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Abstract

In applications of branching processes, usually it is hard to obtain samples of a large size. Therefore, a bootstrap procedure allowing inference based on a small sample size is very useful. Unfortunately, in the critical branching process with a stationary immigration the standard parametric bootstrap is invalid. In this paper we consider the process with non-stationary immigration, whose mean and variance vary regularly with nonnegative exponents α and β, respectively. We prove that $1 + 2\alpha$ is the threshold for the validity of the bootstrap in this model. If $\beta < 1 + 2\alpha$, the standard bootstrap is valid and if $\beta > 1 + 2\alpha$ it is invalid. In the case $\beta = 1 + 2\alpha$ the validity of the bootstrap depends on the slowly varying parts of the immigration mean and variance. These results allow us to develop statistical inferences about the parameters of the process in its early stages.

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1 Introduction

We consider a discrete time branching stochastic process $Z(n), n \geq 0, Z(0) = 0$. It can be defined by two families of independent, nonnegative integer valued random variables $\{X_{ni}, n, i \geq 1\}$ and $\{\xi_n, n \geq 1\}$ recursively as

$$Z(n) = \sum_{i=1}^{Z(n-1)} X_{ni} + \xi_n, \quad n \geq 1. \tag{1.1}$$

Assume that X_{ni} have a common distribution for all n and i, and families $\{X_{ni}\}$ and $\{\xi_n\}$ are independent. Variables X_{ni} will be interpreted as the number of offspring of the ith individual in the $(n-1)$th generation and ξ_n is the number of immigrating individuals in the nth generation. Then $Z(n)$ can be considered as the size of nth generation of the population.

In this interpretation, $a = EX_{ni}$ is the mean number of offspring of a single individual. Process $Z(n)$ is called *subcritical, critical* or *supercritical* depending on $a < 1, a = 1$ or $a > 1$, respectively. The independence assumption of families $\{X_{ni}\}$ and $\{\xi_n\}$ means that reproduction and immigration processes are independent. However, unlike in the classical models, we do not assume that $\xi_n, n \geq 1$, are identically distributed, i. e. the immigration distribution depends on the time of immigration. It is well known that asymptotic behavior of the process with immigration is very sensitive to any changes of the immigration process in time.

If a sample $\{Z(k), k = 1, ..., n\}$ is available, then the weighted conditional least squares estimator of the offspring mean is known to be (see [15])

$$\hat{a}_n = \frac{\sum_{k=1}^{n}(Z(k) - \alpha(k))}{\sum_{k=1}^{n} Z(k-1)}, \ \alpha(k) = E\xi_k \ . \tag{1.2}$$

In the process with a stationary immigration the maximum likelihood estimators (MLE) for the offspring and immigration means, which were derived in [3] for the power series offspring and immigration distributions, are based on the sample of pairs $\{(Z(k), \xi_k), k = 1, ..., n\}$. The MLE for the offspring mean has the same form as $\hat{a}_n$ with ξ_k in place of $\alpha(k)$, and the MLE for the immigration mean is just the average of the number of immigrating individuals.

Sriram [17] investigated validity of the bootstrap estimator of the offspring mean based on MLE and demonstrated that in the critical case the asymptotic validity of the parametric bootstrap does not hold. Similar invalidity of the parametric bootstrap for the first order autoregressive process with autoregressive parameter ± 1 was earlier proved in [2]. The main cause of the failure is the fact that in the critical case the MLE does not have the desired rate of convergence (faster than n^{-1}).

Recently, it was shown [14] that in the process with non-stationary immigration when the immigration mean tends to infinity the conditional least squares estimator (CLSE) has a normal limit distribution and the rate of convergence of the CLSE is faster than n^{-1}. In connection with this the following question is of interest. Will the bootstrap version of the weighted CLSE of the offspring mean have the same limiting distribution as the original CLSE? In other words, will the standard parametric bootstrap be valid in this non-classical model? The present paper addresses this question. It turned out that the validity of the bootstrap depends on the relative rate of the immigration mean and variance. Assuming that the immigration mean and variance vary regularly with nonnegative exponents α and β, respectively, we prove that if $\beta < 1 + 2\alpha$, the bootstrap leads to a valid approximation for the CLSE. If $\beta > 1 + 2\alpha$, i.e. the rate of the immigration variance is relatively large, the conditional distribution of the bootstrap version of the CLSE has a random limit (in distribution). In the threshold case $\beta = 1 + 2\alpha$ the validity depends on slowly varying parts of the mean and variance of the immigration. To prove these results, first we obtain approximation theorems for an array of nearly critical processes, which are also of independent interest.

It follows from the above discussion that the question on criticality of the process is crucial for applications. To answer this question, one may test hypothesis $H_0 : a = 1$ against one of $a \neq 1, a > 1$ or $a < 1$. Our results allow to develop rejection regions for these hypotheses based on bootstrap pivots (equation (2.6)). Sampling distributions of the bootstrap pivots can easily be generated based on a single sample $\{Z(k), k = 1, ..., n\}$ with relatively small generation number n. Thus, the bootstrap procedure is very useful in applications of branching processes, where it is hard to obtain samples of a large size. It allows statistical inferences in early stages of the process. This is important, for example, in epidemic models, which can be approximated by branching processes when initial number of susceptible individuals is large.

Investigation of the problems related to the bootstrap methods and their applications has been an active area of the research since its introduction by Efron [9]. As a result a large number of papers and monographs have been published. We note monographs [8] and [10] and the most recent review articles [7] and [13] as important sources of the literature on bootstrap methods. As it was mentioned before, invalidity of the bootstrap for the critical process with a stationary immigration was shown in [17]. In [6] a modification of the standard bootstrap procedure was proposed, which eliminated the invalidity in the critical case. The second-order correctness of the bootstrap for a studentized version of MLE in subcritical case proved in [18].

The paper is organized as follows. In Section 2 of the paper, we describe the parametric bootstrap and state the main results. Necessary limit theorems for an array of branching processes and their consequences related to CLSE will be derived in Section 3. The proofs of main theorems are given in Section 4. Appendix contains the proofs of limit theorems for the array of nearly critical processes.

2 Main results on the bootstrap

The process with time-dependent immigration is given by the offspring distribution of $X_{ki}, k, i \geq 1$, and by the family of distributions of the number of immigrating individuals $\xi_k, k \geq 1$. We assume that the offspring distribution has the probability mass function

$$p_j(\theta) = P\{X_{ki} = j\},\ j = 0, 1, ..., \tag{2.1}$$

depending on parameter θ, where $\theta \in \Theta \subseteq \mathbb{R}$. Then $a = E_\theta X_{ki} = f(\theta)$ for some function f. We assume throughout the paper that f is one-to-one mapping of Θ to $[0, \infty)$ and is a homeomorphism between its domain and range. It is known that these assumptions are satisfied, for example, by the distributions of the power series family [6].

About the number of immigrating individuals, we assume that ξ_k follows a known distribution with the probability mass function

$$q_j(k) = P\{\xi_k = j\},\ j = 0, 1, ..., \tag{2.2}$$

for any $k \geq 1$.

Throughout the paper " $\xrightarrow{D}$ ", " $\xrightarrow{d}$ " and " $\xrightarrow{P}$ " will denote convergence of random functions in Skorokhod topology and convergence of random variables in distribution and in probability, respectively, and also $X \stackrel{d}{=} Y$ denotes equality of distributions. We assume that $b = VarX_{ni} < \infty$ and $\alpha(k) = E\xi_k,\ \beta(k) = Var\xi_k$ are finite for any $k \geq 1$ and are regularly varying sequences of nonnegative exponents α and β, respectively. Then $A(n) = EZ(n)$ and $B^2(n) = VarZ(n)$ are finite for each $n \geq 0$ and $a = 1$. To provide the asymptotic distribution of $\hat{a}_n$ defined in (1.2), we assume that there exists $c \in [0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\beta(n)}{n\alpha(n)} = c \tag{2.3}$$

and denote for any $\varepsilon > 0$

$$\delta_n(\varepsilon) = \frac{1}{B^2(n)} \sum_{k=1}^{n} E[(\xi_k - \alpha(k))^2; |\xi_k - \alpha(k)| > \varepsilon B(n)].$$

As it was proved in [15], if $a = 1,\ b \in (0, \infty),\ \alpha(n) \to \infty,\ \beta(n) = o(n\alpha^2(n))$, condition (2.3) is satisfied and $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, then as $n \to \infty$

$$\frac{nA(n)}{B(n)}(\hat{a}_n - a) \xrightarrow{d} (2 + \alpha)\mathcal{N}(0, 1). \tag{2.4}$$

Furthermore, under the above conditions, $A(n)/B(n) \to \infty$ as $n \to \infty$ and when $c = 0$ the condition $\delta_n(\varepsilon) \to 0$ is satisfied automatically. More detailed discussion and examples can be seen in [15].

We now describe the bootstrap procedure to approximate the sampling distribution of the pivot

$$V_n = \frac{nA(n)}{B(n)}(\hat{a}_n - a).$$

Given a sample $\mathcal{X}_n = \{Z(k), k = 1, ..., n\}$ of population sizes, we estimate the offspring mean a by the weighted CLSE $\hat{a}_n$. Obtain the estimate of the parameter θ as $\hat{\theta}_n = f^{-1}(\hat{a}_n)$ from equation $a = f(\theta)$. Replace θ in the probability distribution (2.1) by its estimate. Given $\mathcal{X}_n$, let $\{X_{ki}^{*(n)}, k, i \geq 1\}$ be a family of i.i.d. random variables with the probability mass function $\{p_j(\hat{\theta}_n), j = 0, 1, ...\}$. Now we obtain the bootstrap sample $\mathcal{X}_n^* = \{Z^{*(n)}(k), k = 1, ..., n\}$ recursively from

$$Z^{*(n)}(k) = \sum_{i=1}^{Z^{*(n)}(k-1)} X_{ki}^{*(n)} + \xi_k, \quad n, k \geq 1, \tag{2.5}$$

with $Z^{*(n)}(0) = 0$, where $\xi_k, k \geq 1$, are independent random variables with the probability mass functions $\{q_j(k), j = 0, 1, ...\}$. Then, we define the bootstrap analogue of the pivot V_n by

$$V_n^* = \frac{nA(n)}{B(n)}(\hat{a}_n^* - \hat{a}_n), \tag{2.6}$$

where $\hat{a}_n^*$ is the weighted CLSE based on $\mathcal{X}_n^*$, i.e.

$$\hat{a}_n^* = \frac{\sum_{k=1}^n (Z^{*(n)}(k) - \alpha(k))}{\sum_{k=1}^n Z^{*(n)}(k-1)}. \tag{2.7}$$

To state our main result, we need the following conditions be satisfied.

A1. $a = 1$ and moments $E_\theta[(X_{ki})^2]$ and $E_\theta[(X_{ki})^{2+l}]$ are continuous functions of θ for some $l > 0$.
A2. $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$.
A3. $\alpha(n) \to \infty$, $\beta(n) = o(n\alpha^2(n))$ as $n \to \infty$.

Theorem 2.1. *If conditions A1-A3 and (2.3) are satisfied, then*

$$\sup_x |P\{V_n^* \leq x|\mathcal{X}_n\} - \Phi(2+\alpha, x)| \xrightarrow{P} 0 \tag{2.8}$$

as $n \to \infty$, where $\Phi(\sigma, x)$ is the normal distribution with mean zero and variance σ^2.

Remarks. 2.1. Due to (2.4) convergence (2.8) implies the validity of the standard bootstrap procedure to approximate the sampling distribution of V_n, i.e. as $n \to \infty$

$$\sup_x |P\{V_n^* \leq x|\mathcal{X}_n\} - P\{V_n \leq x\}| \xrightarrow{P} 0.$$

2.2. As it was mentioned before, in the case $c = 0$ condition A2 is automatically satisfied. In the case $c > 0$ the condition is equivalent to the Lindeberg condition for the family $\{\xi_n, n \geq 1\}$ of the number of immigrating individuals.

Example 2.1. Let $\xi_k, k \geq 1$, be Poisson with the mean $\alpha(k)$ such that $\alpha(k) \to \infty, k \to \infty$, and regularly varies with exponent α. In this case $\beta(n) = o(n\alpha(n))$ as $n \to \infty$ and condition A3 is satisfied. Moreover, we realize that $c = 0$ in (2.3), which implies that condition A2 is also fulfilled. Thus we have the following result.

Corollary 2.1. *If $\xi_k, k \geq 1$, are Poisson with mean $\alpha(k) \to \infty, k \to \infty$, and $(\alpha(k))_{k=1}^{\infty}$ is regularly varying with exponent α and condition A1 is satisfied, then (2.8) holds.*

Now we consider the case of the fast immigration variance. In other words, we assume that there exists $d \in [0, \infty)$ such that

$$\lim_{n\to\infty} \frac{n\alpha^2(n)}{\beta(n)} = d. \tag{2.9}$$

It is known that in this case the weighted CLSE is not asymptotically normal. More precisely, if $a = 1,\ b \in (0, \infty),\ \alpha(n) \to \infty$, condition (2.9) is satisfied and $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, then as $n \to \infty$

$$n(\hat{a}_n - a) \xrightarrow{d} W_0 =: \frac{W(1)}{\int_0^1 W(t^{1+\beta})dt + \gamma}, \tag{2.10}$$

where $W(t)$ is the standard Wiener process and $\gamma = ((\alpha+1)(\alpha+2))^{-1}\sqrt{d(1+\beta)}$ (see [15], Theorem 3.2).

We denote by $W_n = n(\hat{a}_n - a)$ the pivot corresponding to this result and by $W_n^* = n(\hat{a}_n^* - \hat{a}_n)$ the bootstrap pivot based on the bootstrap sample $\mathcal{X}_n^*$. The next theorem shows that in the case of fast immigration variance the parametric bootstrap is invalid. To formulate corresponding theorem, we need several new notations. Let $\nabla_\beta(t) = \nabla_\beta(c_0, t) = \mu_\beta(2c_0, t)$, where

$$\mu_\alpha(t) = \mu_\alpha(c_0, t) = \int_0^t u^\alpha e^{(t-u)c_0} du,\ \gamma_0 = \gamma_0(c_0) = \left(\frac{d}{\nabla_\beta(1)}\right)^{1/2} \int_0^1 \mu_\alpha(u)du,$$

$$\psi(t) = \psi(c_0, t) = \frac{t^{1+\beta}}{(1+\beta)\nabla_\beta(1)},\ \Lambda(c_0, t) = W(\psi(t)) + c_0 \int_0^t e^{c_0(t-u)} W(\psi(u))du.$$

Further, introduce ratio

$$\nu(c_0) = \frac{W(\psi(c_0, 1))}{\int_0^1 \Lambda(c_0, t)dt + \gamma_0(c_0)} \tag{2.11}$$

and its cumulative distribution function $F(c_0, x) = P\{\nu(c_0) \leq x\}$.

Theorem 2.2. *If conditions A1, A2 and (2.9) are satisfied and $\alpha(n) \to \infty$ as $n \to \infty$, then*

$$P\{W_n^* \le x | \mathcal{X}_n\} \xrightarrow{d} F(W_0, x) \tag{2.12}$$

as $n \to \infty$ for each $x \in \mathbb{R}$.

Remark 2.3. Theorem 2.2 shows that, when the immigration variance is large enough comparatively the mean, the conditional limit distribution of the bootstrap pivot W_n^* does not coincide with the limit distribution of $W_n = n(\hat{a}_n - a)$ for $a = 1$. In other words, as in the case of stationary immigration, the standard bootstrap least squares estimate of the offspring mean is asymptotically invalid. It is not surprising, if we recall that even the bootstrap version of the sample mean is invalid in the infinite variance case [1]. In this case one may develop a modified version of the bootstrap as in [6] (see Section 5 for details).

In order to prove the main theorems, we obtain a series of results for the array of the branching processes in a more general set up, which are of independent interest as well. The scheme of the proofs is as following. Since the bootstrap sample $\mathcal{X}_n^*$ is based on the sequence of branching processes (2.5), we first investigate the array of processes under suitable assumptions of the nearly criticality. In the second step, we derive limit distributions for the CLSE of the offspring mean in the sequence of nearly critical processes. In the third step, we show that conditions of the limit theorems for the CLSE are fulfilled by the bootstrap pivots V_n^* and W_n^*.

3 Approximation of the sequence of processes

We now consider a sequence of branching processes defined as following. Let $\{X_{ki}^{(n)}, k, i \ge 1\}$ and $\{\xi_k^{(n)}, k \ge 1\}$ be two families of independent, non-negative and integer valued random variables for each $n \in \mathbb{N}$. The sequence of branching processes $(Z^{(n)}(k), k \ge 0)_{n \ge 1}$ with $Z^{(n)}(0) = 0, n \ge 1$, is defined recursively as

$$Z^{(n)}(k) = \sum_{i=1}^{Z^{(n)}(k-1)} X_{ki}^{(n)} + \xi_k^{(n)}, \quad n \ge 1. \tag{3.1}$$

Assume that $X_{ki}^{(n)}$ have a common distribution for all k and i, and families $\{X_{ki}^{(n)}\}$ and $\{\xi_k^{(n)}\}$ are independent. As before, the variables $X_{ki}^{(n)}$ will be interpreted as the number of offspring of the ith individual in the $(k-1)$th generation and $\xi_k^{(n)}$ is the number of immigrating individuals in the kth generation. Then $Z^{(n)}(k)$ can be considered as the size of population of kth generation in nth process.

In this scheme, $a(n) = EX_{ki}^{(n)}$ is the criticality parameter of the nth process. The process $\{Z^{(n)}(k), k \geq 0\}$ for each fixed n is called *subcritical, critical* or *supercritical* depending on $a(n) < 1, a(n) = 1$ or $a(n) > 1$, respectively. The sequence of branching processes (3.1) is said to be *nearly critical* if $a(n) \to 1$ as $n \to 1$. In this section we investigate asymptotic behavior of a sum of normalized martingale differences generated by the nearly critical sequence of branching processes with non-stationary immigration, under the assumption that sequences of the immigration means and variances can be approximated by regularly varying sequences with non-negative exponents. We prove that when the immigration mean tends to infinity depending on the time of immigration, the "broken-line" process can be approximated in Skorokhod topology by a time-changed Wiener process.

If a sequence $(f(n))_{n=1}^{\infty}$ is regularly varying with exponent ρ, we will write $(f(n))_{n=1}^{\infty} \in R_{\rho}$. We assume that $a(n) = EX_{ij}^{(n)}$ and $b(n) = VarX_{ij}^{(n)}$ are finite for each $n \geq 1$ and $\alpha(n,i) = E\xi_i^{(n)} < \infty$, $\beta(n,i) = Var\xi_i^{(n)} < \infty$ for all $n, i \geq 1$. Then $A_n(i) = EZ^{(n)}(i)$ and $B_n^2(i) = VarZ^{(n)}(i)$ are finite for each $n \geq 1$, $1 \leq i \leq n$, and by a standard technique we find that

$$A_n(k) = \sum_{i=1}^{k} \alpha(n,i) a^{k-i}(n), \quad B_n^2(k) = \Delta_n^2(k) + \sigma_n^2(k), \tag{3.2}$$

where

$$\Delta_n^2(k) = \sum_{i=1}^{k} \alpha(n,i) Var(X^{(n)}(k-i)), \quad \sigma_n^2(k) = \sum_{i=1}^{k} \beta(n,i) a^{2(k-i)}(n),$$

$$Var(X^{(n)}(i)) = \frac{b(n)}{1-a(n)} a^{i-1}(n)(1-a^i(n)).$$

We also denote by $\Im^{(n)}(k)$ for each $n \geq 1$ the σ-algebra containing all the history of the nth process up to kth generation, i.e. it is generated by $\{Z^{(n)}(0)$, $Z^{(n)}(1)$,..., $Z^{(n)}(k)\}$ and put $M^{(n)}(k) = Z^{(n)}(k) -$

$E[Z^{(n)}(k)|\Im^{(n)}(k)]$. In our proofs we need approximation results for the following processes

$$\mathcal{Z}_n(t) = \frac{Z^{(n)}([nt])}{A_n(n)},\ \mathcal{Y}_n(t) = \frac{1}{B_n(n)} \sum_{k=1}^{[nt]} M^{(n)}(k),\ t \in \mathbb{R}_+.$$

In the approximation we assume the following conditions are satisfied.

C1. There are sequences $(\alpha(i))_{i=1}^{\infty} \in R_\alpha$ and $(\beta(i))_{i=1}^{\infty} \in R_\beta$ with $\alpha, \beta \geq 0$, such that, as $n \to \infty$ for each $s \in \mathbb{R}_+$,

$$\max_{1\leq k\leq ns} |\alpha(n,k) - \alpha(k)| = o(\alpha(n)),\ \max_{1\leq k\leq ns} |\beta(n,k) - \beta(k)| = o(\beta(n)). \quad (3.3)$$

C2. $a(n) = 1 + n^{-1}c_0 + o(n^{-1})$ as $n \to \infty$ for some $c_0 \in \mathbb{R}$.
C3. $b(n) = o(\alpha(n))$ as $n \to \infty$.

Remark 3.1. The condition C2 is the same is in the study of the array of time-homogeneous processes. C3 includes the cases, where the offspring variance tends to infinity or to zero. The first part of condition C1, related to the immigration mean, is satisfied when $\alpha(n) \to \infty$, if just $lim_{n\to\infty} \max_{1\leq k\leq ns} |\alpha(n,k) - \alpha(k)| < \infty$. In general, C1 is satisfied, for example, if there are $\varepsilon_i(n) \to 0$ as $n \to \infty$, $i = 1, 2$, such that $\alpha(n,k) = \alpha(k)(1 + \varepsilon_1(n))$ and $\beta(n,k) = \beta(k)(1 + \varepsilon_2(n))$.

We obtain three different approximations for $\mathcal{Y}_n(t)$, depending on which of the following conditions holds.

C4. $\alpha(n) \to \infty$, $\beta(n) = o(n\alpha(n)b(n))$ as $n \to \infty$ and $\liminf_{n\to\infty} b(n) > 0$.
C5. $\alpha(n) \to \infty$ and $n\alpha(n)b(n) = o(\beta(n))$ as $n \to \infty$.
C6. $\alpha(n) \to \infty$ and $\beta(n) \sim cn\alpha(n)b(n)$ as $n \to \infty$, where $c \in (0, \infty)$.

The following functions appear in the time change of approximating processes:

$$\mu_\alpha(t) = \int_0^t u^\alpha e^{(t-u)c_0} du,\ \nu_\alpha(t) = \int_0^t u^\alpha e^{(t-u)c_0}(1 - e^{(t-u)c_0}) du. \quad (3.4)$$

In particular, it is useful to note that $\mu_\alpha(t) = t^{\alpha+1}/(\alpha + 1)$ when $c_0 = 0$, and $\lim_{c_0\to 0} \nu_\alpha(t)/c_0 = t^{\alpha+2}/(\alpha + 1)(\alpha + 2)$.

We denote $\delta_n^{(1)}(\varepsilon) = E[(X_{ki}^{(n)} - a(n))^2\chi(|X_{ki}^{(n)} - a(n)| > \varepsilon B_n(n))]$, where $\chi(A)$ stands for the indicator of event A. We denote

$$\varphi(t) = \begin{cases} \frac{c_0}{\nu_\alpha(1)} \int_0^t \mu_\alpha(u)du, & \text{if } c_0 \neq 0, \\ \\ t^{2+\alpha}, & \text{if } c_0 = 0. \end{cases}$$

Theorem 3.1. *If conditions C1-C4 are satisfied and* $\delta_n^{(1)}(\varepsilon) \to 0$ *as* $n \to \infty$, *then* $\mathcal{Y}_n \xrightarrow{D} \mathcal{Y}^{(1)}$ *as* $n \to \infty$ *weakly in Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$, *where* $\mathcal{Y}^{(1)}(t) = W(\varphi(t))$ *and* $(W(t),\ t \in \mathbb{R}_+)$ *is a standard Brownian motion.*

Remark 3.2. We note that the Lindeberg-type condition for the family $\{X_{ki}^{(n)}, k, i \geq 1\}$ is needed even in time-homogenous models (see [11], [17]). If $E(X_{ki}^{(n)})^{2+l} < \infty$ for all $n \in \mathbb{N}$ and some $l \in \mathbb{R}_+$, then

$$\delta_n^{(1)}(\varepsilon) \leq \frac{1}{\varepsilon^l B_n^l(n)} E|X_{ki}^{(n)} - a(n)|^{2+l}.$$

Since when C4 is satisfied, $B_n^2(n) \sim Kn^2\alpha(n)b(n)$ as $n \to \infty$, where K is a positive constant, the Lindeberg-type condition is satisfied, for example, if $E|X_{ki}^{(n)} - a(n)|^3 = o(n\sqrt{\alpha(n)b(n)})$ as $n \to \infty$.

To state next theorem, we need the Lindeberg condition for the family of immigrating individuals $\{\xi_k^{(n)}, k \geq 1\}$, which is given by the following ratio

$$\delta_n^{(2)}(\varepsilon) = \frac{1}{\sigma_0^2(n)} \sum_{k=1}^{n} E[(\bar{\xi}_k^{(n)})^2\chi(|\bar{\xi}_k^{(n)}| > \varepsilon\sigma_0(n))],\ \sigma_0^2(n) = \sum_{k=1}^{n} \beta(n,k),$$

where $\bar{\xi}_k^{(n)} = \xi_k^{(n)} - \alpha(n,k)$.

Theorem 3.2. *If conditions C1-C3 and C5 are satisfied and* $\delta_n^{(2)}(\varepsilon) \to 0$ *as* $n \to \infty$, *then* $\mathcal{Y}_n \xrightarrow{D} \mathcal{Y}^{(2)}$ *as* $n \to \infty$ *weakly in Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$, *where* $\mathcal{Y}^{(2)}(t) = W(\psi(t))$,

$$\psi(t) = \frac{t^{1+\beta}}{(1+\beta)\nabla_\beta(1)}.$$

and $\nabla_\beta(t)$ *is defined right before Theorem 2.2.*

Theorem 3.3. *Let conditions C1-C3 and C6 be satisfied. If* $\delta_n^{(i)}(\varepsilon) \to 0$ *as* $n \to \infty$ *for each* $\varepsilon > 0$, $i = 1, 2$, *and* $\liminf_{n\to\infty} b(n) > 0$, *then* $\mathcal{Y}_n \xrightarrow{D} \mathcal{Y}^{(3)}$ *as* $n \to \infty$ *weakly in Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$, *where* $\mathcal{Y}^{(3)}(t) = W(\omega(t))$ *and*

$$\omega(t) = \frac{1}{K(c_0, c)} \int_0^t (\mu_\alpha(u) + cu^\beta)du,$$

$K(c_0, c) = \nu_\alpha(1)/c_0 + c\nabla_\beta(1)$ *for* $c_0 \neq 0$ *and* $K(0, c) = 1/(1+\alpha)(2+\alpha) + c\mu_{\beta(1)}$.

Remark 3.3. It is not difficult to see that for $c_0 = 0$, the limiting process has a simpler time change, namely

$$\omega(t) = (\frac{1}{(1+\alpha)(2+\alpha)} + \frac{c}{1+\beta})^{-1}[\frac{t^{2+\alpha}}{(1+\alpha)(2+\alpha)} + c\frac{t^{1+\beta}}{1+\beta}].$$

Furthermore, if in addition $b(n) \to b$ as $n \to \infty$ with $b \in (0, \infty)$, then $\omega(t) = t^{\alpha+2} = t^{\beta+1}$, i.e. the same time change as in the functional limit theorem for the critical process [14].

The proofs of Theorems 3.1-3.3 are provided in the Appendix. We also need the following theorem on a deterministic approximation of the process $\mathcal{Z}_n(t)$, which is proved in [16].

Theorem 3.4. *Let conditions C1-C3 be satisfied. If* $\alpha(n) \to \infty$ *and* $\beta(n) = o(n\alpha^2(n))$ *as* $n \to \infty$, *then* $\mathcal{Z}_n \xrightarrow{D} \pi_\alpha$ *as* $n \to \infty$ *weakly in Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$, *where* $\pi_\alpha(t) = \mu_\alpha(t)/\mu_\alpha(1)$, $t \in \mathbb{R}_+$.

We now derive two important results from Theorems 3.1-3.5, which are related to the CLSE of the offspring mean in the nearly critical sequence of branching processes. By a standard technique we obtain that the CLSE of the offspring mean $a(n)$ of nth process in (3.1) is

$$\hat{A}_n = \frac{\sum_{k=1}^n (Z^{(n)}(k) - \alpha(n, k))}{\sum_{k=1}^n Z^{(n)}(k-1)}. \tag{3.5}$$

We denote

$$W_n^{(1)} = \frac{nA_n(n)}{B_n(n)}(\hat{A}_n - a(n)),\ W_n^{(2)} = n(\hat{A}_n - a(n)). \tag{3.6}$$

Theorem 3.5. *Let conditions C1-C3 be satisfied and* $\alpha(n) \to \infty$, $\beta(n) = o(n\alpha^2(n))$ *as* $n \to \infty$.

a) If C4 holds and $\delta_n^{(1)}(\varepsilon) \to 0$ *as* $n \to \infty$ *for any* $\varepsilon > 0$, *then*

$$W_n^{(1)} \xrightarrow{d} \frac{W(\varphi(1))}{\int_0^1 \pi_\alpha(u)du}.$$

b) If C5 holds and $\delta_n^{(2)}(\varepsilon) \to 0$ *as* $n \to \infty$ *for any* $\varepsilon > 0$, *then*

$$W_n^{(1)} \xrightarrow{d} \frac{W(\psi(1))}{\int_0^1 \pi_\alpha(u)du}.$$

c) If C6 holds, $\delta_n^{(i)}(\varepsilon) \to 0$, $i = 1, 2$, *as* $n \to \infty$ *for any* $\varepsilon > 0$ *and* $\liminf_{n\to\infty} b(n) > 0$, *then*

$$W_n^{(1)} \xrightarrow{d} \frac{W(\omega(1))}{\int_0^1 \pi_\alpha(u)du}.$$

Proof. We consider the following equality:

$$\hat{A}_n - a(n) = \frac{\sum_{k=1}^n M^{(n)}(k)}{\sum_{k=1}^n Z^{(n)}(k-1)} =: \frac{D(n)}{Q(n)}. \tag{3.7}$$

First we obtain the asymptotic behavior of $Q(n)$. We use the following representation

$$\frac{1}{nA_n(n)} \sum_{k=0}^n Z^{(n)}(k) = \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \mathcal{Z}_n(t)dt. \tag{3.8}$$

Now we consider functionals $\Psi_n : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}$, defined for any $x \in D(\mathbb{R}_+, \mathbb{R})$ and $n \geq 1$ as

$$\Psi_n(x) = \sum_{k=1}^n \int_{(k-1)/n}^{k/n} x(t)dt. \tag{3.9}$$

It is obvious that for all $x, x_n \in D(\mathbb{R}_+, \mathbb{R})$ such that $\|x_n - x\|_\infty \to 0$, we have $|\Psi_n(x_n) - \Psi(x)| \to 0$ as $n \to \infty$, where

$$\Psi(x) = \int_0^1 x(t)dt.$$

Due to Theorem 3.4 we have $\mathcal{Z}_n \xrightarrow{D} \pi_\alpha$ as $n \to \infty$ weakly in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$. Since $\pi_\alpha(t)$ is continuous, this implies that $\|\mathcal{Z}_n - \pi_\alpha\|_\infty \to 0$ as $n \to \infty$. Therefore, due to the extended continuous mapping theorem ([4], Theorem 5.5), we have $\Psi_n(\mathcal{Z}_n) \xrightarrow{d} \Psi(\pi_\alpha)$ as $n \to \infty$, where $\Psi(\pi_\alpha) = \int_0^1 \pi_\alpha(t)dt$. Thus, when conditions C1-C3 are satisfied and $\alpha(n) \to \infty$, $\beta(n) = o(n\alpha^2(n))$ as $n \to \infty$, we have

$$\frac{Q(n)}{nA_n(n)} \xrightarrow{P} \int_0^1 \pi_\alpha(u)du. \tag{3.10}$$

Now we consider $D(n)$. It follows from Theorem 3.1 that when conditions C1-C4 are satisfied and $\delta_n^{(1)}(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ as $n \to \infty$

$$\frac{D(n)}{B_n(n)} \xrightarrow{d} W(\varphi(1)). \tag{3.11}$$

The proof of Part (a) of the theorem follows from (3.7), (3.10) and (3.11) due to Slutsky's theorem.

Using Theorems 3.2 and 3.3, by similar arguments we obtain parts (b) and (c) of Theorem 3.5.

The following consequence of the above theorem is vital in the proof of the main results.

Corollary 3.1. *If conditions of Theorem 3.5 are satisfied and $c_0 = 0$, then $W_n^{(1)} \xrightarrow{d} (2+\alpha)\mathcal{N}(0,1)$ as $n \to \infty$ in all cases (a), (b) and (c).*

The proof of the corollary is straightforward and uses the fact that $\mu_\alpha(u) = u^{1+\alpha}/(1+\alpha)$ when $c_0 = 0$.

The second approximation, related to the pivot $W_n^{(2)}$, holds under the assumption that there exists $d \in [0, \infty)$ such that the condition (2.9) holds.

Theorem 3.6. *Let conditions C1, C2 and (2.9) be satisfied. If $\alpha(n) \to \infty$, $b(n) \to b \in (0, \infty)$ and $\delta_n^{(2)}(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$, then*

$$W_n^{(2)} \xrightarrow{d} \nu(c_0), \tag{3.12}$$

where $\nu(c_0)$ is defined in (2.11).

Proof. We consider relation (3.7) in the proof of Theorem 3.5. It is obvious that conditions C3 and C5 of Theorem 3.2 are satisfied when (2.9) holds and $b(n) \to b \in (0, \infty)$ as $n \to \infty$. Therefore, taking into account that $D(n)/B(n) = \mathcal{Y}_n(1)$, we obtain that as $n \to \infty$

$$\frac{D(n)}{B_n(n)} \stackrel{d}{\Longrightarrow} W(\psi(1)). \tag{3.13}$$

Now we evaluate $Q(n)$. When the condition $\beta(n) = o(n\alpha^2(n))$ as $n \to \infty$ is not valid we can not use the deterministic approximation for $\mathcal{Z}_n(t)$ given in Theorem 3.4. Therefore, first we express the denominator in terms of the process

$$Y_n(t) = \frac{Z^{(n)}([nt]) - A^{(n)}([nt])}{B_n(n)}.$$

Namely, we consider the following equality

$$\frac{Q(n)}{nB_n(n)} = S_1(n) + S_2(n), \tag{3.14}$$

where

$$S_1(n) = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} Y_n(t)dt, \; S_2(n) = \frac{1}{nB_n(n)} \sum_{k=1}^{n} A_n(k-1).$$

To evaluate $S_2(n)$, we use Lemma 2 and Part (c) of Lemma 3, which are proved in the Appendix. Since due to condition (2.9) $B_n^2(n) \sim n\beta(n)\nabla_\beta(1)$ as $n \to \infty$, we easily obtain that

$$\lim_{n\to\infty} S_2(n) = \gamma_0, \tag{3.15}$$

where $\gamma_0 = \gamma_0(c_0)$ is defined just before Theorem 2.2. In order to use the continuous mapping theorem, we need to express $S_1(n)$ in terms of the process $\mathcal{Y}_n(t)$. We obtain using (3.1) that $Z^{(n)}(k) - EZ^{(n)}(k) = M^{(n)}(k) + a(n)(Z^{(n)}(k-1) - EZ^{(n)}(k-1))$ for any $n, k \geq 1$. Thus

$$Z^{(n)}(k) - EZ^{(n)}(k) = \sum_{j=1}^{k} a^{k-j}(n) M^{(n)}(j).$$

From the last equality we easily obtain

$$Y_n(t) = \mathcal{Y}_n(t) + \frac{1}{B_n(n)} \sum_{j=1}^{[nt]} (a^{[nt]-j}(n) - 1) M^{(n)}(j). \tag{3.16}$$

Rearranging the sum on the right side of (3.16), we have

$$\frac{1}{B_n(n)} \sum_{j=1}^{[nt]} (a^{[nt]-j}(n) - 1) M^{(n)}(j) = \frac{a(n)-1}{B_n(n)} \sum_{j=1}^{[nt]} \sum_{i=j+1}^{[nt]} a^{[nt]-i}(n) M^{(n)}(j) =$$

$$= (a(n) - 1) \sum_{i=2}^{[nt]} a^{[nt]-i}(n) \mathcal{Y}_n(\frac{i-1}{n}).$$

Hence, we obtain the following representation

$$S_1(n) = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \left(\mathcal{Y}_n(t) + c_0(n) \frac{1}{n} \sum_{i=2}^{[nt]} a^{[nt]-i}(n) \mathcal{Y}_n(\frac{i-1}{n}) \right) dt, \tag{3.17}$$

where $c_0(n) = n(a(n) - 1)$. Now we consider sequence of functionals $\Phi_n : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}, n \geq 1$, which are defined for any $x \in D(\mathbb{R}_+, \mathbb{R})$ as

$$\Phi_n(x) = \frac{x(1)}{\Omega_n(x) + \gamma_0} \ ,$$

where functionals $\Omega_n : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}, n \geq 1$ are defined by

$$\Omega_n(x) = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \left(x(t) + c_0(n) \frac{1}{n} \sum_{i=2}^{[nt]} a^{[nt]-i}(n) x(\frac{i-1}{n}) \right) dt. \tag{3.18}$$

It is not difficult to see that for any sequence $x_n \in D(\mathbb{R}_+, \mathbb{R}), n \geq 1$, such that $\|x_n - x\|_\infty \to 0$ as $n \to \infty$ with $x \in D(\mathbb{R}_+, \mathbb{R})$, we have $\Phi_n(x_n) \to \Phi(x)$ as $n \to \infty$, where

$$\Phi(x) = x(1) \left(\int_0^1 x(t)dt + c_0 \int_0^1 \int_0^t e^{(t-u)c_0} x(u) du dt + \gamma_0 \right)^{-1}. \tag{3.19}$$

It follows from Theorem 3.2 that $\mathcal{Y}_n \xrightarrow{D} \mathcal{Y}^{(2)}$ as $n \to \infty$ weakly in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$, where $\mathcal{Y}^{(2)}(t) = W(\psi(t))$. Since $\mathcal{Y}^{(2)}(t)$ is continuous, this implies that $\|\mathcal{Y}_n - \mathcal{Y}^{(2)}\|_\infty \to 0$ as $n \to \infty$. Appealing again to the extended continuous mapping theorem ([4], Theorem 5.5), we conclude that $\Phi_n(\mathcal{Y}_n) \xrightarrow{d} \Phi(\mathcal{Y}^{(2)})$ as $n \to \infty$. To obtain the proof of Theorem 3.6, we rewrite (3.7) as

$$n(\hat{A}_n - a(n)) = \frac{\Phi_n(\mathcal{Y}_n)}{1 + \Upsilon_n(\mathcal{Y}_n)},$$

where $\Upsilon_n(\mathcal{Y}_n) = (S_2(n) - \gamma_0)/(\Omega_n(\mathcal{Y}_n) + \gamma_0)$. Since $\Upsilon_n(\mathcal{Y}_n) \xrightarrow{P} 0$ as $n \to \infty$, the assertion of Theorem 3.6 now follows from Slutsky's theorem.

Corollary 3.2. *If conditions of Theorem 3.6 are satisfied and $c_0 = 0$, then*

$$W_n^{(2)} \xrightarrow{d} \frac{W(1)}{\int_0^1 W(t^{1+\beta})dt + \gamma} \tag{3.20}$$

as $n \to \infty$, where $\gamma = ((\alpha+1)(\alpha+2))^{-1}\sqrt{d(1+\beta)}$.

The proof of the corollary is straightforward.

4 Proofs of the main results

Proof of Theorem 2.1. It is easy to see that given the sample $\mathcal{X}_n = \{Z(k), k = 1, ..., n\}$, the bootstrap process $\{Z^{*(n)}(k), k \geq 0\}$ is the sequence of branching processes defined in (3.1), where

$$a(n) = \hat{a}_n,\ b(n) = Var(X_{ki}^{*(n)}),$$

and $\alpha(n,k) = \alpha(k)$ and $\beta(n,k) = \beta(k)$ are the mean and variance of the probability distribution $\{q_j(k),\ j = 0, 1, ...\}$. Therefore, if conditions of Theorem 3.5 are satisfied by the bootstrap process (in probability) and $c_0 = 0$, then

$$\sup_x |H_n(\hat{a}_n, x) - \Phi(2+\alpha, x)| \xrightarrow{P} 0, \tag{4.1}$$

where $H_n(\hat{a}_n, x) = P\{V_n^* \leq x | \mathcal{X}_n\}$. Thus, we need to show that the conditions of Theorem 3.5 are satisfied. Conditions C1 are fulfilled trivially. It follows from (2.4) that $n(\hat{a}_n - 1) \xrightarrow{P} 0$ as $n \to \infty$, i.e. the condition C2 is also satisfied in probability with $c_0 = 0$. Since $\hat{a}_n \xrightarrow{P} 1$ as $n \to \infty$,

$\hat{\theta}_n = f^{-1}(\hat{a}_n) \xrightarrow{P} f^{-1}(1) = \theta$. Since $b(n) = \varphi_1(\theta)$, where φ_1 is a continuous function, we have $Var(X_{ki}^{*(n)}|\mathcal{X}_n) = \varphi_1(\hat{\theta}_n) \xrightarrow{P} \varphi_1(\theta) = b$ as $n \to \infty$, i.e. condition C3 is satisfied.

It follows from Remark 3.2 that if for some $l > 0$

$$(1/B_n^l(n))E[|X_{ki}^{*(n)} - \hat{a}_n|^{2+l}|\mathcal{X}_n] \xrightarrow{P} 0 \tag{4.2}$$

as $n \to \infty$, then

$$\delta_n^{*(1)}(\varepsilon) =: \frac{1}{B_n^2(n)} E[(X_{ki}^{*(n)} - \hat{a}_n)^2 \chi(|X_{ki}^{*(n)} - \hat{a}_n| > \varepsilon B_n(n))|\mathcal{X}_n] \xrightarrow{P} 0$$

as $n \to \infty$ for each $\varepsilon > 0$. We obtain from condition A1 that $E[(X_{ki}^{*(n)})^{2+l}|\mathcal{X}_n] \xrightarrow{P} E[(X_{ki})^{2+l}]$ as $n \to \infty$, which implies (4.2). Thus, all conditions of Theorem 3.5 are satisfied and $c_0 = 0$. It follows from Lemmas 1 and 2 in the Appendix that $A_n(n) \sim A(n)$ and $B_n^2(n) \sim B^2(n)$ as $n \to \infty$ when $c_0 = 0$ and the assertion of (2.8) follows from Corollary 3.1. Theorem 2.1 is proved.

Proof of Theorem 2.2. As in [17], we use quite standard technique based on Skorokhod's theorem (see [5], Theorem 29.6). We have from (2.10) that $n(\hat{a}_n - 1) \xrightarrow{d} W_0$ as $n \to \infty$. Therefore, due to the Skorokhod's theorem there exists a sequence $\{\hat{a}'_n, n \geq 1\}$ of random variables and a random variable W'_0 on a common probability space $(\Omega', \mathcal{F}, Q)$ such that $\hat{a}'_n \stackrel{d}{=} \hat{a}_n$ for all $n \geq 1$, $W'_0 \stackrel{d}{=} W_0$ and $n(\hat{a}'_n(\omega') - 1) \to W'_0$ as $n \to \infty$ for each $\omega' \in \Omega'$.

For any $\omega' \in \Omega'$ we estimate unknown θ by $\hat{\theta}'_n(\omega') = f^{-1}(\hat{a}'_n(\omega'))$. Then we obtain the bootstrap distribution $\{p_j(\hat{\theta}'_n), j \geq 0\}$ substituting θ by $\hat{\theta}'_n(\omega')$. Let now $\{X_{ki}^{\prime(n)}, k, i \geq 1\}$ be a family of i.i.d. random variables such that

$$P\{X_{ki}^{\prime(n)} = j\} = p_j(\hat{\theta}'_n)$$

for each $\omega' \in \Omega'$ and $n \geq 1$ and $\{\xi_k, k \geq 1\}$ be a sequence of random variables with the probability distributions $\{q_j(k), j \geq 0\}$. A new bootstrap sample $\mathcal{X}'_n = \{Z'^{(n)}(k), k = 1, ..., n\}$ will be obtained recursively from the relation

$$Z'^{(n)}(k) = \sum_{i=1}^{Z'^{(n)}(k-1)} X_{ki}^{\prime(n)} + \xi_k, \quad k = 1, 2, ... \tag{4.3}$$

for each $\omega' \in \Omega'$, $n \geq 1$ with $Z'^{(n)}(0) = 0$. We define new pivot $W'_n = n(\tilde{a}_n - \hat{a}'_n(\omega'))$ for each $\omega' \in \Omega'$, where

$$\tilde{a}_n = \frac{\sum_{k=1}^{n}(Z'^{(n)}(k) - \alpha(k))}{\sum_{k=1}^{n} Z'^{(n)}(k-1)}. \tag{4.4}$$

If we denote $F_n(\theta, x) = P\{W_n \leq x\}$, we realize that

$$P\{W_n^* \leq x|\hat{a}_n\} = F_n(\hat{\theta}_n, x),\ P\{W'_n \leq x|\hat{a}'_n\} = F_n(\hat{\theta}'_n, x),$$

where $\hat{\theta}_n = f^{-1}(\hat{a}_n)$ and $\hat{\theta}'_n = f^{-1}(\hat{a}'_n)$. For each $\omega' \in \Omega'$ we apply Theorem 3.6 to W'_n. Under the assumptions of Theorem 2.2 conditions C1 and $\delta_n^{(2)}(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ are trivially satisfied. Condition C2 is also fulfilled for each $\omega' \in \Omega'$ with $c_0 = W'_0$. Due to our assumptions on the moments of the offspring distribution we can write $b(n) = \varphi_1(\theta)$, where φ_1 is a continuous function, as in the proof of Theorem 2.1. Therefore, $Var(X_{ki}'^{(n)}|\hat{\theta}'_n) = \varphi_1(\hat{\theta}'_n) \to \varphi_1(\theta) = b$ as $n \to \infty$ for each $\omega' \in \Omega'$. Thus we obtain from Theorem 3.6 that $F_n(\hat{\theta}'_n, x) \to F(W'_0, x)$ as $n \to \infty$ for each $\omega' \in \Omega'$ and $x \in \mathbb{R}$. The assertion of Theorem 2.1 now follows from this due to $F_n(\hat{\theta}_n, x) \stackrel{d}{=} F_n(\hat{\theta}'_n, x)$ and $F(W_0, x) \stackrel{d}{=} F(W'_0, x)$. The theorem is proved.

5 Conclusions

According to Theorem 2.2, when $n\alpha^2(n) = o(\beta(n))$ as $n \to \infty$ the bootstrap version of CLSE is invalid. The cause of the failure is the same as in the case of stationary immigration [17], namely, in this case the estimator $\hat{a}_n$ does not have the desired rate of convergence to $a = 1$. As in [6], one may consider a modified version of the standard bootstrap procedure. The idea behind the modification is using in the initial estimator of the offspring mean an adaptive shrinkage towards $a = 1$.

If a sample of pairs $\{(Z(k), \xi_k), k = 1, ..., n\}$ is available, then a natural estimator of the offspring mean is

$$\tilde{a_n} = \frac{\sum_{k=1}^{n}(Z(k) - \xi_k)}{\sum_{k=1}^{n} Z(k-1)}.$$

The following questions related to this estimator is of interest. How much improvement in a sense of the rate of convergence we will get because of additional observations of the number of immigrating individuals? Will the standard parametric bootstrap procedure be valid for $\tilde{a_n}$ in the case of large immigration variance? Since

$$\tilde{a_n} - a = \frac{\sum_{k=1}^{n} \sum_{j=1}^{Z(k-1)} (X_{kj} - a)}{\sum_{k=1}^{n} Z(k-1)},$$

one can easily derive asymptotic distributions for the pivot, corresponding to $\tilde{a_n}$ from a martingale central limit theorem. By the arguments as in the proof of Proposition 4.1 in [15], it is possible to prove that $\tilde{a_n}$ is a strongly consistent estimator of a.

The estimation problems and a justification of the validity of the bootstrap for subcritical and supercritical processes with non-stationary immigration are also open. In order to derive the asymptotic distributions for an estimator of the offspring mean, one needs to establish functional limit theorems in these cases. Further, as in the classical models, one may obtain results for the estimator without any assumption of the criticality.

6 Appendix

In the proofs we use several preliminary lemmas. We start with a simple but useful result related to regularly varying sequences.

Lemma 1. *If $(C(n))_{n=1}^{\infty} \in R_\rho$ and $a(n)$ satisfies condition C2, then for any $\rho \in [0, \infty)$ and $\theta \in \mathbb{R}$*

$$\frac{1}{nC(n)} \sum_{k=1}^{[ns]} a^{k\theta}(n) C(k) \to \int_0^s t^\rho e^{t\theta c_0} dt \tag{6.1}$$

as $n \to \infty$ uniformly in $s \in [0, T]$ for each fixed $T > 0$.

Proof. It follows from condition C2 that

$$a^{ns}(n) \to e^{c_0 s} \tag{6.2}$$

as $n \to \infty$ uniformly in $s \in [0, T]$. Since $(C(n))_{n=1}^{\infty} \in R_\rho$, (6.2) implies that

$$\frac{1}{n} \sum_{k=1}^{[ns]} \frac{C(k)}{C(n)} a^{k\theta}(n) - \frac{1}{n} \sum_{k=1}^{[ns]} (\frac{k}{n})^\rho e^{k/n\theta c_0} \to 0$$

as $n \to \infty$ uniformly in $s \in [0, T]$. Now the second sum tends as $n \to \infty$ to the integral

$$\int_0^s t^\rho e^{t\theta c_0} dt,$$

which is a continuous function of s. Hence, the convergence is uniform in $s \in [0, T]$. The Lemma is proved.

The next result is related to the asymptotic behavior of the mean and the variance of the process.

Lemma 2. *If conditions C1 and C2 are satisfied, then uniformly in $s \in [0, T]$ for each fixed $T > 0$*

$$a) \quad \lim_{n\to\infty} \frac{A_n([ns])}{n\alpha(n)} = \mu_\alpha(s), \; \lim_{n\to\infty} \frac{\sigma_n^2([ns])}{n\beta(n)} = \nabla_\beta(s),$$

$$b) \quad \lim_{n\to\infty} \frac{\Delta_n^2([ns])}{n^2\alpha(n)b(n)} = \begin{cases} (1/c_0)\nu_\alpha(s), & \text{if } \; c_0 \neq 0, \\ s^{\alpha+2}/(\alpha+1)(\alpha+2), & \text{if } \; c_0 = 0. \end{cases}$$

Proof. To prove the first relation in Part (a), we consider

$$A_n([ns]) = \sum_{k=1}^{[ns]} \alpha(k) a^{[ns]-k}(n) + \sum_{k=1}^{[ns]} (\alpha(n,k) - \alpha(k)) a^{[ns]-k}(n). \tag{6.3}$$

Applying Lemma 1, we easily obtain that the first term on the right side of (6.3), divided by $n\alpha(n)$, as $n \to \infty$ tends to $\mu_\alpha(s)$ uniformly in $s \in [0, T]$. The second term is dominated by

$$\max_{1\le k\le nT} |\alpha(n,k) - \alpha(k)| \sum_{k=1}^{[ns]} a^{[ns]-k}(n).$$

Due to Lemma 1, the sum in this expression, divided by n, tends to $\int_0^s e^{(1-u)c_0} du$ as $n \to \infty$ uniformly in $s \in [0, T]$. Therefore, taking into account C1, we obtain the assertion.

The proofs of the remaining claims are similar and, therefore, are omitted.

Lemma 3. *If conditions C1 and C2 are satisfied, then for any $\theta \in \mathbb{R}, \; s \in \mathbb{R}_+$*

$$a) \; \lim_{n\to\infty} \frac{1}{n^3\alpha(n)b(n)} \sum_{i=1}^{[ns]} a^{\theta i}(n) \Delta_n^2(i) = \begin{cases} (1/c_0) \int_0^s e^{u\theta c_0} \nu_\alpha(u) du, & \text{if } \; c_0 \neq 0, \\ s^{\alpha+3}/(\alpha+1)(\alpha+2)(\alpha+3), & \text{if } \; c_0 = 0, \end{cases}$$

$$b)\ \lim_{n\to\infty}\frac{1}{n^2\beta(n)}\sum_{i=1}^{[ns]}a^{\theta i}(n)\sigma_n^2(i)=\int_0^s e^{u\theta c_0}\nabla_\beta(u)du,$$

$$c)\ \lim_{n\to\infty}\frac{1}{n^2\alpha(n)}\sum_{i=1}^{[ns]}a^{\theta i}(n)A_n(i)=\int_0^s e^{u\theta c_0}\mu_\alpha(u)du,$$

$$d)\ \lim_{n\to\infty}\frac{1}{n^2\alpha(n)\beta(n)}\sum_{i=1}^{[ns]}a^{\theta i}(n)\beta(i)A_n(i)=\int_0^s e^{u\theta c_0}\mu_\alpha(u)u^\beta du.$$

Proof. We prove Part (a). Let $c_0\neq 0$. In this case due to Part (b) of Lemma 2 we have

$$\frac{1}{n}\sum_{i=1}^{[ns]}a^{\theta i}(n)\frac{\Delta_n^2(i)}{n^2\alpha(n)b(n)}-\frac{1}{nc_0}\sum_{i=1}^{[ns]}e^{(i/n)\theta c_0}\nu_\alpha(\frac{i}{n})\to 0$$

as $n\to\infty$. Since $e^{u\theta c_0}\nu_\alpha(u)$ is bounded in $u\in[0,s]$ for each fixed s,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{[ns]}e^{(i/n)\theta c_0}\nu_\alpha(\frac{i}{n})=\int_0^s e^{u\theta c_0}\nu_\alpha(u)du. \tag{6.4}$$

In the case $c_0=0$ we have $\int_0^s(u^{\alpha+2}/(\alpha+1)(\alpha+2))du$ on the right side of (6.4). The proofs of Parts (b), (c) and (d) are similar. We just use the second relation of Part (a) and Part (b) of the Lemma 2. Lemma 3 is proved.

Lemma 4. *Let $X_{ki}^{(n)}$, $k,j,n\geq 1$, be random variables from (3.1), $\bar{X}_{ki}^{(n)}=X_{ki}^{(n)}-a(n)$ and $T_n(k)=\sum_{i=1}^{Z^{(n)}(k-1)}\bar{X}_{ki}^{(n)}$. Then*
a)

$$E[(T_n(k))^2|\Im^{(n)}(k-1)]=b(n)Z^{(n)}(k-1), \tag{6.5}$$

b)

$$E[(\Sigma'\bar{X}_{ki}^{(n)}\bar{X}_{kj}^{(n)})^2|\Im^{(n)}(k-1)]=2b^2(n)Z^{(n)}(k-1)(Z^{(n)}(k-1)-1), \tag{6.6}$$

where $\sum'$ denotes summation for $i,j=1,2,...,Z^{(n)}(k-1)$ such that $i\neq j$.

Proof. The proof of identities (6.5) and (6.6) follows directly from independence of random variables $X_{ki}^{(n)}$ and $X_{kj}^{(n)}$, $j\neq i$, and properties of the conditional expectations.

Lemma 5. *For the sum* $T_n(k)$ *defined in Lemma 4 and any* $\theta > 0$

$$E[(T_n(k))^2\chi(|T_n(k)| > \theta)|\Im^{(n)}(k-1)] \le I_1 + I_2, \tag{6.7}$$

where

$$I_1 = Z^{(n)}(k-1)E[(\bar{X}_{ki}^{(n)})^2\chi(|\bar{X}_{ki}^{(n)}| > \theta/2)],$$
$$I_2 = \frac{4b^2(n)}{\theta^2}(Z^{(n)}(k-1))^2 + \frac{\sqrt{2b^3(n)}}{\theta}(Z^{(n)}(k-1))^{3/2}.$$

Proof. Using simple inequality

$$\chi(|\xi+\eta| > \varepsilon) \le \chi(|\xi| > \varepsilon/2) + \chi(|\eta| > \varepsilon/2), \tag{6.8}$$

where $\chi(A)$ is the indicator of event A, the left side of (6.7) can be estimated by $R_1 + R_2 + R_3$, with

$$R_1 = E[\sum_{i=1}^{Z^{(n)}(k-1)} (\bar{X}_{ki}^{(n)})^2\chi(|\bar{X}_{ki}^{(n)}| > \theta/2)|\Im^{(n)}(k-1)],$$

$$R_2 = E[\sum_{i=1}^{Z^{(n)}(k-1)} (\bar{X}_{ki}^{(n)})^2\chi(|\tau_{ki}^{(n)}| > \theta/2)|\Im^{(n)}(k-1)],$$

$$R_3 = E[\Sigma'\bar{X}_{ki}^{(n)}\bar{X}_{kj}^{(n)}\chi(|T_n(k)| > \theta)|\Im^{(n)}(k-1)],$$

where Σ' is the same as in Lemma 4 and $\tau_{ki}^{(n)} = T_n(k) - \bar{X}_{ki}^{(n)}$. It is obvious that $R_1 = Z^{(n)}(k-1)E[(\bar{X}_{11}^{(n)})^2\chi(|\bar{X}_{11}^{(n)}| > \theta/2)]$. To estimate R_2 we use independence of $\bar{X}_{ki}^{(n)}$ and $\tau_{ki}^{(n)}$, Chebishev inequality and relation (6.5). As the result we obtain

$$R_2 \le \frac{4b(n)}{\theta^2}\sum_{i=1}^{Z^{(n)}(k-1)} E[(\tau_{ki}^{(n)})^2|\Im^{(n)}(k-1)] = \frac{4b^2(n)}{\theta^2}Z^{(n)}(k-1)(Z^{(n)}(k-1)-1).$$

Using Cauchy-Schwarz and Chebishev inequalities and relations (6.5) and (6.6), we see that R_3 is dominated by

$$\frac{1}{\theta}(E[(\Sigma'\bar{X}_{ki}^{(n)}\bar{X}_{kj}^{(n)})^2|\Im^{(n)}(k-1)]E[(T_n(k))^2|\Im^{(n)}(k-1)])^{1/2} =$$

$$= \frac{\sqrt{2b^3(n)}Z(k-1)}{\theta}\sqrt{Z(k-1)-1}.$$

Lemma is proved.

Proof of Theorem 3.1. In the proof we use the following version of the martingale central limit theorem from [12] (see [12], Theorem VIII, 3.33).

Theorem A. *Let $\{U_k^n, k \geq 1\}$ for each $n \geq 1$ be a sequence of martingale differences with respect to some filtration $\{\Im^{(n)}(k), k \geq 1\}$, such that the conditional Lindeberg condition*

$$\sum_{k=1}^{[nt]} E[(U_k^n)^2 \chi(|U_k^n| > \varepsilon)|\Im^{(n)}(k-1)] \xrightarrow{P} 0 \tag{6.9}$$

holds as $n \to \infty$ for all $\varepsilon > 0$ and $t \in \mathbb{R}_+$. Then

$$\sum_{k=1}^{[nt]} U_k^n \xrightarrow{D} U(t) \tag{6.10}$$

as $n \to \infty$ weakly, where $U(t)$ is a continuous Gaussian martingale with mean zero and covariance function $C(t), t \in \mathbb{R}_+$, if and only if

$$\sum_{k=1}^{[nt]} E[(U_k^n)^2|\Im^{(n)}(k-1)] \xrightarrow{P} C(t) \tag{6.11}$$

as $n \to \infty$ for each $t \in \mathbb{R}_+$.

We show that the conditions of Theorem A are fulfilled by the sum in the definition of $\mathcal{Y}_n(t)$. To show that (6.11) is satisfied, we use simple equality

$$E[(M^{(n)}(k))^2|\Im^{(n)}(k-1)] = b(n)Z^{(n)}(k-1) + \beta(n,k), \tag{6.12}$$

and obtain

$$\sum_{k=1}^{[nt]} E[(U_k^n)^2|\Im^{(n)}(k-1)] = \frac{b(n)}{B_n^2(n)}\sum_{k=1}^{[nt]} Z^{(n)}(k-1)+$$

$$+\frac{1}{B_n^2(n)}\sum_{k=1}^{[nt]}\beta(n,k) = \Gamma_1(n,t) + \Gamma_2(n,t). \tag{6.13}$$

Note that when condition C4 is fulfilled, $B_n^2(n) \sim \Delta_n^2(n)$ as $n \to \infty$. Therefore, using part (b) of Lemma 2 and Part (c) of Lemma 3, we obtain that the mean of $\Gamma_1(n,t)$ tends as $n \to \infty$ for each $t \in \mathbb{R}_+$ to $\varphi(t)$.

Now we show that the variance of $\Gamma_1(n,t)$ tends to zero as $n \to \infty$. Since

$$Var(\Gamma_1(n,t)) = \frac{b^2(n)}{B_n^4(n)} Var(\sum_{k=1}^{[nt]} Z^{(n)}(k-1)),$$

we can use equality

$$Var(\sum_{k=1}^{[nt]} Z^{(n)}(k-1)) = R_1(n,t) + 2R_2(n,t), \tag{6.14}$$

where

$$R_1(n,t) = \sum_{k=1}^{[nt]} B_n^2(k-1),\ R_2(n,t) = \sum_{i=1}^{[nt]-2} \sum_{j=i+1}^{[nt]-1} Cov(Z^{(n)}(i), Z^{(n)}(j)).$$

Let $c_0 \neq 0$. If we use Lemma 3, we obtain that as $n \to \infty$ for each $t \in \mathbb{R}_+$

$$R_1(n,t) \sim \frac{n^3\alpha(n)b(n)}{c_0} \int_0^t \nu_\alpha(u)du + n^2\beta(n) \int_0^t \nabla_\beta(u)du. \tag{6.15}$$

Consider $R_2(n,t)$. Using equality

$$Cov(Z^{(n)}(i),\ Z^{(n)}(i+m)) = a^m(n) Var Z^{(n)}(i), \tag{6.16}$$

we rewrite it as

$$R_2(n,t) = \sum_{i=1}^{[nt]-2} \sum_{j=i+1}^{[nt]-1} a^{j-i}(n) B_n^2(i) = \sum_{i=1}^{[nt]-2} a^{-i}(n) B_n^2(i) \sum_{j=i+1}^{[nt]-1} a^j(n).$$

We obtain due to Lemma 1 that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{[nt]} a^j(n) = \int_0^t e^{uc_0} du. \tag{6.17}$$

The relation (6.17) shows that for any $\varepsilon > 0$ and sufficiently large n

$$R_2(n,t) \leq n \left(\int_0^t e^{uc_0} du + \varepsilon \right) \sum_{i=1}^{[nt]} a^{-i}(n) B_n^2(i). \tag{6.18}$$

Applying Lemma 3, we derive that the sum on the right side of (6.18) as $n \to \infty$ is equivalent to

$$\frac{n^3\alpha(n)b(n)}{c_0}\int_0^t e^{-uc_0}\nu_\alpha(u)du + n^2\beta(n)\int_0^t \nabla_\beta(u)du. \tag{6.19}$$

Using relation (6.15) and part (b) of Lemma 2, we obtain that as $n \to \infty$

$$\frac{b^2(n)R_1(n,t)}{B_n^4(n)} \sim K_1\frac{n^3\alpha(n)b^3(n)}{n^4\alpha^2(n)b^2(n)} + K_2\frac{n^2\beta(n)b^2(n)}{n^4\alpha^2(n)b^2(n)}, \tag{6.20}$$

where $K_i,\ i = 1, 2, ..$ are various positive constants, depending on α, β, c_0 and t. Similarly, using estimate (6.18) and part (b) of Lemma 2 again, we obtain that for sufficiently large n the ratio $b^2(n)R_2(n,t)/B_n^4(n)$ is dominated by

$$K_3\frac{n^4\alpha(n)b^3(n)}{n^4\alpha^2(n)b^2(n)} + K_4\frac{n^3\beta(n)b^2(n)}{n^4\alpha^2(n)b^2(n)}. \tag{6.21}$$

Thus, we deduce from (6.14) that when conditions C3 and C4 are satisfied, the variance of $\Gamma_1(n,t)$ tends to zero as $n \to \infty$ for each $t \in \mathbb{R}_+$.

It is not difficult to see that estimates (6.20) and (6.21) remain true when $c_0 = 0$, with only changes in constants $K_i,\ i = 1, 2, 3, 4$. Therefore, we again obtain that the variance of the first term in (6.13) tends to zero. Hence we conclude that

$$\Gamma_1(n,t) \xrightarrow{P} \varphi(t) \tag{6.22}$$

as $n \to \infty$ for each $t \in \mathbb{R}_+$.

The second term in (6.13) we rewrite as

$$\Gamma_2(n,t) = \frac{1}{B_n^2(n)}\sum_{k=1}^{[nt]}\beta(k) + \frac{1}{B_n^2(n)}\sum_{k=1}^{[nt]}(\beta(n,k) - \beta(k)). \tag{6.23}$$

Since $n\beta(n) = o(\Delta_n^2(n))$ due to condition C4, using Lemma 1 we obtain that the first term in (6.23) tends to zero. The second term is dominated by

$$\frac{[nt]}{\Delta_n^2(n)}\max_{1\le k\le nt}|\beta(n,k) - \beta(k)|,$$

and due to conditions C1 and C4 tends to zero as $n \to \infty$. Consequently, $\Gamma_2(n,t) \to 0$ as $n \to \infty$ for each $t \in \mathbb{R}_+$. Thus, applying Slutsky's theorem, we conclude that

$$\sum_{k=1}^{[nt]} E[(U_k^n)^2|\Im^{(n)}(k-1)] \xrightarrow{P} \varphi(t) \tag{6.24}$$

as $n \to \infty$ for each $t \in \mathbb{R}_+$, i.e. condition (6.11) is satisfied.

Now we show that condition (6.9) is also satisfied. It will be satisfied if for any $\varepsilon > 0$ as $n \to \infty$

$$\frac{1}{B_n^2(n)} \sum_{k=1}^{[nt]} E[(M^{(n)}(k))^2 \chi(|M^{(n)}(k)| > \varepsilon B_n(n)) | \Im^{(n)}(k-1)] \xrightarrow{P} 0. \quad (6.25)$$

We obtain from the definition of the process that

$$M^{(n)}(k) = T_n(k) + \xi_k^{(n)} - \alpha(n,k), \quad (6.26)$$

where $T_n(k)$ is the sum defined in Lemma 4. If we denote the expression in (6.25) by $I(n,t)$, using (6.26) and independence of the reproduction and the immigration processes, it can be represented as

$$I(n,t) = I_1(n,t) + I_2(n,t), \quad (6.27)$$

where

$$I_1(n,t) = \frac{1}{B_n^2(n)} \sum_{k=1}^{[nt]} E[(T_n(k))^2 \chi(|M^{(n)}(k)| > \varepsilon B_n(n)) | \Im^{(n)}(k-1)],$$

$$I_2(n,t) = \frac{1}{B_n^2(n)} \sum_{k=1}^{[nt]} E[(\bar{\xi}_k^{(n)})^2 \chi(|M^{(n)}(k)| > \varepsilon B_n(n)) | \Im^{(n)}(k-1)],$$

and $\bar{\xi}_k^{(n)} = \xi_k^{(n)} - \alpha(n,k)$. First we estimate $I_1(n,t)$. If we use (6.26) and inequality (6.8) for the indicator function, we obtain that it will be dominated by $I_{11}(n,t) + I_{12}(n,t)$, where

$$I_{11}(n,t) = \frac{1}{B_n^2(n)} \sum_{k=1}^{[nt]} E[(T_n(k))^2 \chi(|\bar{\xi}_k^{(n)}| > \frac{\varepsilon B_n(n)}{2}) | \Im^{(n)}(k-1)],$$

$$I_{12}(n,t) = \frac{1}{B_n^2(n)} \sum_{k=1}^{[nt]} E[(T_n(k))^2 \chi(|T_n(k)| > \frac{\varepsilon B_n(n)}{2}) | \Im^{(n)}(k-1)].$$

To estimate $I_{11}(n,t)$, we use independence of $T_n(k)$ and $\xi_k^{(n)}$, part (a) of Lemma 4 and the Chebishev inequality, and obtain

$$I_{11}(n,t) \le \frac{4b(n)}{\varepsilon^2 B_n^4(n)} \sum_{k=1}^{[nt]} \beta(n,k) Z^{(n)}(k-1).$$

Therefore, for any $\delta > 0$

$$P\{I_{11}(n,t) > \delta\} \le \frac{4b(n)}{\delta\varepsilon^2 B_n^4(n)} \sum_{k=1}^{[nt]} \beta(n,k) A_n(k-1). \tag{6.28}$$

The sum on the right side of (6.28) is dominated by

$$\max_{1\le k\le nt} |\beta(n,k) - \beta(k)| \sum_{k=1}^{[nt]} A_n(k-1) + \sum_{k=1}^{[nt]} \beta(k) A_n(k-1).$$

Consequently, using parts (a), (c) and (d) of Lemma 3, we derive that the right side of (6.28) tends to zero as $n \to \infty$, which implies that $I_{11}(n,t) \xrightarrow{P} 0$ as $n \to \infty$ for each $t \in \mathbb{R}_+$.

We now estimate $I_{12}(n,t)$. Applying Lemma 5, we obtain that

$$I_{12}(n,t) \le S_1(n,t) + S_2(n,t) + S_3(n,t), \tag{6.29}$$

where

$$S_1(n,t) = \frac{1}{B_n^2(n)} \sum_{k=1}^{[nt]} \delta_n^{(1)}(\varepsilon/4) Z^{(n)}(k-1),$$

$$S_2(n,t) = \frac{16b^2(n)}{\varepsilon^2 B_n^4(n)} \sum_{k=1}^{[nt]} (Z^{(n)}(k-1))^2,$$

$$S_3(n,t) = \frac{\sqrt{8b^3(n)}}{\varepsilon B_n^3(n)} \sum_{k=1}^{[nt]} (Z^{(n)}(k-1))^{3/2}.$$

Obviously, we have

$$S_1(n,t) = \delta_n^{(1)}(\frac{\varepsilon}{4}) \frac{1}{B_n^2(n)} \sum_{k=1}^{[nt]} Z^{(n)}(k-1). \tag{6.30}$$

The sum on the right side of (6.30) multiplied by $b(n)/B_n^2(n)$ is equal to $\Gamma_1(n,t)$, which is the first term in (6.13), and as $n \to \infty$ converges to $\varphi(t)$ in probability. Taking into account conditions $\delta_n^{(1)}(\varepsilon) \to 0$ as $n \to \infty$ and $\liminf_{n\to\infty} b(n) > 0$ we conclude that

$$S_1(n,t) \xrightarrow{P} 0 \tag{6.31}$$

as $n \to \infty$ for each $t \in \mathbb{R}_+$. We now consider $S_2(n,t)$. For any $\delta > 0$ we have

$$P\{S_2(n,t) \geq \delta\} \leq \frac{16b^2(n)}{\varepsilon^2 \delta B_n^4(n)} R_1(n,t) + \frac{16b^2(n)}{\varepsilon^2 \delta B_n^4(n)} \sum_{k=1}^{[nt]} (A_n(k-1))^2, \quad (6.32)$$

where $R_1(n,t)$ is defined in (6.14). Using (6.15) we immediately obtain that the first term on the right side of (4.32) tends to zero as $n \to \infty$. Using Part (a) of Lemma 2, we have

$$\lim_{n\to\infty} \frac{1}{n^3\alpha^2(n)} \sum_{k=1}^{[nt]} (A_n(k-1))^2 = \int_0^t (\mu_\alpha(u))^2 du. \quad (6.33)$$

Therefore, due to Part (b) of Lemma 2 and the fact that $B_n^2(n) \sim \Delta_n^2(n)$ as $n \to \infty$, one can see that the second term on the right side of (6.32) also tends to zero as $n \to \infty$. Hence

$$S_2(n,t) \xrightarrow{P} 0 \quad (6.34)$$

as $n \to \infty$ for each $t \in \mathbb{R}_+$. To estimate $S_3(n,t)$, we use the following inequality $\sum_{k=1}^n a_k b_k \leq (\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2)^{1/2}$ with

$$a_k = Z^{(n)}(k-1)/B_n^2(n), \; b_k = (Z^{(n)}(k-1))^{1/2}/B_n(n),$$

and obtain that $S_3(n,t) \leq (S_2(n,t)\Gamma_1(n,t)/2)^{1/2}$. Consequently, we have from (6.22) and (6.34) that

$$S_3(n,t) \xrightarrow{P} 0. \quad (6.35)$$

The relations (6.28), (6.29), (6.31), (6.34) and (6.35) imply that

$$I_1(n,t) \xrightarrow{P} 0 \quad (6.36)$$

as $n \to \infty$ for each $t \in \mathbb{R}_+$. For the second term in (6.27) we have $I_2(n,t) \leq \Gamma_2(n,t)$, where $\Gamma_2(n,t)$ is defined in (6.13). Therefore,

$$\lim_{n\to\infty} I_2(n,t) = 0. \quad (6.37)$$

Thus, it follows from (6.27), (6.36) and (6.37) that condition (6.9) is satisfied. The assertion of Theorem 3.1 now follows from Theorem A.

Proof of Theorem 3.2. We again use Theorem A. Note that when C5 is satisfied, $B_n^2(n) \sim \sigma_n^2(n)$ as $n \to \infty$. To prove that condition (6.11) is satisfied, we again use the representation (6.13). Using Part(a) of Lemma 2 and Part(c) of Lemma 3, we obtain that $E[\Gamma_1(n,t)] \to 0$ as $n \to \infty$, due to condition C5. To prove that the variance of $\Gamma_1(n,t)$ tends to zero, we consider ratios in (6.20) and (6.21), with $n^2\beta^2(n)$ in denominators instead of $n^4\alpha^2(n)b^2(n)$. We realize that the ratios tend to zero as $n \to \infty$ when the conditions $n\alpha(n)b(n) = o(\beta(n))$ and $b(n) = o(\alpha(n))$ as $n \to \infty$ are satisfied. Hence we have

$$\Gamma_1(n,t) \xrightarrow{P} 0 \tag{6.38}$$

as $n \to \infty$ for each $t \in \mathbb{R}_+$. To estimate $\Gamma_2(n,t)$, we consider (6.23). Using Part(a) of Lemma 2 and Lemma 1, we immediately obtain that the first term in (6.23) tends to

$$\psi(t) = \frac{1}{\nabla_\beta(1)} \int_0^t u^\beta. \tag{6.39}$$

The second term in (6.23) tends to zero, due to Part(a) of Lemma 2, Lemma 1 and the second relation in the condition C1. Thus, we obtain that (6.24) holds with $\psi(t)$ instead of $\varphi(t)$.

Now we show that (6.25) holds under the conditions of Theorem 3.2, which implies the fulfillment of the condition (6.9). We use (6.27). Using parts (a), (c) and (d) of the Lemma 3 as in the proof of Theorem 3.1, we realize that the right side of (6.28) tends to zero, i.e. $I_{11}(n,t) \xrightarrow{P} 0$ as $n \to \infty$ for each $t \in \mathbb{R}_+$.

Now we estimate $I_{12}(n,t)$ using inequality (6.29). Since $B_n^2(n) \sim \Delta_n^2(n)$ under the conditions of Theorem 3.1, the relation (6.22) remains true, if we replace $B_n^2(n)$ by $\Delta_n^2(n)$ in the definition of $\Gamma_1(n,t)$ in the proof of Theorem 3.1. Therefore, taking into account that $\delta_n^{(1)}(\varepsilon) \le b(n)$, we rewrite (6.30) as

$$S_1(n,t) \le \frac{\Delta_n^2(n)}{B_n^2(n)} \frac{b(n)}{\Delta_n^2(n)} \sum_{k=1}^{[nt]} Z^{(n)}(k-1), \tag{6.40}$$

which implies, due to (6.22) and the fact that $\Delta_n^2(n) = o(B_n^2(n))$ as $n \to \infty$, that (6.31) holds without the condition $\delta_n^{(1)}(\varepsilon) \to 0$.

To estimate $S_2(n,t)$, we consider the inequality (6.32). Using (6.20) again, we obtain that the first term on the right side of (6.32) tends to zero. The second term tends to zero due to (6.33) and the fact that $n^3\alpha^2(n)b^2(n) =$

$o(\sigma_n^2(n))$. It is easy to see that the estimate of $S_3(n,t)$ in (6.28) remains true under the conditions of Theorem 3.2 and tends to zero as $n \to \infty$ for each $t \in \mathbb{R}_+$.

Now we consider the term $I_2(n,t)$ in (6.27). Using inequality (6.8) we obtain

$$I_2(n,t) \le I_{21}(n,t) + I_{22}(n,t), \tag{6.41}$$

where

$$I_{21}(n,t) = \frac{1}{B_n^2(n)} \sum_{k=1}^{[nt]} E[(\bar{\xi}_k^{(n)})^2 \chi(|T_n(k)| > \frac{\varepsilon B_n(n)}{2}) | \Im^{(n)}(k-1)],$$

$$I_{22}(n,t) = \frac{1}{B_n^2(n)} \sum_{k=1}^{[nt]} E[(\bar{\xi}_k^{(n)})^2 \chi(|\bar{\xi}_k^{(n)}| > \frac{\varepsilon B_n(n)}{2})].$$

Taking into account independence of $\xi_k^{(n)}$ and $T_n(k)$ and using the Chebishev inequality and Lemma 4, we have

$$I_{21}(n,t) \le \frac{4b(n)}{\varepsilon^2 B_n^4(n)} \sum_{k=1}^{[nt]} \beta(n,k) Z^{(n)}(k-1). \tag{6.42}$$

Realizing that the right side of (6.42) and the estimate of $I_{11}(n,t)$ in the proof of Theorem 3.1 are the same, we can use inequality (6.28). Therefore, taking into account that $n^2\alpha(n)\beta(n)b(n) = o(\sigma_n^4(n))$ as $n \to \infty$, we conclude that $I_{21}(n,t) \xrightarrow{P} 0$ for each $t \in \mathbb{R}_+$.

Now we consider $I_{22}(n,t)$. It is obvious that

$$I_{22}(n,t) \le \frac{1}{\sigma_n^2(n)} \sum_{k=1}^{[nt]} E[(\bar{\xi}_k^{(n)})^2 \chi(|\bar{\xi}_k^{(n)}| > \frac{\varepsilon \sigma_n(n)}{2})].$$

If we take into account that $\sigma_n^2(n) \sim K\sigma_0^2(n)$ as $n \to \infty$, where K is a positive constant depending on β and t, we see that $I_{22}(n,t) \to 0$ as $n \to \infty$ when the condition $\delta_n^{(2)}(\varepsilon) \to 0$ as $n \to \infty$ is satisfied. Hence $I_2(n,t) \xrightarrow{P} 0$ as $n \to \infty$ for each $t \in \mathbb{R}_+$ and we conclude that (6.25) holds. Theorem 3.2 is thus proved.

Proof of Theorem 3.3. It is easy to see that when condition C6 is satisfied,

$$B_n^2(n) \sim K(c_0, c) n^2 \alpha(n) b(n) \tag{6.43}$$

as $n \to \infty$, where $K(c_0, c) = \nu_\alpha(1)/c_0 + c\nabla_\beta(1)$. To prove that condition (6.11) of Theorem A is satisfied, we consider the representation (6.13). Using the Part(c) of Lemma 3, we have

$$\lim_{n\to\infty} E[\Gamma_1(n,t)] = \frac{1}{K(c_0,c)} \int_0^t \mu_\alpha(u)du. \tag{6.44}$$

Looking the proof of Theorem 3.1, it is not difficult to realize that the variance of $\Gamma_1(n,t)$ tends to zero as $n \to \infty$ when the condition C6 instead of C4 is satisfied.

To estimate $\Gamma_2(n,t)$ we use (6.23). Using Lemma 1 and (6.44), we obtain that the first term in (6.23) tends to

$$\frac{c}{K(c_0,c)} \int_0^t u^\beta du.$$

The second term in (6.23) tends to zero due to the Lemma 1 and the condition C1. Thus, condition (6.11) is satisfied with $C(t) = \omega(t)$, where

$$\omega(t) = \frac{1}{K(c_0,c)} \int_0^t (\mu_\alpha(u) + cu^\beta)du.$$

We now show that condition (6.9) of Theorem A is also satisfied. For this we show that (6.25) remains true under the conditions of Theorem 3.3. Consider the equality (6.27). The first term in (6.27) is dominated by $I_{11}(n,t)+I_{12}(n,t)$ again, where $I_{1i}(n,t),\ i=1,2$ are defined in the proof of Theorem 3.1. By the arguments, similar to those in the proof of Theorem 3.1, we obtain that $I_{11}(n,t) \xrightarrow{P} 0$ as $n \to \infty$ for each $t \in \mathbb{R}_+$. We now consider (6.29). Using the estimate (6.30), we see that (6.31) remains true when $\delta^{(1)}(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$ and $\liminf_{n\to\infty} b(n) > 0$. Taking into account (6.44), we realize that (6.34)-(6.36) also hold under the conditions of Theorem 3.3.

To estimate the second term in (6.27), we use (6.41) in the proof of Theorem 3.2, and see that it tends to zero in probability as $n \to \infty$ when $\delta^{(2)}(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, and consequently $\mathcal{Y}_n(t) \xrightarrow{D} W(\omega(t))$ as $n \to \infty$ weakly in Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$.

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