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Abstract

In this paper, we study some properties of functions with strongly $\alpha$-closed graphs by utilizing $\alpha$-open sets and the $\alpha$-closure operator.

1 Introduction and preliminaries

The notion of $\alpha$-open sets was introduced by O. Njåstad [20] in 1965. Since then it has been widely investigated in the literature (see, [1], [2], [3], [9], [10], [11], [12], [15], [16], [17], [18], [19], [21], [23], [24], [26], [27], [28]). Functions with strongly closed graphs were introduced by Herrington and Long [7] to characterize $H$-closed spaces. Properties of such functions were further investigated by Long and Herrington [14] and Noiri [23]. In this paper, we study some properties of functions with strongly $\alpha$-closed graphs by utilizing $\alpha$-open sets and the $\alpha$-closure operator.

Throughout this paper, by $(X, \tau)$ and $(Y, \sigma)$ (or $X$ and $Y$) we always mean topological spaces. Let $A$ be a subset of $X$. We denote the interior, the closure and the complement of a set $A$ by $Int(A)$, $Cl(A)$ and $X\setminus A$ or $A^c$ respectively. A subset $A$ of a topological space $(X, \tau)$ is called $\alpha$-open [20] (resp. semi-open [13]) if $A \subseteq Int(Cl(Int(A)))$ (resp. $A \subseteq Cl(Int(A))$). The complement of an $\alpha$-open (resp. semi-open) set is called $\alpha$-closed (resp. semi-closed [5]). By $\alpha O(X, \tau)$ (resp. $SO(X, \tau)$, $\alpha C(X, \tau)$), we denote the family of all

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α-open (resp. semi-open, α-closed) sets of $X$. We set $\alpha O(X, x) = \{ U \mid x \in U \in \alpha O(X, \tau) \}$, $O(X, x) = \{ U \mid x \in U \in \tau \}$ and $\alpha C(X, x) = \{ U \mid x \in U \in \alpha C(X, \tau) \}$. The intersection of all α-closed (resp. semi-closed) sets containing $A$ is called the α-closure (resp. semi-closure [4]) of $A$, denoted by $\alpha Cl(A)$ (resp. $sCl(A)$). A set $U$ in a topological space $(X, \tau)$ is an α-neighborhood [16] of a point $x$ if $U$ contains an α-open set $V$ such that $x \in V$.

**Lemma 1.1** The intersection of an arbitrary collection of α-closed sets in $(X, \tau)$ is α-closed.

**Corollary 1.2** [15]. Let $A$ be a subset of $X$. Then, $x \in \alpha Cl(A)$ if and only if for any α-open set $U$ in $X$ containing $x$, $A \cap U \neq \emptyset$.

**Lemma 1.3** Let $A$ and $B$ be subsets of a space $(X, \tau)$, then the following properties hold:

1. $A \subseteq \alpha Cl(A)$.
2. If $A \subseteq B$, then $\alpha Cl(A) \subseteq \alpha Cl(B)$.
3. $\alpha Cl(A)$ is α-closed.
4. $\alpha Cl(\alpha Cl(A)) = \alpha Cl(A)$.
5. $A$ is α-closed if and only $A = \alpha Cl(A)$.

**Corollary 1.4** Let $A_i (i \in I)$ be a subset of a space $(X, \tau)$, then the following properties hold:

1. $\alpha Cl(\cap \{A_i : i \in I\}) \subseteq \cap \{\alpha Cl(A_i) : i \in I\}$.
2. $\alpha Cl(\cup \{A_i : i \in I\}) \supseteq \cup \{\alpha Cl(A_i) : i \in I\}$.

**Definition 1** A topological space $(X, \tau)$ is said to be:

1. $\alpha-T_1$ [17], if for any pair of distinct points $x$ and $y$ in $X$, there exist an α-open set $U$ in $X$ containing $x$ but not $y$ and an α-open set $V$ in $X$ containing $y$ but not $x$.
2. $\alpha-T_2$ [15], if for any pair of distinct points $x$ and $y$ in $X$, there exist $U \in \alpha O(X, x)$ and $V \in \alpha O(X, y)$ such that $U \cap V = \emptyset$.

**Lemma 1.5** A topological space $(X, \tau)$ is $\alpha-T_2$ if and only if it is $T_2$.

**Proof.** This is shown in [27] and a simple proof is given in [[24], Corollary 4.7].
Definition 2 A function $f : X \to Y$ is said to be
(1) $\alpha$-continuous \cite{19} if $f^{-1}(V) \in \alpha O(X)$ for each open set $V$ of $Y$;
(2) weakly $\alpha$-continuous \cite{23} if for each $x \in X$ and each $V \in O(Y, f(x))$, there exists $U \in \alpha O(X, x)$ such that $f(U) \subset Cl(V)$.

Lemma 1.6 Let $(X, \tau)$ be a topological space. Then $\alpha Cl(V) = Cl(V)$ for each $V \in SO(X)$.

Proof. For any $V \in SO(X)$, $\alpha Cl(V) = V \cup Cl(Int(Cl(V))) = V \cup Cl(Int(V)) = V \cup Cl(V) = Cl(V)$.

Lemma 1.7 A function $f : X \to Y$ is weakly $\alpha$-continuous if and only if for each $x \in X$ and each $V \in \alpha O(Y, f(x))$, there exists $U \in \alpha O(X, x)$ such that $f(U) \subset \alpha Cl(V)$.

Proof. Necessity. Let $x \in X$ and $V \in \alpha O(Y, f(x))$. Then $f(x) \in V \subset Int(Cl(Int(V)))$ and there exists $U \in \alpha O(X, x)$ such that $f(U) \subset Cl(Int(Cl(Int(V))))$. By Lemma 1.6, we have $Cl(Int(Cl(Int(V)))) = Cl(Int(V)) = Cl(V) = \alpha Cl(V)$. Therefore, $f(U) \subset \alpha Cl(V)$.

Sufficiency. Let $x \in X$ and $V \in O(Y, f(x))$. There exists $U \in \alpha O(X, x)$ such that $f(U) \subset \alpha Cl(V)$. By Lemma 1.6, we obtain $f(U) \subset Cl(V)$.

2 Strongly $\alpha$-closed graphs

If $f : (X, \tau) \to (Y, \sigma)$ is any function, then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of $f$ \cite{8}.

Definition 3 A function $f : X \to Y$ has a strongly $\alpha$-closed (resp. strongly closed \cite{7})
graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \alpha O(X, x)$ (resp. $U \in O(X, x)$)
and $V \in O(Y, y)$ such that $(U \times Cl(V)) \cap G(f) = \emptyset$.

Lemma 2.1 For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:
(1) $G(f)$ is strongly $\alpha$-closed;
(2) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \alpha O(X, x)$ and $V \in O(Y, y)$ such that
f(U) \cap \text{Cl}(V) = \emptyset;

(3) For each \((x,y) \in (X \times Y) \setminus G(f)\), there exist \(U \in \alpha O(X,x)\) and \(V \in \alpha O(Y,y)\) such that \((U \times \alpha \text{Cl}(V)) \cap G(f) = \emptyset\);

(4) For each \((x,y) \in (X \times Y) \setminus G(f)\), there exist \(U \in \alpha O(X,x)\) and \(V \in \alpha O(Y,y)\) such that \(f(U) \cap \alpha \text{Cl}(V) = \emptyset\).

Proof. It is obvious that (1) \iff (2) and (3) \iff (4).

(1) \Rightarrow (3): Since \(\tau \subset \alpha O(X) \subset SO(X)\), by Lemma 1.6 the proof is obvious.

(3) \Rightarrow (1): Let \((x,y) \in (X \times Y) \setminus G(f)\). There exist \(U \in \alpha O(X,x)\) and \(V \in \alpha O(Y,y)\) such that \((U \times \alpha \text{Cl}(V)) \cap G(f) = \emptyset\). Put \(G = \text{Int}(\text{Cl}(\text{Int}(V)))\). Then \(y \in V \subset G \in \sigma\) and \(\text{Cl}(G) = \text{Cl}(V) = \alpha \text{Cl}(V)\). Therefore, we obtain \((U \times \text{Cl}(G)) \cap G(f) = (U \times \alpha \text{Cl}(V)) \cap G(f) = \emptyset\). This shows that \(G(f)\) is strongly \(\alpha\)-closed.

**Theorem 2.2** If \(f : X \to Y\) is a function with the strongly \(\alpha\)-closed graph, then for each \(x \in X\), \(f(x) = \cap \{\alpha \text{Cl}(f(U)) : U \in \alpha O(X,x)\}\).

Proof. Suppose the theorem is false. Then there exists a \(y \neq f(x)\) such that \(y \in \cap \{\alpha \text{Cl}(f(U)) : U \in \alpha O(X,x)\}\). This implies that \(y \in \alpha \text{Cl}(f(U))\) for every \(U \in \alpha O(X,x)\). So \(V \cap f(U) \neq \emptyset\) for every \(V \in \alpha O(Y,y)\). This, in its turn, indicates that \(\alpha \text{Cl}(V) \cap f(U) \supset V \cap f(U) \neq \emptyset\) which contradicts the hypothesis that \(f\) is a function with strongly \(\alpha\)-closed graph. Hence the theorem holds.

**Theorem 2.3** If \(f : X \to Y\) is \(\alpha\)-continuous and \(Y\) is \(T_2\), then \(G(f)\) is strongly \(\alpha\)-closed.

Proof. Let \((x,y) \in (X \times Y)\setminus G(f)\). The \(T_2\)-ness of \(Y\) gives the existence of a set \(V \in O(Y,y)\) such that \(f(x) \notin \text{Cl}(V)\). Now \(Y \setminus \text{Cl}(V) \in O(Y,f(x))\). Therefore, by the \(\alpha\)-continuity of \(f\) there exists \(U \in \alpha O(X,x)\) such that \(f(U) \subset Y \setminus \text{Cl}(V)\). Consequently, \(f(U) \cap \text{Cl}(V) = \emptyset\) and therefore \(G(f)\) is strongly \(\alpha\)-closed.

It is shown in ([14], Theorem 3) and ([22], Theorem 2) that if \(f : X \to Y\) is surjective and \(G(f)\) is strongly closed, then \(Y\) is Hausdorff. The following theorem is a slight improvement of this result.
Theorem 2.4 If \( f : X \to Y \) is surjective and has a strongly \( \alpha \)-closed graph \( G(f) \), then \( Y \) is both \( T_2 \) and \( \alpha \)-\( T_1 \).

Proof. Let \( y_1, y_2 (y_1 \neq y_2) \in Y \). The surjectivity of \( f \) gives a \( x_1 \in X \) such that \( f(x_1) = y_1 \). Now \( (x_1, y_2) \in (X \times Y) \setminus G(f) \). The strongly \( \alpha \)-closedness of \( G(f) \) provides \( U \in \alpha O(X, x_1) \), \( V \in O(Y, y_2) \) such that \( f(U) \cap \text{Cl}(V) = \emptyset \), whence one infers that \( y_1 \not\in \text{Cl}(V) \). This means that there exists \( W \in O(Y, y_1) \) such that \( W \cap V = \emptyset \). So, \( Y \) is \( T_2 \) and \( T_2 \)-ness always guarantees \( \alpha \)-\( T_1 \)-ness. Hence \( Y \) is \( \alpha \)-\( T_1 \).

Theorem 2.5 A space \( X \) is \( T_2 \) if and only if the identity function \( \text{id} : X \to X \) has a strongly \( \alpha \)-closed graph \( G(\text{id}) \).

Proof. Necessity. Let \( X \) be \( T_2 \). Since the identity function \( \text{id} : X \to X \) is continuous, it follows from Theorem 2.4 that \( G(\text{id}) \) is strongly \( \alpha \)-closed.

Sufficiency. Let \( G(\text{id}) \) be a strongly \( \alpha \)-closed graph. Then the surjectivity of \( \text{id} \) and strong \( \alpha \)-closedness of \( G(\text{id}) \) together imply, by Theorem 2.4, that \( X \) is \( T_2 \).

Theorem 2.6 If \( f : X \to Y \) is an injection and \( G(f) \) is strongly \( \alpha \)-closed, then \( X \) is \( \alpha \)-\( T_1 \).

Proof. Since \( f \) is injective, for any pair of distinct points \( x_1, x_2 \in X \), \( f(x_1) \neq f(x_2) \). Then \( (x_1, f(x_2)) \in (X \times Y) \setminus G(f) \). Since \( G(f) \) is strongly \( \alpha \)-closed, there exist \( U \in \alpha O(X, x_1) \), \( V \in O(Y, f(x_2)) \) such that \( f(U) \cap \text{Cl}(V) = \emptyset \). Therefore \( x_2 \not\in U \). Pursuing the same reasoning as before we obtain a set \( W \in \alpha O(X, x_2) \) such that \( x_1 \not\in W \). Hence \( Y \) is \( \alpha \)-\( T_1 \).

Theorem 2.7 If \( f : X \to Y \) is a bijection with the strongly \( \alpha \)-closed graph, then both \( X \) and \( Y \) are \( \alpha \)-\( T_1 \).

Proof. The proof is an immediate consequence of Theorems 2.4 and 2.6.

Theorem 2.8 If a function \( f : X \to Y \) is a weakly \( \alpha \)-continuous injection with the strongly \( \alpha \)-closed graph \( G(f) \), then \( X \) is \( T_2 \).
Proof. Since $f$ is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. Therefore $(x_1, f(x_2)) \in (X \times Y) \setminus G(f)$. Since $G(f)$ is strongly $\alpha$-closed, there exist $U \in \alpha O(X, x_1)$, $V \in O(Y, f(x_2))$ such that $f(U) \cap Cl(V) = \emptyset$; hence $U \cap f^{-1}(Cl(V)) = \emptyset$. Consequently, $f^{-1}(Cl(V)) \subset X \setminus U$. Since $f$ is weakly $\alpha$ continuous, there exists $W \in \alpha O(X, x_2)$ such that $f(W) \subset Cl(V)$. From this and the foregoing it follows that $W \subset f^{-1}(Cl(V)) \subset X \setminus U$; hence $W \cap U = \emptyset$. Thus for the pair of distinct points $x_1, x_2 \in X$, there exist $U \in \alpha O(X, x_1)$, $W \in \alpha O(X, x_2)$ such that $W \cap U = \emptyset$. By Lemma 1.5, this guarantees the $T_2$-ness of $X$.

Corollary 2.9 If a function $f : X \to Y$ is an $\alpha$-continuous injection with the strongly $\alpha$-closed graph, then $X$ is $T_2$.

Proof. The proof follows from Theorem 2.8 and the fact that every $\alpha$-continuous is weakly $\alpha$-continuous.

Remark 2.10 If $f$ is not $T_2$ in Corollary 2.9, then even $\alpha$-continuity need not imply a strongly $\alpha$-closed graph. For example, let $X$ be a topological space containing more than one point with the indiscrete topology and let $id : X \to X$ the identity function. Then $id$ is certainly $\alpha$-continuous, but the graph of $id$ is not strongly $\alpha$-closed because $X \times X$ has the indiscrete topology and hence the graph of $id$ being the diagonal set, which is different from the whole space, is not strongly $\alpha$-closed.

Theorem 2.11 If $f : X \to Y$ is a weakly $\alpha$-continuous bijection with the strongly $\alpha$-closed graph, then both $X$ and $Y$ are $T_2$.

Proof. The proof follows from Theorems 2.8 and 2.4.

Lemma 2.12 Every clopen subset of a quasi $H$-closed space $X$ is quasi $H$-closed relative to $X$. 

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Proof. Let $B$ be any clopen subset of a quasi $H$-closed space $X$. Let $\{O_\lambda : \lambda \in \Omega \}$ be any cover of $B$ by open sets in $X$. Then the family $F = \{O_\lambda : \lambda \in \Omega \} \cup \{X \setminus B\}$ is a cover of $X$ by open sets in $X$. Because of quasi $H$-closedness of $X$ there exists a finite subfamily $F^* = \{O_{\lambda_i} : 1 \leq i \leq n\} \cup \{X \setminus B\}$ of $F$ whose closure covers $X$. So, because of clopenness of $B$ we now infer that the family $\{Cl(O_{\lambda_i}) : 1 \leq i \leq n\}$ covers $B$. Therefore, $B$ is quasi $H$-closed relative to $X$.

**Theorem 2.13** If $Y$ is a quasi $H$-closed extremally disconnected space, then a function $f : X \to Y$ with the strongly $\alpha$-closed graph $G(f)$ is weakly $\alpha$-continuous.

**Proof.** Let $x \in X$ and $V \in O(Y, f(x))$. Take any $y \in Y \setminus Cl(V)$. Then $(x, y) \in (X \times Y) \setminus G(f)$. Now the strong $\alpha$-closedness of $G(f)$ induces the existence of $U_y(x) \in \alpha O(X, x)$, $V_y \in O(Y, y)$ such that $f(U_y(x)) \cap Cl(V_y) = \emptyset$...(*).

Now extremal disconnectedness of $Y$ induces the clopenness of $Cl(V)$ and hence $Y \setminus Cl(V)$ is also clopen. Now $\{V_y : y \in Y \setminus Cl(V)\}$ is a cover of $Y \setminus Cl(V)$ by open sets in $Y$. By Lemma 2.12, there exists a finite subfamily $\{V_{y_i} : 1 \leq i \leq n\}$ such that $Y \setminus Cl(V) \subset \bigcup_{i=1}^{n} Cl(V_{y_i})$. Let $W = \bigcap_{i=1}^{n} U_{y_i}(x)$, where $U_{y_i}(x)$ are $\alpha$-open sets in $X$ satisfying (*). Also, $W \in \alpha O(X, x)$.

Now $f(W) \cap (Y \setminus Cl(V)) \subset f[\bigcap_{i=1}^{n} U_{y_i}(x)] \cap \bigcup_{i=1}^{n} Cl(V_{y_i}) \subset \bigcup_{i=1}^{n} (f[U_{y_i}(x)] \cap Cl(V_{y_i})) = \emptyset$, by (*).

Therefore, $f(W) \subset Cl(V)$ and this indicates that $f$ is weakly $\alpha$-continuous.

Noiri [22] showed that if $G(f)$ is strongly closed then $f$ has the following property:

(P) For every set $B$ which is quasi $H$–closed relative to $Y$, $f^{-1}(B)$ is a closed set of $X$.

Analogously, we have the following theorem.

**Theorem 2.14** If a function $f : X \to Y$ has a strongly $\alpha$-closed graph $G(f)$, then $f$ enjoys the following property:

(P*’) For every set $F$ which is quasi $H$-closed relative to $Y$, $f^{-1}(F)$ is $\alpha$-closed in $X$.

**Proof.** Let $f^{-1}(F)$ be not $\alpha$-closed in $X$. Then there exists $x \in \alpha Cl(f^{-1}(F)) \setminus f^{-1}(F)$. Let $y \in F$. Then $(x, y) \in (X \times Y) \setminus G(f)$. Strong $\alpha$-closedness of $G(f)$ gives the existence of
$U_y(x) \in \alpha O(X, x)$ and $V_y \in O(Y, y)$ such that $f(U_y(x)) \cap Cl(V_y) = \emptyset$. (*)

Clearly $\{V_y : y \in F\}$ is a cover of $F$ by open sets in $Y$. Since $F$ is quasi $H$-closed relative to $Y$, there exist a finite number of open sets $V_{y_1}, V_{y_2}, ..., V_{y_n}$ in $Y$ such that $F \subset \bigcup_{i=1}^{n} Cl(V_{y_i})$.

Let $U = \bigcap_{i=1}^{n} U_{y_i}(x)$, where $U_{y_i}(x)$ are the $\alpha$-open sets in $X$ satisfying (*). Also $U \in \alpha O(X, x)$.

Now $f(U) \cap F \subset f[\bigcap_{i=1}^{n} U_{y_i}(x)] \cap (\bigcup_{i=1}^{n} Cl(V_{y_i})) \subset \bigcup_{i=1}^{n} (f[U_{y_i}(x)] \cap Cl(V_{y_i})) = \emptyset$. But since $x \in \alpha Cl(f^{-1}(F))$, $U \cap f^{-1}(F) \neq \emptyset$; hence $f(U) \cap F \neq \emptyset$. This is a contradiction. Hence the result holds.

3 Additional properties

**Lemma 3.1** For a topological space $X$, the following properties are equivalent:

1. $X$ is Urysohn;
2. For every pair of distinct points $x, y \in X$, there exist $U \in \alpha O(X, x), V \in \alpha O(X, y)$ such that $Cl(U) \cap Cl(V) = \emptyset$;
3. For every pair of distinct points $x, y \in X$, there exist $U \in \alpha O(X, x), V \in \alpha O(X, y)$ such that $\alpha Cl(U) \cap \alpha Cl(V) = \emptyset$.

**Proof.** (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (3): Since $\alpha Cl(U) = Cl(U)$ for each $U \in \alpha (X)$ by Lemma 1.6, this is obvious.

(3) $\Rightarrow$ (1): Suppose that (3) holds. For every pair of distinct points $x, y$, there exist $U \in \alpha O(X, x), V \in \alpha O(X, y)$ such that $\alpha Cl(U) \cap \alpha Cl(V) = \emptyset$. Now, put $G = Int(Cl(Int(U)))$ and $H = Int(Cl(Int(V)))$, then $G$ and $H$ are open sets containing $x$ and $y$, respectively. Furthermore, $Cl(G) \cap Cl(H) = Cl(U) \cap Cl(V) = \alpha Cl(U) \cap \alpha Cl(V) = \emptyset$. Therefore, $X$ is Urysohn.

Recall, that a function $f : X \to Y$ is said to be $\alpha$-open [19] if $f(A) \in \alpha O(Y)$ for all open set $A$ of $Y$.

**Lemma 3.2** Let a bijection $f : X \to Y$ be $\alpha$-open. Then for any closed set $B$ of $X$, $f(B) \in \alpha C(Y)$. 

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Urysohn spaces remain invariant under certain bijective function as is shown in the next theorem.

**Theorem 3.3** If a bijection $f : X \rightarrow Y$ is $\alpha$-open and $X$ is Urysohn, then $Y$ is Urysohn.

**Proof.** Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Since $f$ is bijective, $f^{-1}(y_1), f^{-1}(y_2) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$. The Urysohn property of $X$ gives the existence of sets $U \in O(X, f^{-1}(y_1)), V \in O(X, f^{-1}(y_2))$ such that $Cl(U) \cap Cl(V) = \emptyset$. As $Cl(U)$ is a closed set in $X$, then by the bijectivity and $\alpha$-openness of $f$ together then indicate, by Lemma 3.2 that $f(Cl(U)) \in \alpha C(Y)$. Therefore by the injectivity of $f$, $\alpha Cl(f(U)) \cap \alpha Cl(f(V)) \subset f(Cl(U)) \cap f(Cl(V)) = f(Cl(U) \cap Cl(V)) = \emptyset$. Thus $\alpha$-openness of $f$ gives the existence of two sets $f(U) \in \alpha O(Y, y_1)$, $f(V) \in \alpha O(Y, y_2)$, with $\alpha Cl(f(U)) \cap \alpha Cl(f(V)) = \emptyset$. By Lemma 3.1, $Y$ is Urysohn.

**Theorem 3.4** If $f : X \rightarrow Y$ is weakly $\alpha$-continuous and $Y$ is Urysohn, then $G(f)$ is strongly $\alpha$-closed.

**Proof.** Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since $Y$ is Urysohn, there exist $V \in O(Y, y), W \in O(Y, f(x))$ such that $Cl(V) \cap Cl(W) = \emptyset$. Since $f$ is weakly $\alpha$-continuous, there exists $U \in \alpha O(X, x)$ such that $f(U) \subset Cl(W)$. This, therefore, implies that $f(U) \cap Cl(V) = \emptyset$. So by Lemma 2.2, $G(f)$ is strongly $\alpha$-closed.

**Theorem 3.5** Let $X$ be a Urysohn space. Then any $\alpha$-open bijection $f : X \rightarrow Y$ has a strongly $\alpha$-closed graph.

**Proof.** Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ and $y \neq f^{-1}(y)$, where $f^{-1}(y)$ is a singleton. Since $X$ is Urysohn, there exist open sets $U_x$ and $U_y$ such that $x \in U_x, f^{-1}(y) \in U_y$ and $Cl(U_x) \cap Cl(U_y) = \emptyset$. Since $f$ is $\alpha$-open, $f(U_x) \in \alpha O(Y, f(x)), f(U_y) \in \alpha O(Y, y)$ and $f(U_x) \cap \alpha Cl(f(U_y)) \subset \alpha Cl(f(U_x)) \cap \alpha Cl(f(U_y)) \subset f(Cl(U_x)) \cap f(Cl(U_y)) = \emptyset$. Therefore, by Lemma 2.2, $G(f)$ is strongly $\alpha$-closed.

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