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Abstract

The initial inverse problem in heat equation arise when experimental measurements of temperature distribution at a particular time are used to calculate the temperature distribution at some particular time in the past. Such problem can be reduced to integral equation of the first kind and Picard's criterion can be applied to solve the inverse problem with the help of the associated singular system. Initial inverse heat conduction problems are generally ill-posed and the results without taking into account the inherint ill-posedness are missleading. An alternative model with a small parameter is presented which addresses the issue of ill-posedness. The initial inverse problem in a sphere with radial symmetry and in a general spherical body are separately considered.

Key Words: Heat conduction, Initial profile, Inverse problem

1. Introduction.—

Initial inverse problems are much less encountered in the literature than some other types of inverse problems. However, one of the earliest studies on inverse problems by Fourier and Kelvin [3] were concerned with initial inverse problems, that is, they tried to estimate the initial temperature distribution of the earth from current temperature measurements. Recently Nakamura et al. [13] used transformation techniques to solve the initial inverse problem in heat conduction and Al-Khalidi [9] dealt with the problem numerically. For comprehensive review of the literature and summary of various approaches in the field of inverse heat conduction problems, one can consult the books by Beck et al. [1] and by Hensel [8]. The inverse heat conduction problems are ill-posed [4, 6, 7, 15], so the slightest error in the measurements can give meaningless results.

We consider the regularization of the heat conduction problem by introducing the model based upon the damped wave equation. The application of this idea to some interesting inverse problems in heat conduction have appeared in Masood and Zaman [11, 12], Masood, Messaoudi and Zaman [10]. The need to consider the alternate formulation has some physical advantages. In many applications, one encounters a situation where the usual parabolic heat equation does not serve as a realistic model. Since the speed of propagation of the thermal signal is finite, e.g. for short-pulse laser applications, the hyperbolic differential equation correctly models the problem; see Vedavarz et al. [14] and Gratzke et al. [5] among others.

The initial inverse problem in the hyperbolic heat equation is stable and well posed. Moreover, numerical methods for hyperbolic problems are efficient and accurate. We will utilize the small value of the parameter and apply the WKBJ method to solve the initial inverse problem, see Bender and Orszag [2]. In the second section we will solve the inverse problem by considering the usual parabolic heat equation and its regularization with the help of hyperbolic heat equation. In the third section the inverse problem in a spherical body with radial symmetry is reduced to a hyperbolic heat equation and then it can be solved using the method presented in section 2. A more general heat equation of a spherical body is considered in the fourth section. Finally in the last section the results are summarized.

2. Demonstration of ill-posed nature of inverse problem in a parabolic model and its regularization

Consider the parabolic heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in [0, \pi], \ t \geq 0, \tag{1}$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi, t) = 0. \tag{2}$$

The initial inverse problem in classical parabolic heat conduction model is to use the final temperature profile for a given time T as follows:

$$f(x) \doteq u(x, T). \tag{3}$$

We want to determine the initial temperature profile $v_0(x)$

$$v_0(x) \doteq u(x, 0), \tag{4}$$

The functions $\phi_n(x) \doteq \sqrt{\frac{2}{\pi}} \sin(nx)$ are complete orthonormal system in $L^2[0,\pi]$ and eigenfunctions of $\frac{d^2}{dx^2}$ on $[0, \pi]$. Thus $v_0(x) \in L^2[0,\pi]$ can be expanded as

$$v_0(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \ , \ \ x \in [0, \pi], \tag{5}$$

where

$$c_n = \sqrt{\frac{2}{\pi}} \int_0^{\pi} v_0(\tau) \sin(n\tau) d\tau. \tag{6}$$

Now by separation of variables suppose solution of direct problem (1), (2) and (3) is

$$u(x,t) \doteq \sum_{n=1}^{\infty} a_n(t)\phi_n(x), \qquad x \in [0,\pi], \tag{7}$$

where $a_n(t)$ have to solve the initial value problem

$$\frac{da_n(t)}{dt} = -n^2 a_n(t), \qquad t \geq 0, \tag{8}$$

$$a_n(0) = c_n, \tag{9}$$

Therefore (7) can be written as

$$u(x,t) = \sum_{n=1}^{\infty} c_n \phi_n(x) \exp[-n^2 t], \tag{10}$$

Using condition (3) to write

$$f(x) = \int_0^{\pi} k(x,\tau) v_0(\tau) d\tau, \tag{11}$$

where

$$k(x,\tau) \doteq \frac{2}{\pi} \sum_{n=1}^{\infty} \exp\left[-n^2 T\right] \sin(n\tau)\sin(nx). \tag{12}$$

Thus the inverse problem is reduced to solving integral equation of the first kind. The singular system for the integral operator in (11) is given by

$$\left\{ \exp\left[-n^2 T\right]; \sqrt{\frac{2}{\pi}}\sin(nx), \sqrt{\frac{2}{\pi}}\sin(nx) \right\}. \tag{13}$$

It now follows from Picard's theorem [7] that our inverse problem is solvable if and only if

$$\sum_{n=1}^{\infty} e^{2n^2} |f_n|^2 < \infty, \tag{14}$$

where

$$f_n \doteq \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(\tau) \sin(n\tau)\, d\tau, \tag{15}$$

are classical Fourier coefficients of f. In this case solution is given by

$$v_0(x) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} e^{n^2} f_n \sin(nx)\,. \tag{16}$$

From (14) and (16) it is clear that this problem is extremely ill-posed i.e. solution exist only if Fourier coefficients decay much faster than $\exp[-n^2T]$. A small error in the n-th Fourier coefficient is amplified by the factor $\exp[n^2T]$ as is evident from (16). These are serious problems which we will try to address by introducing a modified model through the use of the hyperbolic heat equation.

2.1 Regularization of the heat conduction model by a hyperbolic model.—

We introduce a higher order term $\epsilon\frac{\partial^2 u}{\partial t^2}$ (Masood et al [10, 11, 12]) with a small positive parameter ϵ. By controlling the size of small parameter ε in the so called hyperbolic model of heat equation, we can regularize the inverse problem. Thus instead of equation (1), we consider

$$\epsilon\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad \epsilon > 0, \tag{17}$$

together with conditions $(2) - (4)$. However, in this case we require an additional condition

$$\frac{\partial u}{\partial t}(x, 0) = 0. \tag{18}$$

Following the same procedure as before, in this case, $a_n(t)$ have to solve the following initial value problem

$$\epsilon \frac{d^2 a_n(t)}{dt^2} + \frac{d a_n(t)}{dt} + n^2 a_n(t) = 0, \qquad t > 0, \tag{19}$$

$$a_n(0) = c_n, \tag{20}$$

$$\frac{d a_n(0)}{dt} = 0, \tag{21}$$

where $\epsilon \to 0^+$. This is a singular perturbation problem, so we can seek WKBJ solution to this problem [2]. The WKBJ solution to (19) is

$$a_n(t) = \left(\frac{\epsilon n^2 - 1}{2\epsilon n^2 - 1}\right) c_n \exp\left[-n^2 t\right] + \left(\frac{\epsilon n^2 c_n}{2\epsilon n^2 - 1}\right) \exp\left[n^2 t - \frac{t}{\epsilon}\right]. \tag{22}$$

The singular system for this problem is

$$\left\{ \begin{array}{c} \left(\frac{\epsilon n^2-1}{2\epsilon n^2-1}\right) \exp\left[-n^2 T\right] + \left(\frac{\epsilon n^2}{2\epsilon n^2-1}\right) \exp\left[n^2 T - \frac{T}{\epsilon}\right] .; \ \sqrt{\frac{2}{\pi}} \sin(nx), \\ \sqrt{\frac{2}{\pi}} \sin(nx) \end{array} \right\}. \tag{23}$$

The solution exists if and only if

$$\sum_{n=1}^{\infty} \frac{|f_n|^2}{\left\{\left(\frac{\epsilon n^2-1}{2\epsilon n^2-1}\right) \exp\left[-n^2 T\right] + \left(\frac{\epsilon n^2}{2\epsilon n^2-1}\right) \exp\left[n^2 T - \frac{T}{\epsilon}\right]\right\}^2} < \infty. \tag{24}$$

The solution is given by

$$v_0(x) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \frac{f_n \sin(nx)}{\left\{\left(\frac{\epsilon n^2-1}{2\epsilon n^2-1}\right) \exp\left[-n^2 T\right] + \left(\frac{\epsilon n^2}{2\epsilon n^2-1}\right) \exp\left[n^2 T - \frac{T}{\epsilon}\right]\right\}}. \tag{25}$$

We note that by letting $\epsilon \longrightarrow 0^+$ in the expressions (24) and (25), we get the solutions given by (14) and (16). We also observe that a small change in the n-th Fourier coefficient is no longer multiplied by the factor $\exp\left[n^2 T\right]$, but it has a controlling parameter ϵ which can be chosen to obtain stability.

3 Initial inverse problem in a sphere with radial symmetry

Consider the time dependent temerature distribution in a sphere of unit radius with radial symmetry. The hyperbolic form of heat edquation, with an appropriate choice of units can be written as

$$\epsilon\frac{\partial^2 w}{\partial t^2}+\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial r^2}+\frac{2}{r}\frac{\partial w}{\partial r},\qquad 0<r<1,\quad t,\epsilon>0, \tag{26}$$

with Dirichlet boundary condition

$$w\left(1,t\right)=w_0,\ t>0. \tag{27}$$

The initial inverse problem concerns with the recovery of the initial profile which is given by

$$w\left(r,0\right)=v_0\left(r\right),\quad 0<r<1, \tag{28}$$

and we also require that

$$\frac{\partial w\left(r,0\right)}{\partial t}=0. \tag{29}$$

We assume that the final profile is given as

$$w\left(r,T\right)=f\left(r\right). \tag{30}$$

The problem (26) can be reduced to problem (17) by using the following transformation

$$rw\left(r,t\right)=u\left(r,t\right)+\psi\left(r\right), \tag{31}$$

with the requirement $d^2\psi\left(r\right)/dr^2=0$, which gives $\psi\left(r\right)=c_1r+c_2$. Since $w\left(r,t\right)$ is bounded as $r\to 0$, which implies $c_2=0$. By using the condition (27), we obtain $\psi\left(r\right)=w_0r$ and $u\left(1,t\right)=0$. Hence the transformed problem can be written as

$$\epsilon\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}, \tag{32}$$

$$u(0,t) = 0, \tag{33}$$

$$u(1,t) = 0, \tag{34}$$

$$\frac{\partial u(r,0)}{\partial t} = 0, \tag{35}$$

$$u(r,0) = F(r), \tag{36}$$

where $F(r)$ is related to the initial profile by the relation

$$v_0(r) = \frac{1}{r}F(r) + w_0 \tag{37}$$

The final temperature distribution in this case takes the form

$$u(r,T) = rf(r) - w_0 r \doteq g(r). \tag{38}$$

Now we follow the same procedure as followed for the equation (17)to arrive at similar results given by (24) and (25) as follows:

$$\sum_{n=1}^{\infty} \frac{|g_n|^2}{\left\{\left[\frac{\epsilon(n\pi)^2-1}{2\epsilon(n\pi)^2-1}\right]\exp\left[-(n\pi)^2 T\right] + \left[\frac{\epsilon(n\pi)^2}{2\epsilon(n\pi)^2-1}\right]\exp\left[(n\pi)^2 T - \frac{T}{\epsilon}\right]\right\}^2} < \infty, \tag{39}$$

$$F(r) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \frac{g_n \sin(n\pi r)}{\left\{\left[\frac{\epsilon(n\pi)^2-1}{2\epsilon(n\pi)^2-1}\right]\exp\left[-(n\pi)^2 T\right] + \left[\frac{\epsilon(n\pi)^2}{2\epsilon(n\pi)^2-1}\right]\exp\left[(n\pi)^2 T - \frac{T}{\epsilon}\right]\right\}}, \tag{40}$$

where g_n are classical Fourier coefficients of g and are given by the following relation

$$g_n \doteq \sqrt{\frac{2}{\pi}} \int_0^1 g(\tau) \sin(n\pi\tau)\, d\tau. \tag{41}$$

Equation (37) can then be used to find the initial profile $v_0(r)$ by substituting $F(r)$ from equation (40).

4 Initial inverse problem in a more general spherical body

Cionsider the three-dimensional, hyperbolic heat conduction equation in the spherical coordinate system given as

$$\epsilon\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}. \tag{42}$$

We consider a homogenious spherical body in which the symmetry over the polar angle is assumed. In this case, the above equation can be written as

$$\epsilon\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right),\quad 0<r<1,\quad 0<\theta<\pi,\quad t>0, \tag{43}$$

where, $u=u\left(r,\theta,t\right)$. This equation can be put in a more convenient form by defining a new independent variable μ as

$$\mu=\cos\theta. \tag{44}$$

With application of (44) the equation (43) becomes

$$\epsilon\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial u}{\partial\mu}\right],\quad 0<r<1,\quad -1<\mu<1,\quad t>0, \tag{45}$$

together with the following initial and boundary conditions

$$u\left(1,\mu,t\right)=0,\ t>0. \tag{46}$$

The initial inverse problem concerns with the recovery of the initial profile which is given by

$$u\left(r,\mu,0\right)=v_0\left(r,\mu\right), \tag{47}$$

and we also require that

$$\frac{\partial u\left(r,\mu,0\right)}{\partial t}=0. \tag{48}$$

We assume that the final profile is given as

$$u(r,\mu,T)=f(r,\mu). \tag{49}$$

We define a new dependent variable as

$$v(r,\mu,t)=\sqrt{r}u(r,\mu,t). \tag{50}$$

Equations (45) – (49) under the transformation (50) takes the following form

$$\epsilon\frac{\partial^2 v}{\partial t^2}+\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{4}\frac{v}{r^2}+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial v}{\partial\mu}\right],\quad 0<r<1,\quad -1<\mu<1,\quad t>0, \tag{51}$$

together with the following initial and boundary conditions

$$v(1,\mu,t)=0, \tag{52}$$

$$v(0,\mu,t)=0, \tag{53}$$

$$v(r,\mu,0)=\sqrt{r}v_0(r,\mu), \tag{54}$$

$$\frac{\partial v(r,\mu,0)}{\partial t}=0, \tag{55}$$

$$v(r,\mu,T)=\sqrt{r}f(r,\mu). \tag{56}$$

The corresponding eigenvalue problems are

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left[\lambda^2-\left(n+\frac{1}{2}\right)^2\frac{1}{r^2}\right]R=0, \tag{57}$$

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right]+n(n+1)M=0. \tag{58}$$

The elementary solutions of equation (57) are $J_{n+\frac{1}{2}}(\lambda_{nm}r)$ and $Y_{n+\frac{1}{2}}(\lambda_{nm}r)$, and of equation (58) are $P_n(\mu)$ and $Q_n(\mu)$. The solution $Q_n(\mu)$ becomes infinite at $\mu=\pm1$ and $Y_{n+\frac{1}{2}}(\lambda_{nm}r)$ becomes infinite at $r=0$; therefore they are inadmissible solutions on the physical grounds. So the admissible elementary solutions are $J_{n+\frac{1}{2}}(\lambda_{nm}r)$ and $P_n(\mu)$. Where $P_n(\mu)$ is the Legendre polynomials with $n = 1, 2, 3, ...,$ which is convergent in the interval $-1 \leq \mu \leq 1$. The application of the condition (52) gives the eigenvalues λ_{nm} as positive roots of

$$J_{n+\frac{1}{2}}(\lambda_{nm}) = 0. \tag{59}$$

The Legendre polynomials and Bessel functions are orthogonal and the orthogonality relations for these functions are given by

$$\int_{-1}^{1} P_n(\mu) P_m(\mu) \, d\mu = \begin{cases} 0 & \text{for } n \neq m, \\ N(n) = \frac{2}{2n+1} & \text{for } n = m, \end{cases} \tag{60}$$

$$\int_0^1 r J_{n+\frac{1}{2}}(\lambda_{nm}r) J_{n+\frac{1}{2}}(\lambda_{np}r) \, dr = \begin{cases} 0 & \text{for } m \neq p, \\ N(\lambda_{nm}) = \int_0^1 r J_{n+\frac{1}{2}}^2(\lambda_{nm}r) \, dr & \text{for } m = p. \end{cases} \tag{61}$$

The functions $\psi_{nm}(r,\mu) \doteq \left(1/\sqrt{N(n) N(\lambda_{nm})}\right) J_{n+\frac{1}{2}}(\lambda_{nm}r) P_n(\mu)$ are complete orthonormal system in $L^2[0,1] \times [-1,1]$. Thus $\sqrt{r}v_0(r,\mu) \in L^2[0,1] \times [-1,1]$ can be expanded as

$$\sqrt{r}v_0(r,\mu) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c_{nm}\psi_{nm}(r,\mu) \ , \ \ r \in [0,1], \ \mu \in [-1,1], \tag{62}$$

where

$$c_{nm} = \frac{1}{\sqrt{N(n) N(\lambda_{nm})}} \int_0^1 \int_{-1}^1 r^{\frac{3}{2}} v_0(r,\mu) J_{n+\frac{1}{2}}(\lambda_{nm}r) P_n(\mu) \, dr d\mu. \tag{63}$$

Now by separation of variables suppose solution of direct problem (51) is

$$v(r,\mu,t) \doteq \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_{nm}(t)\psi_{nm}(r,\mu) \ , \ \ r \in [0,1], \ \mu \in [-1,1], \tag{64}$$

where $a_{nm}(t)$ have to solve the initial value problem

$$\epsilon\frac{d^2a_{nm}(t)}{dt^2}+\frac{da_{nm}(t)}{dt}+\lambda_{nm}^2a_{nm}(t)=0, \qquad t>0, \tag{65}$$

$$a_{nm}(0)=c_{nm}, \tag{66}$$

$$\frac{da_{nm}(0)}{dt}=0, \tag{67}$$

The WKBJ solution to (65) is

$$a_{nm}(t)=\left(\frac{\epsilon\lambda_{nm}^2-1}{2\epsilon\lambda_{nm}^2-1}\right)c_{nm}\exp\left[-\lambda_{nm}^2t\right]+\left(\frac{\epsilon\lambda_{nm}^2c_{nm}}{2\epsilon\lambda_{nm}^2-1}\right)\exp\left[\lambda_{nm}^2t-\frac{t}{\epsilon}\right]. \tag{68}$$

So the solution given by (64) takes the form

$$v(r,\mu,t) \doteq \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{N(n)N(\lambda_{nm})}J_{n+\frac{1}{2}}(\lambda_{nm}r)P_n(\mu)\left\{\left(\frac{\epsilon\lambda_{nm}^2-1}{2\epsilon\lambda_{nm}^2-1}\right)\exp\left[-\lambda_{nm}^2t\right]+\left(\frac{\epsilon\lambda_{nm}^2}{2\epsilon\lambda_{nm}^2-1}\right)\exp\left[\lambda_{nm}^2t-\frac{t}{\epsilon}\right]\right\}\int_0^1\int_{-1}^1 r^{\frac{3}{2}}v_0(r,\mu)J_{n+\frac{1}{2}}(\lambda_{nm}r)P_n(\mu)\,drd\mu \tag{69}$$

Now using condition (56) in (69), the singular system for this problem is

$$\left\{\begin{array}{c}\left(\frac{\epsilon\lambda_{nm}^2-1}{2\epsilon\lambda_{nm}^2-1}\right)\exp\left[-\lambda_{nm}^2T\right]+\left(\frac{\epsilon\lambda_{nm}^2}{2\epsilon\lambda_{nm}^2-1}\right)\exp\left[\lambda_{nm}^2T-\frac{T}{\epsilon}\right];\\ \frac{1}{\sqrt{N(n)N(\lambda_{nm})}}J_{n+\frac{1}{2}}(\lambda_{nm}r)P_n(\mu),\frac{1}{\sqrt{N(n)N(\lambda_{nm})}}J_{n+\frac{1}{2}}(\lambda_{nm}r)P_n(\mu)\end{array}\right\}. \tag{70}$$

The solution exists if and only if

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{|F_{nm}|^2}{\left\{\left(\frac{\epsilon\lambda_{nm}^2-1}{2\epsilon\lambda_{nm}^2-1}\right)\exp\left[-\lambda_{nm}^2T\right]+\left(\frac{\epsilon\lambda_{nm}^2}{2\epsilon\lambda_{nm}^2-1}\right)\exp\left[\lambda_{nm}^2T-\frac{T}{\epsilon}\right]\right\}^2}<\infty. \tag{71}$$

where

$$F_{nm}\doteq\frac{1}{\sqrt{N(n)N(\lambda_{nm})}}\int_0^1\int_{-1}^1 r^{\frac{3}{2}}f(r,\mu)J_{n+\frac{1}{2}}(\lambda_{nm}r)P_n(\mu)\,drd\mu, \tag{72}$$

are the classical Fourier coefficients of $\sqrt{r}f(r,\mu)$. The solution in this case is given by

$$v_0(r,\mu) = \frac{1}{\sqrt{r}\sqrt{N(n)\,N(\lambda_{nm})}} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{F_{nm} J_{n+\frac{1}{2}}(\lambda_{nm} r)\, P_n(\mu)}{\left\{\left(\frac{\epsilon n^2-1}{2\epsilon n^2-1}\right)\exp\left[-n^2T\right] + \left(\frac{\epsilon n^2}{2\epsilon n^2-1}\right)\exp\left[n^2T - \frac{T}{\epsilon}\right]\right\}}. \tag{73}$$

This solution will reduce to the usual parabolic model solution by evaluating the limit $\epsilon \to 0^+$. The solution to the parabolic model is given as

$$v_0(r,\mu) = \frac{1}{\sqrt{r}\sqrt{N(n)\,N(\lambda_{nm})}} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} F_{nm} J_{n+\frac{1}{2}}(\lambda_{nm} r)\, P_n(\mu)\exp\left[n^2T\right]. \tag{74}$$

5 Conclusions

The inverse solution of the heat conduction model is characterized by discontinuous dependence on the data. A small error in the nth Fourier coefficient is amplified by the factor $\exp[\lambda_{nm}T]$. Thus it depends on the rate of decay of singular values and this rate of decay also depends on the size of the parameter T. Since the singular values are strictly decreasing, one has to remove smaller singular values to get some meaningful result.

The hyperbolic model with a small parameter closely approximates the heat conduction equation. The small parameter introduced in the parabolic model act as a damping filter for noise as well as the actual signal. As the magnitude of noise increases, the solution with parabolic model becomes highly unstable and no meaningful information about the signal can be extracted. By choosing the size of the parameter ϵ, some meaningful information about the solution can be recovered.

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