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# CHARACTERIZATIONS OF FUNCTIONS WITH STRONGLY $\alpha$ -CLOSED GRAPHS

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**Abstract**

In this paper, we study some properties of functions with strongly $\alpha$-closed graphs by utilizing $\alpha$-open sets and the $\alpha$-closure operator.

## 1 Introduction and preliminaries

The notion of $\alpha$-open sets was introduced by O. Njåstad [20] in 1965. Since then it has been widely investigated in the literature (see, [1], [2], [3], [9], [10], [11], [12], [15], [16], [17], [18], [19], [21], [23], [24], [26], [27], [28]). Functions with strongly closed graphs were introduced by Herrington and long [7] to characterize $H$-closed spaces. Properties of such functions were further investigated by Long and Herrington [14] and Noiri [23]. In this paper, we study some properties of functions with strongly $\alpha$-closed graphs by utilizing $\alpha$-open sets and the $\alpha$-closure operator.

Throughout this paper, by $(X,\tau)$ and $(Y,\sigma)$ (or $X$ and $Y$) we always mean topological spaces. Let $A$ be a subset of $X$. We denote the interior, the closure and the complement of a set $A$ by $Int(A)$, $Cl(A)$ and $X\backslash A$ or $A^c$ respectively. A subset $A$ of a topological space $(X,\tau)$ is called *$\alpha$-open* [20] (resp. *semi-open* [13]) if $A \subseteq Int(Cl(Int(A)))$ (resp. $A \subseteq Cl(Int(A))$). The complement of an $\alpha$-open (resp. semi-open) set is called *$\alpha$-closed* (resp. *semi-closed* [5]). By $\alpha\mathrm{O}(X,\tau)$ (resp. $SO(X,\tau)$, $\alpha\mathrm{C}(X,\tau)$), we denote the family of all

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$\alpha$-open (resp. semi-open, $\alpha$-closed) sets of $X$. We set $\alpha O(X,x)=\{U \mid x\in U\in \alpha O(X,\tau)\}$, $O(X,x)=\{U \mid x\in U\in \tau\}$ and $\alpha C(X,x)=\{U \mid x\in U\in \alpha C(X,\tau)\}$. The intersection of all $\alpha$-closed (resp. semi-closed) sets containing $A$ is called the *$\alpha$-closure* (resp. *semi-closure* [4]) of $A$, denoted by $\alpha Cl(A)$ (resp. $sCl(A)$). A set $U$ in a topological space $(X,\tau)$ is an *$\alpha$-neighborhood* [16] of a point $x$ if $U$ contains an $\alpha$-open set $V$ such that $x\in V$.

**Lemma 1.1** *The intersection of an arbitrary collection of $\alpha$-closed sets in $(X,\tau)$ is $\alpha$-closed*

**Corollary 1.2** *[15]. Let $A$ be a subset of $X$. Then, $x\in \alpha Cl(A)$ if and only if for any $\alpha$-open set $U$ in $X$ containing $x$ , $A\bigcap U\neq \phi$.*

**Lemma 1.3** *Let $A$ and $B$ be subsets of a space $(X,\tau)$, then the following properties hold:*
*(1) $A\subset \alpha Cl(A)$.*
*(2) If $A\subset B$, then $\alpha Cl(A)\subset \alpha Cl(B)$.*
*(3) $\alpha Cl(A)$ is $\alpha$-closed.*
*(4) $\alpha Cl(\alpha Cl(A)) = \alpha Cl(A)$.*
*(5) $A$ is $\alpha$-closed if and only $A=\alpha Cl(A)$.*

**Corollary 1.4** *Let $A_i$ $(i\in I)$ be a subset of a space $(X,\tau)$, then the following properties hold:*
*(1) $\alpha Cl(\cap\{A_i : i\in I\})\subset \cap\{\alpha Cl(A_i) : i\in I\}$.*
*(2) $\alpha Cl(\cup\{A_i : i\in I\})\supset \cup\{\alpha Cl(A_i) : i\in I\}$.*

**Definition 1** *A topological space $(X,\tau)$ is said to be:*
*(1) $\alpha$-$T_1$ [17], if for any pair of distinct points $x$ and $y$ in $X$, there exist an $\alpha$-open set $U$ in $X$ containing $x$ but not $y$ and an $\alpha$-open set $V$ in $X$ containing $y$ but not $x$.*
*(2) $\alpha$-$T_2$ [15], if for any pair of distinct points $x$ and $y$ in $X$, there exist $U\in \alpha O(X,x)$ and $V\in \alpha O(X,y)$ such that $U\cap V=\emptyset$.*

**Lemma 1.5** *A topological space $(X,\tau)$ is $\alpha$-$T_2$ if and only if it is $T_2$.*

*Proof.* This is shown in [27] and a simple proof is given in [[24], Corollary 4.7].

**Definition 2** *A function $f : X \to Y$ is said to be*
*(1) $\alpha$-continuous [19] if $f^{-1}(V) \in \alpha O(X)$ for each open set $V$ of $Y$;*
*(2) weakly $\alpha$-continuous [23]if for each $x \in X$ and each $V \in O(Y, f(x))$, there exists $U \in \alpha O(X, x)$ such that $f(U) \subset Cl(V)$.*

**Lemma 1.6** *Let $(X, \tau)$ be a topological space. Then $\alpha Cl(V) = Cl(V)$ for each $V \in SO(X)$.*

*Proof.* For any $V \in SO(X)$, $\alpha Cl(V) = V \cup Cl(Int(Cl(V))) = V \cup Cl(Int(V)) = V \cup Cl(V) = Cl(V)$.

**Lemma 1.7** *A function $f : X \to Y$ is weakly $\alpha$-continuous if and only if for each $x \in X$ and each $V \in \alpha O(Y, f(x))$, there exists $U \in \alpha O(X, x)$ such that $f(U) \subset \alpha Cl(V)$.*

*Proof.* Necessity. Let $x \in X$ and $V \in \alpha O(Y, f(x))$. Then $f(x) \in V \subset Int(Cl(Int(V)))$ and there exists $U \in \alpha O(X, x)$ such that $f(U) \subset Cl(Int(Cl(Int(V))))$. By Lemma 1.6, we have $Cl(Int(Cl(Int(V)))) = Cl(Int(V)) = Cl(V) = \alpha Cl(V)$. Therefore, $f(U) \subset \alpha Cl(V)$.

Sufficiency. Let $x \in X$ and $V \in O(Y, f(x))$. There exists $U \in \alpha O(X, x)$ such that $f(U) \subset \alpha Cl(V)$. By Lemma 1.6, we obtain $f(U) \subset Cl(V)$.

# 2 Strongly $\alpha$-closed graphs

If $f : (X, \tau) \to (Y, \sigma)$ is any function, then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of $f$ [8].

**Definition 3** *A function $f : X \to Y$ has a strongly $\alpha$-closed (resp. strongly closed [7]) graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \alpha O(X, x)$ (resp. $U \in O(X, x)$) and $V \in O(Y, y)$ such that $(U \times Cl(V)) \cap G(f) = \emptyset$.*

**Lemma 2.1** *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*
*(1) $G(f)$ is strongly $\alpha$-closed;*
*(2) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \alpha O(X, x)$ and $V \in O(Y, y)$ such that*

*$f(U) \cap Cl(V) = \emptyset$;*
*(3) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \alpha O(X, x)$ and $V \in \alpha O(Y, y)$ such that $(U \times \alpha Cl(V)) \cap G(f) = \emptyset$;*
*(4) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \alpha O(X, x)$ and $V \in \alpha O(Y, y)$ such that $f(U) \cap \alpha Cl(V) = \emptyset$.*

*Proof.* It is obvious that (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4).
(1) $\Rightarrow$ (3): Since $\tau \subset \alpha O(X) \subset SO(X)$, by Lemma 1.6 the proof is obvious.
(3) $\Rightarrow$ (1): Let $(x, y) \in (X \times Y) \setminus G(f)$. There exist $U \in \alpha O(X, x)$ and $V \in \alpha O(Y, y)$ such that $(U \times \alpha Cl(V)) \cap G(f) = \emptyset$. Put $G = Int(Cl(Int(V)))$. Then $y \in V \subset G \in \sigma$ and $Cl(G) = Cl(V) = \alpha Cl(V)$. Therefore, we obtain $(U \times Cl(G)) \cap G(f) = (U \times \alpha Cl(V)) \cap G(f) = \emptyset$. This shows that $G(f)$ is strongly $\alpha$-closed.

**Theorem 2.2** *If $f : X \to Y$ is a function with the strongly $\alpha$-closed graph, then for each $x \in X$, $f(x) = \cap\{\alpha Cl(f(U)) : U \in \alpha O(X, x)\}$.*

*Proof.* Suppose the theorem is false. Then there exists a $y \neq f(x)$ such that $y \in \cap\{\alpha Cl(f(U)) : U \in \alpha O(X, x)\}$. This implies that $y \in \alpha Cl(f(U))$ for every $U \in \alpha O(X, x)$. So $V \cap f(U) \neq \emptyset$ for every $V \in \alpha O(Y, y)$. This, in its turn, indicates that $\alpha Cl(V) \cap f(U) \supset V \cap f(U) \neq \emptyset$ which contradicts the hypothesis that $f$ is a function with strongly $\alpha$-closed graph. Hence the theorem holds.

**Theorem 2.3** *If $f : X \to Y$ is $\alpha$-continuous and $Y$ is $T_2$, then $G(f)$ is strongly $\alpha$-closed.*

*Proof.* Let $(x, y) \in (X \times Y) \setminus G(f)$. The $T_2$-ness of $Y$ gives the existence of a set $V \in O(Y, y)$ such that $f(x) \notin Cl(V)$. Now $Y \setminus Cl(V) \in O(Y, f(x))$. Therefore, by the $\alpha$-continuity of $f$ there exists $U \in \alpha O(X, x)$ such that $f(U) \subset Y \setminus Cl(V)$. Consequently, $f(U) \cap Cl(V) = \emptyset$ and therefore $G(f)$ is strongly $\alpha$-closed.

It is shown in ([14], Theorem 3) and ([22], Theorem 2) that if $f : X \to Y$ is surjective and $G(f)$ is strongly closed, then $Y$ is Hausdorff. The following theorem is a slight improvement of this result.

**Theorem 2.4** *If $f : X \to Y$ is surjective and has a strongly $\alpha$-closed graph $G(f)$, then $Y$ is both $T_2$ and $\alpha$-$T_1$.*

*Proof.* Let $y_1, y_2$ $(y_1 \neq y_2) \in Y$. The surjectivity of $f$ gives a $x_1 \in X$ such that $f(x_1) = y_1$. Now $(x_1, y_2) \in (X \times Y) \backslash G(f)$. The strongly $\alpha$-closedness of $G(f)$ provides $U \in \alpha O(X, x_1)$, $V \in O(Y, y_2)$ such that $f(U) \cap Cl(V) = \emptyset$, whence one infers that $y_1 \notin Cl(V)$. This means that there exists $W \in O(Y, y_1)$ such that $W \cap V = \emptyset$. So, $Y$ is $T_2$ and $T_2$-ness always guarantees $\alpha$-$T_1$-ness. Hence $Y$ is $\alpha$-$T_1$.

**Theorem 2.5** *A space $X$ is $T_2$ if and only if the identity function $id : X \to X$ has a strongly $\alpha$-closed graph $G(id)$.*

*Proof.* Necessity. Let $X$ be $T_2$. Since the identity function $id : X \to X$ is continuous, it follows from Theorem 2.4 that $G(id)$ is strongly $\alpha$-closed.
Sufficiency. Let $G(id)$ be a strongly $\alpha$-closed graph. Then the surjectivity of $id$ and strong $\alpha$-closedness of $G(id)$ together imply, by Theorem 2.4, that $X$ is $T_2$.

**Theorem 2.6** *If $f : X \to Y$ is an injection and $G(f)$ is strongly $\alpha$-closed, then $X$ is $\alpha$-$T_1$.*

*Proof.* Since $f$ is injective, for any pair of distinct points $x_1, x_2 \in X$, $f( x_1) \neq f(x_2)$. Then $(x_1, f(x_2)) \in (X \times Y) \backslash G(f)$. Since $G(f)$ is strongly $\alpha$-closed, there exist $U \in \alpha O(X, x_1)$, $V \in O(Y, f(x_2))$ such that $f(U) \cap Cl(V) = \emptyset$. Therefore $x_2 \notin U$. Pursuing the same reasoning as before we obtain a set $W \in \alpha O(X, x_2)$ such that $x_1 \notin W$. Hence $Y$ is $\alpha$-$T_1$.

**Theorem 2.7** *If $f : X \to Y$ is a bijection with the strongly $\alpha$-closed graph, then both $X$ and $Y$ are $\alpha$-$T_1$.*

*Proof.* The proof is an immediate consequence of Theorems 2.4 and 2.6.

**Theorem 2.8** *If a function $f : X \to Y$ is a weakly $\alpha$-continuous injection with the strongly $\alpha$-closed graph $G(f)$, then $X$ is $T_2$.*

*Proof.* Since $f$ is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. Therefore $(x_1, f(x_2)) \in (X \times Y) \backslash G(f)$. Since $G(f)$ is strongly $\alpha$-closed, there exist $U \in \alpha O(X, x_1)$, $V \in O(Y, f(x_2))$ such that $f(U) \cap Cl(V) = \emptyset$; hence $U \cap f^{-1}(Cl(V)) = \emptyset$. Consequently, $f^{-1}(Cl(V)) \subset X \backslash U$. Since $f$ is weakly $\alpha$ continuous, there exists $W \in \alpha O(X, x_2)$ such that $f(W) \subset Cl(V)$. From this and the foregoing it follows that $W \subset f^{-1}(Cl(V)) \subset X \backslash U$; hence $W \cap U = \emptyset$. Thus for the pair of distinct points $x_1, x_2 \in X$, there exist $U \in \alpha O(X, x_1)$, $W \in \alpha O(X, x_2)$ such that $W \cap U = \emptyset$. By Lemma 1.5, this guarantees the $T_2$-ness of $X$.

**Corollary 2.9** *If a function $f : X \to Y$ is an $\alpha$-continuous injection with the strongly $\alpha$-closed graph, then $X$ is $T_2$.*

*Proof.* The proof follows from Theorem 2.8 and the fact that every $\alpha$-continuous is weakly $\alpha$-continuous.

**Remark 2.10** *If $f$ is not $T_2$ in Corollary 2.9, then even $\alpha$-continuity need not imply a strongly $\alpha$-closed graph. For example, let $X$ be a topological space containing more than one point with the indiscrete topology and let $id : X \to X$ the identity function. Then $id$ is certainly $\alpha$-continuous, but the graph of $id$ is not strongly $\alpha$-closed because $X \times X$ has the indiscrete topology and hence the graph of $id$ being the diagonal set, which is different from the whole space, is not strongly $\alpha$-closed.*

**Theorem 2.11** *If $f : X \to Y$ is a weakly $\alpha$-continuous bijection with the strongly $\alpha$-closed graph, then both $X$ and $Y$ are $T_2$.*

*Proof.* The proof follows from Theorems 2.8 and 2.4.

**Lemma 2.12** *Every clopen subset of a quasi $H$-closed space $X$ is quasi $H$-closed relative to $X$.*

*Proof.* Let $B$ be any clopen subset of a quasi $H$-closed space $X$. Let $\{O_\lambda : \lambda \in \Omega\}$ be any cover of $B$ by open sets in $X$. Then the family $F = \{O_\lambda : \lambda \in \Omega\} \cup \{X \backslash B\}$ is a cover of $X$ by open sets in $X$. Because of quasi $H$-closedness of $X$ there exists a finite subfamily $F^* = \{O_{\lambda_i} : 1 \leq i \leq n\} \cup \{X \backslash B\}$ of $F$ whose closure covers $X$. So, because of clopenness of $B$ we now infer that the family $\{Cl(O_{\lambda_i}) : 1 \leq i \leq n\}$ covers $B$. Therefore, $B$ is quasi $H$-closed relative to $X$.

**Theorem 2.13** *If $Y$ is a quasi $H$-closed extremally disconnected space, then a function $f : X \to Y$ with the strongly $\alpha$-closed graph $G(f)$ is weakly $\alpha$-continuous.*

*Proof.* Let $x \in X$ and $V \in O(Y, f(x))$. Take any $y \in Y \backslash Cl(V)$. Then $(x, y) \in (X \times Y) \backslash G(f)$. Now the strong $\alpha$-closedness of $G(f)$ induces the existence of $U_y(x) \in \alpha O(X, x)$, $V_y \in O(Y, y)$ such that $f(U_y(x)) \cap Cl(V_y) = \emptyset$....(*).
Now extremal disconnectedness of $Y$ induces the clopenness of $Cl(V)$ and hence $Y \backslash Cl(V)$ is also clopen. Now $\{V_y : y \in Y \backslash Cl(V)\}$ is a cover of $Y \backslash Cl(V)$ by open sets in $Y$. By Lemma 2.12, there exists a finite subfamily $\{V_{y_i} : 1 \leq i \leq n\}$ such that $Y \backslash Cl(V) \subset \bigcup\limits_{i=1}^{n} Cl(V_{y_i})$. Let $W = \bigcap\limits_{i=1}^{n} U_{y_i}(x)$, where $U_{y_i}(x)$ are $\alpha$-open sets in $X$ satisfying (*). Also, $W \in \alpha O(X, x)$. Now $f(W) \cap (Y \backslash Cl(V)) \subset f[\bigcap\limits_{i=1}^{n} U_{y_i}(x)] \cap (\bigcup\limits_{i=1}^{n} Cl(V_{y_i})) \subset \bigcup\limits_{i=1}^{n} (f[U_{y_i}(x)] \cap Cl(V_{y_i})) = \emptyset$, by (*). Therefore, $f(W) \subset Cl(V)$ and this indicates that $f$ is weakly $\alpha$-continuous.

Noiri [22] showed that if $G(f)$ is strongly closed then $f$ has the following property:
$(P)$ For every set $B$ which is quasi H–closed relative to $Y$, $f^{-1}(B)$ is a closed set of $X$.

Analogously, we have the following theorem.

**Theorem 2.14** *If a function $f : X \to Y$ has a strongly $\alpha$-closed graph $G(f)$, then $f$ enjoys the following property:*

$(P^*)$ *For every set $F$ which is quasi $H$-closed relative to $Y$, $f^{-1}(F)$ is $\alpha$-closed in $X$.*

*Proof.* Let $f^{-1}(F)$ be not $\alpha$-closed in $X$. Then there exists $x \in \alpha Cl(f^{-1}(F)) \backslash f^{-1}(F)$. Let $y \in F$. Then $(x, y) \in (X \times Y) \backslash G(f)$. Strong $\alpha$-closedness of $G(f)$ gives the existence of

$U_y(x) \in \alpha O(X, x)$ and $V_y \in O(Y, y)$ such that $f(U_y(x)) \cap Cl(V_y) = \emptyset$....(*).
Clearly $\{V_y : y \in F\}$ is a cover of $F$ by open sets in $Y$. Since $F$ is quasi $H$-closed relative to $Y$, there exist a finite number of open sets $V_{y_1}, V_{y_2}, ..., V_{y_n}$ in $Y$ such that $F \subset \bigcup_{i=1}^{n} Cl(V_{y_i})$. Let $U = \bigcap_{i=1}^{n} U_{y_i}(x)$, where $U_{y_i}(x)$ are the $\alpha$-open sets in $X$ satisfying (*). Also $U \in \alpha O(X, x)$. Now $f(U) \cap F \subset f[\bigcap_{i=1}^{n} U_{y_i}(x)] \cap (\bigcup_{i=1}^{n} Cl(V_{y_i})) \subset \bigcup_{i=1}^{n}(f[U_{y_i}(x)] \cap Cl(V_{y_i})) = \emptyset$. But since $x \in \alpha Cl(f^{-1}(F))$, $U \cap f^{-1}(F) \neq \emptyset$; hence $f(U) \cap F \neq \emptyset$. This is a contradiction. Hence the result holds.

# 3 Additional properties

**Lemma 3.1** *For a topological space $X$, the following properties are equivalent:*
*(1) $X$ is Urysohn;*
*(2) For every pair of distinct points $x, y \in X$, there exist $U \in \alpha O(X, x)$, $V \in \alpha O(X, y)$ such that $Cl(U) \cap Cl(V) = \emptyset$;*
*(3) For every pair of distinct points $x, y \in X$, there exist $U \in \alpha O(X, x)$, $V \in \alpha O(X, y)$ such that $\alpha Cl(U) \cap \alpha Cl(V) = \emptyset$.*

*Proof.* (1) $\Rightarrow$ (2): This is obvious.
(2) $\Rightarrow$ (3): Since $\alpha Cl(U) = Cl(U)$ for each $U \in \alpha(X)$ by Lemma 1.6, this is obvious.
(3) $\Rightarrow$ (1): Suppose that (3) holds. For every pair of distinct points $x$, $y$, there exist $U \in \alpha O(X, x)$, $V \in \alpha O(X, y)$ such that $\alpha Cl(U) \cap \alpha Cl(V) = \emptyset$. Now, put $G = Int(Cl(Int(U)))$ and $H = Int(Cl(Int(V)))$, then $G$ and $H$ are open sets containing $x$ and $y$, respectively. Furthermore, $Cl(G) \cap Cl(H) = Cl(U) \cap Cl(V) = \alpha Cl(U) \cap \alpha Cl(V) = \emptyset$. Therefore, $X$ is Urysohn.

Recall, that a function $f : X \to Y$ is said to be $\alpha$-open [19] if $f(A) \in \alpha O(Y)$ for all open set $A$ of $Y$.

**Lemma 3.2** *Let a bijection $f : X \to Y$ be $\alpha$-open. Then for any closed set $B$ of $X$, $f(B) \in \alpha C(Y)$.*

Urysohn spaces remain invariant under certain bijective function as is shown in the next theorem.

**Theorem 3.3** *If a bijection $f : X \to Y$ is $\alpha$-open and $X$ is Urysohn, then $Y$ is Urysohn.*

*Proof.* Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Since $f$ is bijective, $f^{-1}(y_1)$, $f^{-1}(y_2) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$. The Urysohn property of $X$ gives the existence of sets $U \in O(X, f^{-1}(y_1))$, $V \in O(X, f^{-1}(y_2))$ such that $Cl(U) \cap Cl(V) = \emptyset$. As $Cl(U)$ is a closed set in $X$, then by the bijectivity and $\alpha$-openness of $f$ together then indicate, by Lemma 3.2 that $f(Cl(U)) \in \alpha C(Y)$. Therefore by the injectivity of $f$, $\alpha Cl(f(U)) \cap \alpha Cl(f(V)) \subset f(Cl(U)) \cap f(Cl(V)) = f(Cl(U) \cap Cl(V)) = \emptyset$. Thus $\alpha$-openness of $f$ gives the existence of two sets $f(U) \in \alpha O(Y, y_1)$, $f(V) \in \alpha O(Y, y_2)$, with $\alpha Cl(f(U)) \cap \alpha Cl(f(V)) = \emptyset$. By Lemma 3.1, $Y$ is Urysohn.

**Theorem 3.4** *If $f : X \to Y$ is weakly $\alpha$-continuous and $Y$ is Urysohn, then $G(f)$ is strongly $\alpha$-closed.*

*Proof.* Let $(x, y) \in (X \times Y) \backslash G(f)$. Then $y \neq f(x)$. Since $Y$ is Urysohn, there exist $V \in O(Y, y)$, $W \in O(Y, f(x))$ such that $Cl(V) \cap Cl(W) = \emptyset$. Since $f$ is weakly $\alpha$-continuous, there exists $U \in \alpha O(X, x)$ such that $f(U) \subset Cl(W)$. This, therefore, implies that $f(U) \cap Cl(V) = \emptyset$. So by Lemma 2.2, $G(f)$ is strongly $\alpha$-closed.

**Theorem 3.5** *Let $X$ be a Urysohn space. Then any $\alpha$-open bijection $f : X \to Y$ has a strongly $\alpha$-closed graph.*

*Proof.* Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ and $y \neq f^{-1}(y)$, where $f^{-1}(y)$ is a singleton. Since $X$ is Urysohn, there exist open sets $U_x$ and $U_y$ such that $x \in U_x$, $f^{-1}(y) \in U_y$ and $Cl(U_x) \cap Cl(U_y) = \emptyset$. Since $f$ is $\alpha$-open, $f(U_x) \in \alpha O(Y, f(x))$, $f(U_y) \in \alpha O(Y, y)$ and $f(U_x) \cap \alpha Cl(f(U_y)) \subset \alpha Cl(f(U_x)) \cap \alpha Cl(f(U_y)) \subset f(Cl(U_x)) \cap f(Cl(U_y)) = \emptyset$. Therefore, by Lemma 2.2, $G(f)$ is strongly $\alpha$-closed.

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