x \) there exists \( B \in \mathcal{F} \) such that \( B \subseteq Cl(U) \).
Definition 5 A θ-observative net \((x_i)_i \in I\) strongly θ-converges to a point \(x\) in \(X\) if it θ-converges to \(x\) with respect to \(\tau_\theta\), and also if it satisfies that \(x\) is an element of every θ-open set which eventually contains the net. We denote it by \(x_i \xrightarrow{\theta} x\).

Lemma 2.2 If \((x_i)_i \in I\) is a θ-observative net in a space \((X, \tau)\), then \(x_i \xrightarrow{\theta} x\) if and only if \(x_i \xrightarrow{\tau_\theta} x\) with respect to \(\tau_\theta\).

Proof. Let \(x_i \xrightarrow{\theta} x\) and \(x \in A\) for some θ-closed set \(A\). If the net is not eventually contained in \(A\), then it is frequently in the θ-open set \(X - A\). By hypothesis, the net is θ-observative and therefore \([x]_{\geq i} \subseteq X - A\) for some tail. Hence \(x \in X - A\) as a consequence of strong θ-convergence. But this is a contradiction and hence the claim.

Now suppose that \(x_i \xrightarrow{\tau_\theta} x\) with respect to the \(\tau_\theta\). Then a θ-open set which eventually contains the net but does not contain \(x\) establishes a θ-neighbourhood \(X - U\) of \(x\) which has been forgotten by the net. Therefore strong θ-convergence follows readily.

Here we establish the θ-derived filter \(\mathcal{F}(x)\mathcal{I}\) for a net \((x_i)_i \in I\) as follows:

\[
\mathcal{F}(x)\mathcal{I} = \{ U \in \tau_\theta \mid \exists i \in I \text{ such that } [x]_{\geq i} \subseteq U \}.
\]

Definition 6 A filter \(\mathcal{F} \subseteq \tau\) is called θ-completely prime if for every \(O \in \mathcal{F}\) and for any family of θ-open sets \((O_i)_{i \in I}\) such that \(O \subseteq \bigcup_I O_i\), then \(O_k \in \mathcal{F}\) for some \(k \in I\).

Theorem 2.3 A filter derived from a θ-observative net is θ-completely prime.

Proof. Let the net \((x_i)_i \in I\) be θ-observative and \([x]_{\geq i} \subseteq \bigcup_{j \in J} U_j\) for some collection of θ-open sets and some index \(i \in I\). Hence \(x_i \in \bigcup_{j \in J} U_j\). Therefore there is some \(j_0 \in J\) with \(x_j \in U_{j_0}\). Since the net is θ-observative, then it follows that some tail is contained in \(U_{j_0}\). Thus the set is a filter.

Proposition 2.4 If \((x_i)_i \in I\) is a θ-observative net, then \(x_i \xrightarrow{\theta} x\) if and only if \(\mathcal{F}(x)\mathcal{I} = \Omega_\theta(x)\).
Proof. Obvious; since \((x_i)_{i \in I}\) strongly \(\theta\)-converges to \(x\) if and only if it is true that \(x \in U\) is equivalent to the existence of some \(i \in I\) with \([x]_{\geq i} \subseteq U\).

But how can we deal with the situation where a space is \(\theta\)-sober if all its \(\theta\)-observative nets strongly \(\theta\)-converge?

In such situation, we need the following construction:

Assign to each \(\theta\)-completely prime filter \(\mathcal{F}\) a \(\theta\)-observative net such that \(\mathcal{F}(x)\mathcal{I} = \mathcal{F}\).

**Theorem 2.5** Let \(\mathcal{F}\) be a filter of \(\theta\)-open subsets of the space \((X, \tau)\). Then \(\mathcal{F}\) is \(\theta\)-completely prime if and only if for all \(U \in \mathcal{F}\), there exists \(x \in U\) with the property that \(x \in G\) implies \(G \in \mathcal{F}\) for every \(G \in \tau_{\theta}\).

**Proof.** Suppose that \(\mathcal{F}\) has this property and \(\bigcup_{j \in I} U_j \in \mathcal{F}\). Take \(x \in \bigcup_{j \in I} U_j\) with \(x \in G \in \tau_{\theta} \Rightarrow G \in \mathcal{F}\). Clearly, \(x \in U_{j_0}\) for some \(j_0 \in J\). Therefore \(U_{j_0} \in \mathcal{F}\). This means that the filter \(\mathcal{F}\) is \(\theta\)-completely prime.

Conversely, assume that \(U \in \mathcal{F}\) has not this property. It follows that for each \(x \in U\), there is \(G_x \in \tau_{\theta}\) with \(G_x \notin \mathcal{F}\). Put \(U_x := G_x \cap U\). Now we have \(U_x \notin \mathcal{F}\) for all \(x \in U\) and \(U = \bigcup_{x \in U} U_x \in \mathcal{F}\). But this is against our hypothesis that \(\mathcal{F}\) is \(\theta\)-completely prime and hence the claim.

Now we give a new appropriate construction. Let \(\mathcal{F}\) be a \(\theta\)-completely prime filter of \(\theta\)-open sets on \((X, \tau)\). Take \(\mathcal{F}\) with reserved set inclusion as order to be the index set of our net. If \(U \in \mathcal{F}\), pick \(x_U \in U\) with the property that \(x_U \in G\) implies \(G \in \mathcal{F}\). This is possible by the previous Theorem. A net established in this way is called a \(\theta\)-derived net from the filter.

**Lemma 2.6** A \(\theta\)-derived net from a \(\theta\)-completely prime filter is \(\theta\)-observative.

**Proof.** Let \(U \in \mathcal{F}\) and \(x_U \in G \in \tau_{\theta}\). Then \(G \in \mathcal{F}\) by choice of \(x_U\). If \(V \subseteq G\), then \(x_V \in V \subseteq G\). Hence \([x]_{\geq i} \subseteq G\). Therefore the net is \(\theta\)-observative.
Theorem 2.7 Every $\theta$-completely prime filter equals the $\theta$-derived filter of any of its $\theta$-derived nets.

Proof. Clearly, $[x]_{\geq i} \subseteq U$ for $U \in \mathcal{F}$. Thus $U \in \mathcal{F} \implies U \in \mathcal{F}(x)\mathcal{I}$. Conversely, $U \in \mathcal{F}(x)\mathcal{I} \implies [x]_{\geq i} \subseteq U$ for some $G \in \mathcal{F}$. Therefore, $x_U \in U$ which implies that $U \in \mathcal{F}$ by choice of $x_G$.

Theorem 2.8 A topological space is $\theta$-sober if and only if every $\theta$-observative net strongly $\theta$-converges to a unique point.

Proof. Obvious.

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