King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 398

July 2008

Wave Equation in some Lorentzian Metrics

Ashfaque H. Bokhari, F. D. Zaman and A. H. Kara

DHAHRAN 31261 ● SAUDI ARABIA ● www.kfupm.edu.sa/math/ ● E-mail: c-math@kfupm.edu.sa

Wave Equation in some Lorentzian Metrics

Ashfaque H. Bokhari
Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

F. D. Zaman
Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

A. H. Kara
School of Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa

Abstract

In this note we discuss symmetries of a nonlinear wave equation that arises as a consequence of a Riemannian metric of signature -2. The objective of this study is to show how geometry can be responsible in giving rise to an expression of the wave equation without having any inhomogeneity put in by hand. We find Lie point symmetries of this wave equation and compare them with some well known symmetries of the Riemannian metric in which the wave equation is embedded. Some interesting physical conclusions relating to solutions and conservation laws arise.

1 Introduction

The wave equation has extensively been studied in literature from the point of view of its Lie point symmetries. A comprehensive symmetry analysis of the equation is discussed by Cantwell [1], Ibragimov [2] and Bluman and Kumei [3]. It is well known that in three space dimensions the linear wave equation admits 16-dimensional Lie algebra of point symmetries excluding the 'infinite symmetry' [4]. The dimension of the algebra of the Lie point symmetry of the wave equation reduces with introduction of nonlinearities there. Realistically speaking, one would expect that a genuinely interesting wave equation will possess nonlinearities. Whereas nonlinearities make the wave equation represent physically plausible situations, the difficult part is to justify introduction of nonlinearities? Generally nonlinearities in the wave equation are introduced by keeping in mind physical considerations such as properties of material. In this study we try a geometric approach to introduce nonlinearity in the wave equation. With this point in mind and the fact that geometric considerations may be of interest to other areas of applied sciences, we use two back ground geometries to write the wave equation in; one a plane symmetric static geometry and other spherically symmetric non-static one. Both geometries are of Lorentzian nature with signature -2 in 4-spacetime dimensions. We will write the wave equation in these two geometries one by one and obtain a 'nonlinear' form of the wave equations. We will then discuss Lie point symmetries of these wave equations and compare them with some other conventional symmetries possessed by the background Lorentzian metrics [5, 6].

The plan of the paper is as follows. In the next section we derive the wave equation in a plane symmetric static spacetime metric, find its Lie point symmetries and perform some reductions. In section 3 we derive the wave equation in spherically symmetric non-static metric and present our analysis there. We conclude the work the last section by giving a brief comparison of the symmetries of the wave equations with that of some well known conventional symmetries of the back ground geometries.

2 Wave equation in a plane symmetric static spacetime background

In order to write the wave equation in a Lorentzian geometry, we use the well know d'alembertian operator $\Box$ to write the wave equation. In four Lorentzian metric g_{ab} in which a time translational invariance exists, this operator acting on a the wave (or mode) function $u(t, \mathbf{x})$ is given by,

$$\Box u(\mathbf{x}, t) = g^{00}\partial_0^2 + \frac{1}{2}g^{ij}[g^{00}(\partial_i g_{00})\partial_j + \partial_i\partial_j - \Gamma^k_{ij}\partial_k]u(\mathbf{x}, t) = 0, \tag{2.1}$$

where $\Gamma^k_{ij} = \frac{1}{2}g^{km}(\partial_j g_{im} + \partial_i g_{jm} - \partial_m g_{ij})$ represents Christoffel symbol, g^{ij} inverse of the metric g_{ij} and $\mathbf{x}$ three space variables x, y, z. Since we are interested in the present study to motivate to studying wave equation in some given back ground Lorentzian geometries, we chose a particular metric,

$$g_{ij} = ((x/a)^2, -1, -1, -1), \tag{2.2}$$

with i and j take values from $0, ..., 3$ respectively. Further, 0 is used to represent t coordinate whereas $1, ..., 3$ respectively represent x, y and z coordinates. In the metric given by (2.2), the wave equation (2.1) takes the form,

$$u_{tt} = \frac{x^2}{a^2}(u_{xx} + u_{yy} + u_{zz}) + \frac{x}{a^2}u_x \tag{2.3}$$

We now derive infinitesimal symmetry generators. The one parameter Lie point transformations which leave (2.1) invariant are given by [7, 1]

$$\widetilde{\mathbf{x}} = \mathbf{x} + \epsilon\xi_i(\mathbf{x}, u) + O(\epsilon^2) \ for \ i = 0, \dots, 3 \tag{2.4}$$

where $\mathbf{x}$ and ξ_i respectively represent variables on which the wave equation depends and corresponding components of the tangent vector X. Using (2.4), the expressions for the derivatives (of the transformed 'dependent' variables with respect to the transformed 'independent' variables) become:

$$\widetilde{u_j} = u_j + \epsilon\phi^j(\mathbf{x}, u) + O(\epsilon^2) \ for \ j = 1, 2, \ldots \tag{2.5}$$

where $u_j = \frac{\partial u}{\partial x^1}$ and $u_j = \frac{\partial^2 u}{\partial x^1 \partial x^2}$ respectively, for $j = 1$ and $j = 1, 2$. To solve (2.3), we start by writing symmetry generator corresponding to the variables x, y, t and u

$$X = m\frac{\partial}{\partial x} + n\frac{\partial}{\partial y} + p\frac{\partial}{\partial z} + q\frac{\partial}{\partial t} + s\frac{\partial}{\partial u} \tag{2.6}$$

where $m,\ n,\ p$ and q and s are the components of the tangent vector X computed for $i = 0, \ldots, 3$ at $\epsilon = 0$. Now prolonging the above generator to second order [3, 1, 7] and using symmetry criterion $X^2[u_{tt} - \frac{x^2}{a^2}(u_{xx} + u_{yy} + u_{zz}) - \frac{x}{a^2}u_x] \mid_{(2.3)} = 0$ for partial differential equations to the wave equation. Using this condition, replacing u_{tt} into the resulting expression and then comparing coefficients of all possible derivatives and products of derivative of u, gives rise to an over-determined system of partial differential equations given in appendix **1**.

One way to solve the above system is to follow the ab-initio method [8]. However, keeping in mind that this is a routine calculation, we use an algebraic software [9] to solve this system. From crack one immediately finds a partial solution of the system given in appendix 1 that is spanned by 15 linearly independent Lie point symmetries with one satisfying the wave equation for an arbitrary function.

Reducing (2.3) to an ordinary differential equation (ode) using similarity methods would require three-dimensional subalgebras of (4.17). From this equation one can notice that the three generators $\{X_5, X_6, X_{12}\}$ generate a subalgebra whose commutators are,

$$[X_5, X_6] = 0 \quad [X_6, X_{12}] = X_6, \quad [X_5, X_{12}] = 0.$$

The invariants of X_5 can be shown to be $\alpha = y^2 + z^2, t, x, u$ by which (2.3) becomes

$$u_{tt} = \frac{x^2}{a^2}(u_{xx} + 4\alpha u_{\alpha\alpha} + 4u_\alpha) + \frac{x}{a^2}u_x \tag{2.7}$$

and X_6 in these variables remain with invariants $\beta = t + a\ln x, \alpha, u$. Equation (2.7) then reduces to

$$\alpha u_{\alpha\alpha} + u_\alpha = 0. \tag{2.8}$$

Lastly, X_{12} in the final set of variables is $a\partial_\beta + 2\alpha\partial_\alpha + u\partial_u$ with invariants $\gamma = \ln\alpha - \frac{2}{a}\beta, u$ so that (2.8) becomes the ode

$$u_{\gamma\gamma} = 0. \tag{2.9}$$

Thus $u = A\gamma + B$ for some constant A and B so that

$$u = A[\ln(y^2 + z^2) - \frac{2}{a}(t + a\ln x)] + B$$

which is invariant under the subalgebra formed by rotation X_5, X_6 and dilation X_{12}.

3 Wave equation in flat Friedmann spacetime background

The metric of the flat Friedmann metric in Cartesian coordinates is given by [5, 6]

$$ds^2 = dt^2 - t^{4/3}(dx^2 + dy^2 + dz^2). \tag{3.10}$$

The wave equation, in Cartesian coordinates, on this manifold lead to

$$t^{4/3}u_{tt} + 2t^{1/3}u_t - (u_{xx} + u_{yy} + u_{zz}) = 0. \tag{3.11}$$

The lie point symmetry generator

$$X = p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y} + r\frac{\partial}{\partial z} + n\frac{\partial}{\partial t} + f\frac{\partial}{\partial u} \tag{3.12}$$

is obtained by $X[(t^{4/3}u_{tt} + 2t^{1/3}u_t - (u_{xx} + u_{yy} + u_{zz})|_{(3.11)} = 0$. Following a procedure similar to the first case, the terms in the resulting expression can be separated. First separating by quadratic and cubic terms in the second derivatives of u, it is immediately found that p, q, r are n are independent of u whilst f is linear in u. Proceeding with further separations one obtains the over estimated system given in appendix 3.

The solution of this over determined system leads to a 10-dimensional algebra of symmetry generators (excluding the infinite one)whose generators are listed in appendix 4.

We consider a reduction to an ordinary differential equation via the three-dimensional sub-algebra representing rotation in xy, dilation and translation in z, viz., $X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, $X_9 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t}$ and $X_2 = \frac{\partial}{\partial z}$ whose commutators are zero. The generator X has invariants $\alpha = x^2 + y^2$, t, z and u so that, with simultaneously applying the translation in z, (3.11) becomes

$$t^{4/3}u_{tt} + 2t^{1/3}u_t - (4u_{\alpha\alpha} + 4u_\alpha) = 0. \tag{3.13}$$

Equation (3.13) inherits the generator Y which in the new coordinates is $2\alpha\frac{\partial}{\partial\alpha}+3t\frac{\partial}{\partial t}$ with invariants $\gamma=\frac{\alpha}{t^{2/3}}$ and u which reduces (3.13) to the linear ode

$$-(\frac{2}{9}\gamma+4)u_{\gamma}+4(\frac{1}{9}\gamma^{2}-\gamma)u_{\gamma\gamma}=0 \tag{3.14}$$

which, for further analysis, may be written

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}[4(\frac{1}{9}\gamma^{2}-\gamma)u_{\gamma}]=\frac{10}{9}\gamma u_{\gamma}. \tag{3.15}$$

4 Discussion and conclusion

We have considered the classical wave equation in two Lorentzian spacetime spacetime background with a point in mind that the wave equation there may inherit nonlinearity from geometry naturally. In this connection we have considered two spacetime metrics which respectively represent a plane symmetric static metric [15] and flat Friedmann metric of signature -2. For both cases we have given one solution each to show how wave equation there can be either solved or reduced to ordinary differential equations by using symmetry methods. Of the two metrics considered, the second metric is of more interest than the first as its Lie as well as Noether symmetry analysis has already been discussed in literature yielding additional conservation laws that were not given previously by some conventional symmetries of the same metric [14]. In his book [4] Ibragimov suggests that in three flat space dimensions the linear wave equation admits 16-dimensional Lie algebra of point symmetries excluding the 'infinite symmetry'. In this study we show that the wave equations admits fewer symmetries when it is solved on general Lorentzian manifolds. In particular we have shown that the wave equations in plane symmetric static spacetime admits 15 Lie point symmetries which are one less than 16 Lie pint symmetries of the wave equation in 3 cartesian space dimensions suggested in [4]. Our hunch is that it is presumably a shift away effect from 'flatness' from Minkowski manifold to other manifolds leading to reduction in symmetry as well as solutions of the wave equation. This shift away from flatness of Minkowski manifolds is more clear in flat Friedmann metric case where only 10 Lie point symmetries of the wave equation are recovered. There the shift away from flatness is by 7 symmetry generators [4] and 7 solutions at least. In fact, the alternatives to the Minkowski case need not lend themselves to the variational case as conveniently as does the Minkowski case. For relationship between symmetries and conservation laws, we refer the reader to [13]. Also, it should be noted that these alternatives to the wave equation on the Minkowski manifold are not achievable via a simple point transformation of variables on the Minkowski version. It is hoped that solving wave equation in curved spacetime background may provide some other useful relationship of solutions with those of gravitational waves solutions in general relativity.

Acknowledgements

Two of the authors (AHB and FDZ) thank King Fahd University of Petroleum and Minerals project number FT080004 for support and funds provided to complete this work.AHK thanks the NRF for support under programme FA2007041200006.

References

[1] B. J. Cantwell, *An Introduction to Symmetry Analysis*, Cambridge University Press, 2002.

[2] N. H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, Wiley, 1999.

[3] G. Bluman and S. Kumei, *Symmetries and Differential Equations*, New York, 1989.

[4] N. H. Ibragimov, in Handbook , CRC vol. I, .

[5] A. Z. Petrov, (1969) *Einstein Spaces*, Pergamon, Oxford.

[6] H. Stephani, D. Kramer, M. A. H. MacCallum,C. Hoenselaers, *Exact Solutions of Eistein Equation's Fiedl Equation*, Cambridge University Press, (2003).

[7] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York 1986.

[8] A. H. Bokhari, (1992) *International Journal of Theoretical Physics* **31** 2091.

[9] T. Wolf, Crack, LiePDE, ApplySym and ConLaw, section 4.3.5 and computer program on CD-ROM in: Grabmeier, J., Kaltofen, E. and Weispfenning, V. (Eds.): Computer Algebra Handbook, Springer (2002) 465.

[10] E. A. Saied and M. M. Hussain, Similarity solutions for a nonlinear model of the heat equation, *Nonlinear Mathematical Physics*, 3(1-2), 219–225, 1996.

[11] P. A. Clarkson and E. L. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear heat equations, *Physica D*, 70, 250–288, 1993.

[12] M. I. Servo, Conditional and nonlocal symmetry of nonlinear heat equation, *Nonlinear Mathematical Physics*, 3(1-2), 63–67, 1996.

[13] A. H. Kara and F. M. Mahomed, The relationship between symmetries and conservation laws, *Int Journal of Theoretical Physics*, 39(1), 23–40, 2000.

[14] A. H. Bokhari and A. H. Kara, Noether versus Killing symmetry of conformally flat Friedmann metric, *Gen Relativ Gravit* 39, 2053-2059, 2007.

[15] A. R. Kashif, Curvature collineations of some spacetimes and their physical interpretation, Ph. D thesis, Quaid-i-Azam University Preprint, 2003.

Appendix 1

Over determined system of partial differential equations (2.3)

$$\begin{aligned}
&m_u = 0 = n_u = p_u = q_u = q_{u,u} = s_{u,u}\\
&a^2 m_t - x^2 q_x = 0, \quad a^2 n_t - x^2 q_y = 0, \quad a^2 p_t - x^2 q_z = 0\\
&m - x m_x + x q_t = 0, \quad m - x n_y + x q_t = 0, \quad m - x p_z + x q_t = 0\\
&m_y + n_x = 0, \quad m_z + p_x = 0, \quad n_z + p_y = 0\\
&m + 2x q_t - x m_x + a^2 m_{t,t} - x^2 (m_{x,x} + m_{y,y} + m_{z,z} - 2 s_{x,u}) = 0\\
&x n_x - a^2 n_{t,t} + x^2 (n_{x,x} + n_{y,y} + n_{z,z} - 2 s_{y,u}) = 0\\
&x p_x - a^2 p_{t,t} + x^2 (p_{x,x} + p_{y,y} + p_{z,z} - 2 s_{z,u}) = 0\\
&x q_x - a^2 q_{t,t} + x^2 (q_{x,x} + q_{y,y} + q_{z,z} - 2 s_{t,u}) = 0\\
&x s_x - a^2 s_{t,t} + x^2 (s_{x,x} + s_{y,y} + s_{z,z}) = 0
\end{aligned} \tag{4.16}$$

Appendix 2

Symmetry generators corresponding to (4.16)

$$\begin{aligned}
&X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = u\frac{\partial}{\partial u}\\
&X_5 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad X_6 = e^{\frac{t}{a}}\left(\frac{\partial}{\partial x} - \frac{a}{x}\frac{\partial}{\partial t}\right), \quad X_7 = e^{-\frac{t}{a}}\left(\frac{\partial}{\partial x} + \frac{a}{x}\frac{\partial}{\partial t}\right)\\
&X_8 = e^{\frac{t}{a}}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} - a\frac{z}{x}\frac{\partial}{\partial t}\right), \quad X_9 = e^{-\frac{t}{a}}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + a\frac{z}{x}\frac{\partial}{\partial t}\right)\\
&X_{10} = e^{\frac{t}{a}}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - a\frac{y}{x}\frac{\partial}{\partial t}\right), \quad X_{11} = e^{-\frac{t}{a}}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + a\frac{y}{x}\frac{\partial}{\partial t}\right)\\
&X_{12} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}\\
&X_{13} = 2xy\frac{\partial}{\partial x} + (y^2 - x^2 - z^2)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} - 2uy\frac{\partial}{\partial u}\\
&X_{14} = 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (z^2 - x^2 - y^2)\frac{\partial}{\partial z} - 2uz\frac{\partial}{\partial u}
\end{aligned} \tag{4.17}$$

Appendix 3

Over determined system of partial differential equation (3.13)

$$
\begin{aligned}
& nt^{1/3} - t^{4/3}n_t - t^{7/3}f_{tu} + \tfrac{1}{2}t^{7/3}n_{tt} - \tfrac{1}{2}tn_{xx} - \tfrac{1}{2}tn_y - \tfrac{1}{2}tn_{zz} = 0 \\
& \tfrac{1}{3}\tfrac{n}{t} - n_t + 2p_x + tf_{tu} - \tfrac{1}{2}tn_{tt} + \tfrac{1}{2t^{1/3}}(n_{xx} + n_{yy} + n_{zz}) = 0 \\
& \tfrac{1}{3}\tfrac{n}{t} - n_t + 2q_y + tf_{tu} - \tfrac{1}{2}tn_{tt} + \tfrac{1}{2t^{1/3}}(n_{xx} + n_{yy} + n_{zz}) = 0 \\
& \tfrac{1}{3}\tfrac{n}{t} - n_t + 2r_z + tf_{tu} - \tfrac{1}{2}tn_{tt} + \tfrac{1}{2t^{1/3}}(n_{xx} + n_{yy} + n_{zz}) = 0 \\
& 2n_y - 2t^{4/3}q_t = 0 \\
& 2n_x - 2t^{4/3}p_t = 0 \\
& 2n_z - 2t^{4/3}r_t = 0 \\
& 2p_z + 2r_x = 0 \\
& 2p_y + 2q_x = 0 \\
& 2q_z + 2r_y = 0 \\
& -2t^{1/3}p_t - 2f_{xu} - t^{4/3}p_{tt} + p_{xx} + p_{yy} + p_{zz} = 0 \\
& -2t^{1/3}q_t - 2f_{yu} - t^{4/3}q_{tt} + q_{xx} + q_{yy} + q_{zz} = 0 \\
& -2t^{1/3}r_t - 2f_{zu} - t^{4/3}r_{tt} + r_{xx} + r_{yy} + r_{zz} = 0 \\
& t^{4/3}f_{tt} + 2t^{1/3}f_t - (f_{xx} + f_{yy} + f_{zz}) = 0
\end{aligned}
\tag{4.18}
$$

Appendix 4

Symmetry generators corresponding to (4.18)

$$\begin{aligned}
&X_o = \tfrac{\partial}{\partial x}, \quad X_1 = \tfrac{\partial}{\partial y}, \quad X_2 = \tfrac{\partial}{\partial z},\\
&X_3 = x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}, \quad X_3 = x\tfrac{\partial}{\partial z} - z\tfrac{\partial}{\partial x}, \quad X_5 = y\tfrac{\partial}{\partial z} - z\tfrac{\partial}{\partial y},\\
&X_6 = -2xy\tfrac{\partial}{\partial x} + (-9t^{2/3} + x^2 - y^2 - z^2)\tfrac{\partial}{\partial y} - 2zy\tfrac{\partial}{\partial z} - 6ty\tfrac{\partial}{\partial t} + 6yu\tfrac{\partial}{\partial u},\\
&X_7 = (-9t^{2/3} - x^2 + y^2 + z^2)\tfrac{\partial}{\partial x} - 2xy\tfrac{\partial}{\partial y} - 2zx\tfrac{\partial}{\partial z} - 6tx\tfrac{\partial}{\partial t} + 6xu\tfrac{\partial}{\partial u},\\
&X_8 = -2xz\tfrac{\partial}{\partial x} - 2yz\tfrac{\partial}{\partial y} + (-9t^{2/3} + x^2 + y^2 - z^2)\tfrac{\partial}{\partial z} - 6tz\tfrac{\partial}{\partial t} + 6zu\tfrac{\partial}{\partial u},\\
&X_9 = x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + z\tfrac{\partial}{\partial z} + 2t\tfrac{\partial}{\partial t}.
\end{aligned} \tag{4.19}$$