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Topological Properties of $\delta-$Open Sets*

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Abstract

In 1968 Velicko [29] introduced the concepts of $\delta - closure$ and $\delta - interior$ operations. We introduce and study topological properties of $\delta - derived$, $\delta - border$, $\delta - frontier$ and $\delta - exterior$ of a set using the concept of $\delta - open$ sets and study also other properties of the well-known notions of $\delta - closure$ and $\delta - interior$. We also introduce some new classes of topological spaces in terms of the concept of delta-D-sets and investigate some of their fundamental properties. Moreover, we investigate and study some further topological properties of the well-known notions of $\delta - closure$ and $\delta - interior$ of a set in a topological space. We also introduce $\delta - R_0$ space and study its characteristics.

1 Introduction

Velicko [29] introduced the notion of $\delta - closure$ and $\delta - interior$ operations. Throughout this paper, (X, τ) (simply X) always mean topological space on

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which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure (resp., *interior*) of S will be denoted by $Cl(S)$ (resp., $Int(S)$). A subset S of X is called a semi-open set [20] if $S \subseteq Cl[Int(S)]$. The complement of a semi-open set is called a semi-closed set. The intersection of all semi-closed sets containing A is called the semi-closure of A and is denoted by $sCl(A)$. The family of all semi-open sets in a topological space (X, τ) will be denoted by $SO(X)$. A subset $M(x)$ of a space X is called a semi-neighborhood of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subseteq M(x)$. In [18] Latif introduced the notion of semi-convergence of filters and investigated some characterizations related to semi-open continuous function. A point $x \in X$ is called the $\delta-cluster$ point of $A \subseteq X$ if $A \cap Int[Cl(U)] \neq \phi$ for every open set U of X containing x. The set of all $\delta-cluster$ points of A is called the $\delta-closure$ of A, denoted by $Cl_{\delta}(A)$. A subset $A \subseteq X$ is called $\delta-closed$ if $A = Cl_{\delta}(A)$. The complement of a $\delta-closed$ set is called $\delta-open$. The collection of all $\delta-open$ sets in a topological space (X, τ) forms a topology τ_{δ} on X, called the semigeneralization topology of τ, weaker than τ and the class of all regular open sets in τ forms an open basis for τ_{δ}. In this paper, we introduce and study topological properties of $\delta-derived$, $\delta-border$, $\delta-frontier$ and $\delta-exterior$ of a set using the concept of $\delta-open$ sets and study also other properties of the well-known notions of $\delta-closure$ and $\delta-interior$. The notion of $\theta-open$ subsets, $\theta-closed$ subsets and $\theta-closure$ were introduced by Velicko [29] for the purpose of studying the important class of $H-closed$ spaces in terms of arbitrary filterbases. A point $x \in X$ is called a $\theta-adherent$ point of A [7], if $A \cap Cl(V) \neq \phi$ for every open set V containing x. The set of all $\theta-adherent$ points of A is called the $\theta-closure$ of A and is denoted by $Cl_{\theta}(A)$. A subset A of X is called $\theta-closed$ if $A = Cl_{\theta}(A)$. Dontchev and Maki [[7], Lemma 3.9] have shown that if A and B are subsets of a space (X, τ), then

$Cl_\theta(A \cup B) = Cl_\theta(A) \cup Cl_\theta(B)$ and $Cl_\theta(A \cap B) = Cl_\theta(A) \cap Cl_\theta(B)$. Note also that the $\theta-$*closure* of a given set need not be a $\theta-$*closed* set. But it is always closed. The complement of a $\theta-$*closed* set is called a $\theta-$*open* set. The $\theta-$*interior* of set A in X, written $Int_\theta(A)$, consists of those points x of A such that for some open set U containing x, $Cl(U) \subseteq A$. A set A is $\theta-$*open* if and only if $A = Int_\theta(A)$, or equivalently, $X - A$ is $\theta-$*closed*. The collection of all $\theta-$*open* sets in a topological space (X, τ) forms a topology τ_θ on X, weaker than τ. We observe that for any topological space (X, τ) the relation $\tau_\theta \subseteq \tau_\delta \subseteq \tau$ always holds. We also have $A \subseteq Cl(A) \subseteq Cl_\delta(A) \subseteq Cl_\theta(A)$, for any subset A of X.

2 Properties of $\delta-$Open Sets

Definition 2.1. Let A be a subset of a space X. A point $x \in A$ is said to be a $\delta-$*limit* point of A if for each $\delta-$*open* set U containing x, $U \cap (A - \{x\}) \neq \phi$. The set of all $\delta-$*limit* points of A is called the $\delta-$*derived* set of A and is denoted by $D_\delta(A)$.

Theorem 2.2. For subsets A, B of a space X, the following statements hold:

(1) $D(A) \subseteq D_\delta(A)$, where $D(A)$ is the derived set of A;

(2) if $A \subseteq B$, then $D_\delta(A) \subseteq D_\delta(B)$;

(3) $D_\delta(A) \cup D_\delta(B) = D_\delta(A \cup B)$ and $D_\delta(A \cap B) \subseteq D_\delta(A) \cap D_\delta(B)$;

(4) $[D_\delta(D_\delta(A)) - A] \subseteq D_\delta(A)$;

(5) $D_\delta[A \cup D_\delta(A)] \subseteq A \cup D_\delta(A)$.

Proof. (1) It suffices to observe that every $\delta-$*open* set is an open set.

(2) Obvious.

(3) $D_\delta(A) \cup D_\delta(B) = D_\delta(A \cup B)$ is a modification of the standard proof for D, where open sets are replaced by $\delta-$*open* sets. $D_\delta(A \cap B) \subseteq D_\delta(A) \cap D_\delta(B)$ follows by (2).

(4) If $x \in [D_\delta(D_\delta(A)) - A]$ and U is a $\delta - open$ set containing x, then $U \cap [D_\delta(A) - \{x\}] \neq \phi$. Let $y \in U \cap [D_\delta(A) - \{x\}]$. Then, since $y \in D_\delta(A)$ and $y \in U$, so $U \cap [A - \{y\}] \neq \phi$. Let $z \in U \cap [A - \{y\}]$. Then, $z \neq x$ for $z \in A$ and $x \notin A$. Hence, $U \cap [A - \{x\}] \neq \phi$. Therefore, $x \in D_\delta(A)$.

(5) Let $x \in D_\delta[A \cup D_\delta(A)]$. If $x \in A$, the result is obvious. So, let $x \in [D_\delta(A \cup D_\delta(A)) - A]$, then, for $\delta-open$ set U containing x, $U \cap [A \cup D_\delta(A) - \{x\}] \neq \phi$. Thus, $U \cap (A - \{x\}) \neq \phi$ or $U \cap [D_\delta(A) - \{x\}] \neq \phi$. Now, it follows similarly from (4) that $U \cap [A - \{x\}] \neq \phi$. Hence, $x \in D_\delta(A)$. Therefore, in any case, $D_\delta[A \cup D_\delta(A)] \subseteq [A \cup D_\delta(A)]$.

Theorem 2.3. For any subset A of a space X, $Cl_\delta(A) = A \cup D_\delta(A)$.

Proof. Since $D_\delta(A) \subseteq Cl_\delta(A)$, $A \cup D_\delta(A) \subseteq Cl_\delta(A)$. On the other hand, let $x \in Cl_\delta(A)$. If $x \in A$, then the proof is complete. If $x \notin A$, each $\delta-$ *open* set U containing x intersects A at a point distinct from x; so $x \in D_\delta(A)$. Thus, $Cl_\delta(A) \subseteq [A \cup D_\delta(A)]$, which completes the proof.

Corollary 2.4. A subset A is $\delta - closed$ if and only if it contains the set of its $\delta - Iimit$ points.

Definition 2.5. A point $x \in X$ is said to be a $\delta - interior$ point of A if there exists a $\delta - open$ set U containing x such that $U \subseteq A$. The set of all $\delta - interior$ points of A is said to be $\delta - interior$ of A and is denoted by $Int_\delta(A)$.

Theorem 2.6. For subsets A, B of a space X, the following statements are true:

(1) $Int_\delta(A)$ is the largest $\delta - open$ set contained in A;

(2) A is $\delta - open$ if and only if $A = Int_\delta(A)$;

(3) $Int_\delta[Int_\delta(A)] = Int_\delta(A)$;

(4) $Int_\delta(A) = [A - D_\delta(X - A)]$;

(5) $[X - Int_\delta(A)] = Cl_\delta(X - A)$;

(6) $[X - Cl_\delta(A)] = Int_\delta(X - A)$;

(7) $A \subseteq B$, then $Int_\delta (A) \subseteq Int_\delta (B)$;

(8) $Int_\delta(A) \cup Int_\delta (B) \subseteq Int_\delta (A \cup B)$;

(9) $Int_\delta(A \cap B) = Int_\delta (A) \cap Int_\delta (B)$.

Proof. (4) If $x \in [A - D_\delta (X - A)]$, then $x \notin D_\delta (X - A)$ and so there exists a $\delta-open$ set U containing x such that $U \cap (X - A) = \phi$. Then, $x \in U \subseteq A$ and hence $x \in Int_\delta (A)$, that is, $[A - D_\delta (X - A)] \subseteq Int_\delta (A)$. On the other hand, if $x \in Int_\delta (A)$, then $x \notin D_\delta (X - A)$ since $Int_\delta (A)$ is $\delta - open$ and $[Int_\delta (A) \cap (X - A)] = \phi$. Hence, $Int_\delta (A) = [A - D_\delta (X - A)]$.

(5) $X - Int_\delta (A) = X - [A - D_\delta (X - A)]$

$= (X - A) \cup D_\delta (X - A) = Cl_\delta (X - A)$.

Definition 2.7. $Bd_\delta (A) = A - Int_\delta (A)$ is said to be the $\delta - border$ of A.

Theorem 2.8. For a subset A of a space X, the following statements hold:

(1) $Bd (A) \subseteq Bd_\delta (A)$ where $Bd (A)$ denotes the border of A;

(2) $A = Int_\delta (A) \cup Bd_\delta (A)$;

(3) $Int_\delta (A) \cap Bd_\delta (A) = \phi$;

(4) A is a $\delta - open$ set if and only if $Bd_\delta (A) = \phi$;

(5) $Bd_\delta [Int_\delta (A)] = \phi$;

(6) $Int_\delta [Bd_\delta (A)] = \phi$;

(7) $Bd_\delta [Bd_\delta (A)] = Bd_\delta (A)$;

(8) $Bd_\delta (A) = A \cap [Cl_\delta (X - A)]$;

(9) $Bd_\delta (A) = D_\delta (X - A)$.

Proof. (6) If $x \in Int_\delta [Bd_\delta (A)]$, then $x \in Bd_\delta (A)$. On the other hand, since $Bd_\delta (A) \subseteq A$, $x \in Int_\delta [Bd_\delta (A)] \subseteq Int_\delta (A)$. Hence, $x \in Int_\delta (A) \cap Bd_\delta (A)$, which contradicts (3). Thus, $Int_\delta [Bd_\delta (A)] = \phi$.

(8) $Bd_\delta (A) = A - Int_\delta (A)$

$= A - [X - Cl_\delta (X - A)] = A \cap Cl_\delta (X - A)$.

(9) $Bd_\delta (A) = A - Int_\delta (A)$

$= A - [A - D_\delta(X - A)] = D_\delta(X - A)$.

Definition 2.9. $Fr_\delta(A) = Cl_\delta(A) - Int_\delta(A)$ is said to be the $\delta - frontier$ of A.

Theorem 2.10. For a subset A of a space X, the following statements hold:

(1) $Fr(A) \subseteq Fr_\delta(A)$ where $Fr(A)$ denotes the frontier of A;

(2) $Cl_\delta(A) = Int_\delta(A) \cup Fr_\delta(A)$;

(3) $Int_\delta(A) \cap Fr_\delta(A) = \phi$;

(4) $Bd_\delta(A) \subseteq Fr_\delta(A)$;

(5) $Fr_\delta(A) = Bd_\delta(A) \cup D_\delta(A)$;

(6) A is a $\delta - open$ set if and only if $Fr_\delta(A) = D_\delta(A)$;

(7) $Fr_\delta(A) = Cl_\delta(A) \cap Cl_\delta(X - A)$;

(8) $Fr_\delta(A) = Fr_\delta(X - A)$;

(9) $Fr_\delta(A)$ is $\delta - closed$;

(10) $Fr_\delta[Fr_\delta(A)] \subseteq Fr_\delta(A)$;

(11) $Fr_\delta[Int_\delta(A)] \subseteq Fr_\delta(A)$;

(12) $Fr_\delta[Cl_\delta(A)] \subseteq Fr_\delta(A)$;

(13) $Int_\delta(A) = A - Fr_\delta(A)$.

Proof. (2) $Int_\delta(A) \cup Fr_\delta(A)$

$= Int_\delta(A) \cup [Cl_\delta(A) - Int_\delta(A)] = Cl_\delta(A)$.

(3) $Int_\delta(A) \cap Fr_\delta(A)$

$= Int_\delta(A) \cap [Cl_\delta(A) - Int_\delta(A)] = \phi$.

(5) Since $Int_\delta(A) \cup Fr_\delta(A)$

$= Int_\delta(A) \cup Bd_\delta(A) \cup D_\delta(A)$,

$Fr_\delta(A) = Bd_\delta(A) \cup D_\delta(A)$.

(7) $Fr_\delta(A) = Cl_\delta(A) - Int_\delta(A)$

$= Cl_\delta(A) \cap Cl_\delta(X - A)$.

(9) $Cl_\delta[Fr_\delta(A)] = Cl_\delta[Cl_\delta(A) \cap Cl_\delta(X - A)]$

$\subseteq Cl_\delta\left[Cl_\delta\left(A\right)\right]\cap Cl_\delta\left[Cl_\delta\left(X-A\right)\right]$

$=Cl_\delta\left(A\right)\cap Cl_\delta\left(X-A\right)=Fr_\delta\left(A\right).$

Hence, $Fr_\delta\left(A\right)$ is $\delta-closed$.

(10) $Fr_\delta\left[Fr_\gamma\left(A\right)\right]=Cl_\delta\left[Fr_\delta\left(A\right)\right]\cap Cl_\delta\left[X-Fr_\delta\left(A\right)\right]$

$\subseteq Cl_\delta\left[Fr_\delta\left(A\right)\right]=Fr_\delta\left(A\right).$

(12) $Fr_\delta\left(Cl_\delta\left(A\right)\right)=Cl_\delta\left[Cl_\delta\left(A\right)\right]-Int_\delta\left[Cl_\delta\left(A\right)\right]$

$=Cl_\delta\left(A\right)-Int_\delta\left[Cl_\delta\left(A\right)\right]\subseteq\left[Cl_\delta\left(A\right)-Int_\delta\left(A\right)\right]=Fr_\delta\left(A\right).$

(13) $A-Fr_\delta\left(A\right)=A-\left[Cl_\delta\left(A\right)-Int_\delta\left(A\right)\right]=Int_\delta\left(A\right).$

Remark 2.11. Let A and B be subsets of X. Then $A\subseteq B$ does not imply that either $Fr_\delta\left(B\right)\subseteq Fr_\delta\left(A\right)$ or $Fr_\delta\left(A\right)\subseteq Fr_\delta\left(B\right).$

Definition 2.12. $Ext_\delta\left(A\right)=Int_\delta\left(X-A\right)$ is said to be a $\delta-exterior$ of A.

Theorem 2.13. For a subset A of a space X, the following statements hold:

(1) $Ext_\delta\left(A\right)\subseteq Ext\left(A\right)$ where $Ext\left(A\right)$ denotes the exterior of A;

(2) $Ext_\delta\left(A\right)$ is $\delta-open$;

(3) $Ext_\delta\left(A\right)=Int_\delta\left(X-A\right)=X-Cl_\delta\left(A\right);$

(4) $Ext_\delta\left[Ext_\delta\left(A\right)\right]=Int_\delta\left[Cl_\delta\left(A\right)\right];$

(5) If $A\subseteq B$, then $Ext_\delta\left(A\right)\supseteq Ext_\delta\left(B\right);$

(6) $Ext_\delta\left(A\cup B\right)=Ext_\delta\left(A\right)\cup Ext_\delta\left(B\right);$

(7) $Ext_\delta\left(A\right)\cap Ext_\delta\left(B\right)\subseteq Ext_\delta\left(A\cap B\right);$

(8) $Ext_\delta\left(X\right)=\phi$;

(9) $Ext_\delta\left(\phi\right)=X$;

(10) $Ext_\delta\left(A\right)=Ext_\delta\left[X-Ext_\delta\left(A\right)\right];$

(11) $Int_\delta\left(A\right)\subseteq Ext_\delta\left[Ext_\delta\left(A\right)\right];$

(12) $X=Int_\delta\left(A\right)\cup Ext_\delta\left(A\right)\cup Fr_\delta\left(A\right);$

(13) $Ext_\delta\left(A\right)\cup Ext_\delta\left(B\right)\subseteq Ext_\delta\left(A\cap B\right).$

Proof. (4) $Ext_\delta\left[Ext_\delta\left(A\right)\right]=Ext_\delta\left[X-Cl_\delta\left(A\right)\right]$

$=Int_\delta\left[X-\left(X-Cl_\delta\left(A\right)\right)\right]=Int_\delta\left[Cl_\delta\left(A\right)\right].$

(10) $Ext_\delta[X - Ext_\delta(A)] = Ext_\delta[X - Int_\delta(X - A)]$

$= Int_\delta[X - (X - Int_\delta(X - A))]$

$= Int_\delta[Int_\delta(X - A)] = Int_\delta(X - A) = Ext_\delta(A)$.

(11) $Int_\delta(A) \subseteq Int_\delta[Cl_\delta(A)] = Int_\delta[X - Int_\delta(X - A)]$

$= Int_\delta[X - Ext_\delta(A)] = Ext_\delta[Ext_\delta(A)]$.

(13) $Ext_\delta(A) \cup Ext_\delta(B) = Int_\delta(X - A) \cup Int_\delta(X - B)$

$\subseteq Int_\delta[(X - A) \cup (X - B)]$

$= Int_\delta[X - (A \cap B)] = Ext_\delta(A \cap B)$.

3 Applications of $\delta-$Open Sets

Definition 3.1.Let X be a topological space. A set $A \subseteq X$ is said to be $\delta-$*saturated* if for every $x \in A$ it follows $Cl_\delta(\{x\}) \subseteq A$. The set of all $\delta-$*saturated* sets in X we denote by $B_\delta(X)$.

Theorem 3.2. Let X be a topological space. Then $B_\delta(X)$ is a complete Boolean set Algebra.

Proof. We will prove that all the unions and complements of elements of $B_\delta(X)$ are members of $B_\delta(X)$. Obviously, only the proof regarding the complements is not trivial. Let $A \in B_\delta(X)$ and suppose that $Cl_\delta(\{x\}) \nsubseteq (X - A)$ for some $x \in (X - A)$. Then there exists $y \in A$ such that $y \in Cl_\delta(\{x\})$. It follows that x, y have no disjoint neighborhoods. Then $x \in Cl_\delta(\{y\})$. But this is a contradiction, because by the definition of $B_\delta(X)$ we have $Cl_\delta(\{y\}) \subseteq A$. Hence, $Cl_\delta(\{x\}) \subseteq (X - A)$ for every $x \in (X - A)$, which implies $(X - A) \in B_\delta(X)$.

Corollary 3.3. $B_\delta(X)$ contains every union and every intersection of $\delta-$*closed* and $\delta - $*open* sets in X.

Definition 3.4. A space X is said to be $\delta-$Hausdorff if for every $x \neq y \in X$, there exist $\delta - $*open* sets U_x, V_y such that $x \in U_x$, $y \in V_y$ and $U_x \cap V_y = \phi$.

Theorem 3.5. The following four properties are equivalent:

(1) X is $\delta - T_2$;

(2) Let $x \in X$. For each $y \neq x$, there exists a $\delta - open$ set U such that $x \in U$ and $y \notin Cl_\delta(U)$;

(3) For each $x \in X$, $\cap \{Cl_\delta(U) | U \in \tau_\delta$ and $x \in U\} = \{x\}$;

(4) The diagonal $\Delta = \{(x, x) | x \in X\}$ is $\delta - closed$ in $X \times X$.

Proof. $(1) \Longrightarrow (2)$. Let $x \in X$ and $y \neq x$. Then there are disjoint $\delta - open$ sets U and V such that $x \in U$ and $y \in V$. Clearly, V^c is $\delta - closed$, $Cl_\delta(U) \subseteq V^c$, $y \notin V^c$ and therefore $y \notin Cl_\delta(U)$.

$(2) \Longrightarrow (3)$. If $y \neq x$, there exists a $\delta - open$ set U such that $x \in U$ and $y \notin Cl_\delta(U)$. So $y \notin \cap \{Cl_\delta(U) | U \in \tau_\delta$ and $x \in U\}$.

$(3) \Longrightarrow (4)$. We prove that Δ^c is $\delta - open$. Let $(x, y) \notin \Delta$. Then $y \neq x$ and since $\cap \{Cl_\delta(U) | U \in \tau_\delta$ and $x \in U\} = \{x\}$ there is some $U \in \tau_\delta$ with $x \in U$ and $y \notin Cl_\delta(U)$. Since $U \cap [Cl_\delta(U)]^c = \phi$, $U \times [Cl_\delta(U)]^c$ is a $\delta - open$ set such that $(x, y) \in U \times [Cl_\delta(U)]^c \subseteq \Delta^c$.

$(4) \Longrightarrow (1)$. If $y \neq x$, then $(x, y) \notin \Delta$ and thus there exist $\delta - open$ sets U and V such that $(x, y) \in U \times V$ and $(U \times V) \cap \Delta = \phi$. Clearly, for the $\delta - open$ sets U and V we have: $x \in U, y \in V$ and $U \cap V = \phi$.

Definition 3.6. A subset A of a space X is said to be Delta-compact (briefly, $\delta - compact$) if every cover of $\delta - open$ sets has a finite subcover.

It is well-known that every closed subset of a compact space is compact. The next theorem approximates this result for $\delta - compactness$.

Theorem 3.7. A $\delta - compact$ subset of a $\delta-$Hausdorff space is $\delta - closed$.

Proof. Let A be a $\delta - compact$ subset of a $\delta-$Hausdorff space X. We will show that $(X - A)$ is $\delta - open$. Let $x \in (X - A)$ then for each $a \in A$ there exist $\delta - open$ sets $U_{x,a}$ and V_a such that $x \in U_{x,a}$ and $a \in V_a$ and $U_{x,a} \cap V_a = \phi$. The collection $\{V_a : a \in A\}$ is a $\delta - open$ cover of A. Therefore, there exists a finite subcollection $V_{a_1}, V_{a_2}, \ldots, V_{a_n}$ that covers A. Let $U_x = U_{x,a_1} \cap U_{x,a_1} \cap$.

$. . \cap U_{x,a_n}$. Then $x \in U_x$, U_x is $\delta - open$ and $U_x \cap A = \phi$. This proves that A is $\delta - closed$.

Theorem 3.8. A $\delta - closed$ subset of a $\delta-$Haudorff space is $\delta - compact$.

Proof. Let X be $\delta - compact$ and let A be a $\delta - closed$ subset of X. Let Γ be a $\delta - open$ cover of A. Then $\Gamma^* = \Gamma \cup \{X - A\}$ is a $\delta - open$ cover of X. Since X is $\delta - compact$, this collection Γ^* has a finite collection Λ^* that covers X. But then Γ ha a finite subcollection $\Lambda = \Lambda^* - \{X - A\}$ that covers A as we need.

Definition 3.9. Let A be a subset of a topological space X. Then $\delta - \ker nel$ of A, denoted by $Ker_\delta (A) = \cap \{O \in \tau_\delta | A \subseteq O\}$.

Definition 3.10. Let x be a point of a topological space X. Then $\delta - \ker nel$ of x, denoted by $Ker_\delta (\{x\})$ *is* defined to be the set $Ker_\delta (\{x\}) = \cap \{O \in \tau_\delta | x \in O\}$.

Lemma 3.11. Let (X, τ) be a topological space and $x \in X$. Then $Ker_\delta (A) = \{x \in X | Cl_\delta (\{x\}) \cap A \neq \phi\}$.

Proof. Let $x \in Ker_\delta (A)$ and $Cl_\delta (\{x\}) \cap A = \phi$. Hence $x \notin [X - Ker_\delta (\{x\})]$ which is a $\delta - open$ set containing A. This is impossible, since $x \in Ker_\delta (A)$. Consequently, $Ker_\delta (\{x\}) \cap A \neq \phi$. Let $Cl_\delta (\{x\}) \cap A \neq \phi$ and $x \notin Ker_\delta (A)$. Then there exists a $\delta - open$ set D containing A and $x \notin D$. Let $y \in Cl_\delta (\{x\}) \cap A$. Hence, D is a $\delta - open$ neighborhood of y with $x \notin D$. By this contradiction, $x \in Ker_\delta (A)$ and the claim.

Definition 3.12. A topological space (X, τ) is said to be a $\delta - R_0$ space if every $\delta - open$ set contains the $\delta - closure$ of each of its singletons.

Lemma 3.13. Let (X, τ) be a topological space and $x \in X$. Then $y \in Ker_\delta (\{x\})$ if and only if $x \in Ker_\delta (\{y\})$.

Proof. Suppose that $y \notin Ker_\delta (\{x\})$. Then there exists a $\delta - open$ set V containing x such that $y \notin V$. Therefore we have $x \notin Cl_\delta (\{y\})$. The proof of the converse case can be done similarly.

Lemma 3.14. The following statements are equivalent for any points x and y in a topological space (X,τ) :

(1) $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$;

(2) $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$.

Proof. (1) $\Longrightarrow$ (2) : Suppose that $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$. Then there exists a point z in X such that $z \in Ker_\delta(\{x\})$ and $z \notin Ker_\delta(\{y\})$. It follows from $z \in Ker_\delta(\{x\})$ that $\{x\} \cap Cl_\delta(\{x\}) \neq \phi$. This implies that $x \in Cl_\delta(\{z\})$. By $z \notin Ker_\delta(\{y\})$, we have $\{y\} \cap Cl_\delta(\{z\}) = \phi$. Since $x \in Cl_\delta(\{z\})$ and $Cl_\delta(\{x\}) \subseteq Cl_\delta(\{z\})$. Hence $\{y\} \cap Cl_\delta(\{x\}) = \phi$. Therefore, $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$.

(2) $\Longrightarrow$ (1) : Suppose that $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$. Then there exists a point $z \in X$ such that $z \in Cl_\delta(\{x\})$ and $z \notin Cl_\delta(\{y\})$. Then, there exists a $\delta-$ *open* set containing z and therefore x but not $y, i.e.,$ $y \notin Ker_\delta(\{x\})$. Hence $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$.

Theorem 3.15. A topological space (X,τ) is a $\delta - R_0$ space if and only if for every x and y in $X, Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$ implies $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) = \phi$.

Proof. **Necessity**. Suppose that (X,τ) is $\delta - R_0$ and $x, y \in X$ such that $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$. Then, there exists $z \in Cl_\delta(\{x\})$ such that $z \notin Cl_\delta(\{y\})$ (or $z \in Cl_\delta(\{y\})$ such that $z \notin Cl_\delta(\{x\})$). There exists $V \in \tau_\delta$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin Cl_\delta(\{y\})$. Thus $x \in [X - Cl_\delta(\{y\})] \in \tau_\delta$, which implies $Cl_\delta(\{x\}) \subseteq [X - Cl_\delta(\{y\})]$ and $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) = \phi$. The proof for otherwise is similar.

Sufficiency. Let $V \in \tau_\delta$ and let $x \in V$. We will show that $Cl_\delta(\{x\}) \subseteq V$. Let $y \notin V$, i.e., $y \in (X - V)$. Then $x \neq y$ and $x \notin Cl_\delta(\{y\})$. This shows that $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$. By assumption, $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) = \phi$. Hence $y \notin Cl_\delta(\{x\})$ and therefore $Cl_\delta(\{x\}) \subseteq V$.

Theorem 3.16. A topological space (X,τ) is a $\delta - R_0$ space if and only if for any points x and y in X, $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$ implies $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) = \phi$.

Proof. Suppose that (X,τ) is a $\delta - R_0$ space. Thus by Lemma 3.14, for any points x and y in X if $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$ then $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$. Now we prove that $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) = \phi$. Assume that $z \in Ker_\delta(\{x\}) \cap Ker_\delta(\{y\})$. By $z \in Ker_\delta(\{x\})$ and Lemma 3.13, it follows that $x \in Ker_\delta(\{z\})$. Since $x \in Ker_\delta(\{x\})$, by Theorem 3.15, $Cl_\delta(\{x\}) = Cl_\delta(\{z\})$. Si$milarly$, we have $Cl_\delta(\{y\}) = Cl_\delta(\{z\}) = Cl_\delta(\{x\})$. This is a contradiction. Therefore, we have $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) = \phi$.

Conversely, let (X,τ) be a topological space such that for any points x and y in X, $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$ implies $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) = \phi$. If $Cl_\delta(\{x\}) \neq Cl_\delta(\{y\})$, then by Lemma 3.14, $Ker_\delta(\{x\}) \neq Ker_\delta(\{y\})$. Hence $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) = \phi$ which implies $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) = \phi$. Because $z \in Ker_\delta(\{x\})$ implies that $x \in Ker_\delta(\{z\})$. Therefore $Ker_\delta(\{x\}) \cap Ker_\delta(\{y\}) \neq \phi$. By hypothesis, we have $Ker_\delta(\{x\}) = Ker_\delta(\{z\})$. Then $z \in Cl_\delta(\{x\}) \cap Cl_\delta(\{y\})$ implies that $Ker_\delta(\{x\}) = Ker_\delta(\{z\}) = Ker_\delta(\{y\})$. This is a contradiction. Hence, $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) = \phi$. By Theorem 3.15 (X,τ) is a $\delta - R_0$ space.

Theorem 3.17. For a topological space (X,τ), the following properties are equivalent:

(1) (X,τ) is a $\delta - R_0$ space;

(2) For any $A \neq \phi$ and $G \in \tau_\delta$ such that $A \cap G \neq \phi$, there exists $F \in C_\delta(X,\tau)$ such that $A \cap F \neq \phi$ and $F \subseteq G$;

(3) Any $G \in \tau_\delta$, $G = \cup\{F \in C_\delta(X,\tau) | F \subseteq G\}$;

(4) Any $F \in C_\delta(X,\tau)$, $F = \cap\{G \in \tau_\delta | F \subseteq G\}$;

(5) For any $x \in X$, $Cl_\delta(\{x\}) \subseteq Ker_\delta(\{x\})$.

Proof. (1) $\Longrightarrow$ (2) : Let A be a nonempty subset of X and $G \in \tau_\delta$ such that $A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in \tau_\theta$, $Cl_\delta(\{x\}) \subseteq G$. Set $F = Cl_\delta(\{x\})$. Then F is a $\delta - closed$ subset X such that $F \subseteq G$ and $A \cap F \neq \phi$.

(2) $\Longrightarrow$ (3) : Let $G \in \tau_\delta$. Then $\cup \{F \in C_\delta(X,\tau) | F \subseteq G\} \subseteq G$. Let x be any point of G. There exists $F \in C_\delta(X,\tau)$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup \{F \in C_\delta(X,\tau) | F \subseteq G\}$ and hence $G = \cup \{F \in C_\delta(X,\tau) | F \subseteq G\}$.

(3) $\Longrightarrow$ (4) : This is obvious.

(4) $\Longrightarrow$ (5) : Let x be any point of X and $y \notin Ker_\delta(\{x\})$. There exists $V \in \tau_\delta$ such that $x \in V$ and $y \notin V$; hence $Cl_\delta(\{x\}) \cap V = \phi$. By (4) $(\cap \{G \in \tau_\delta | Cl_\delta(\{y\}) \subseteq G\}) \cap V = \phi$. There exists $G \in \tau_\delta$ such that $x \notin G$ and $Cl_\delta(\{y\}) \subseteq G$. Therefore $Cl_\delta(\{x\}) \cap G = \phi$ and $y \notin Cl_\delta(\{x\})$. Consequently, we obtain $Cl_\delta(\{x\}) \subseteq Ker_\delta(\{x\})$.

(5) $\Longrightarrow$ (1) : Let $G \in \tau_\delta$ and $x \in G$. Suppose $y \in Ker_\delta(\{x\})$. Then $x \in Cl_\delta(\{y\})$ and $y \in G$. This implies that $Cl_\delta(\{x\}) \subseteq Ker_\theta(\{x\}) \subseteq G$. Therefore, (X,τ) is a $\delta - R_0$ space.

Corollary 3.18. For a topological space (X,τ), the following properties are equivalent:

(1) (X,τ) is a $\delta - R_0$ space;

(2) $Cl_\delta(\{x\}) = Ker_\delta(\{x\})$ for all $x \in X$.

Proof. (1) $\Longrightarrow$ (2) : Suppose that (X,τ) is a $\delta - R_0$ space. By Theorem 3.17, $Cl_\delta(\{x\}) \subseteq Ker_\delta(\{x\})$ for each $x \in X$. Let $y \in Ker_\delta(\{x\})$. Then $x \in Cl_\delta(\{y\})$ and so $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$. Therefore, $y \in Cl_\delta(\{x\})$ and hence $Ker_\delta(\{x\}) \subseteq Cl_\delta(\{x\})$. This shows that $Cl_\delta(\{x\}) = Ker_\delta(\{x\})$.

(2) $\Longrightarrow$ (1) : This is obvious by Theorem 3.17.

Theorem 3.19.For a topological space (X,τ), the following properties are equivalent:

(1) (X,τ) is a $\delta - R_0$ space;

(2) $x \in Cl_\delta(\{y\})$ if and only if $y \in Cl_\delta(\{x\})$, for any points x and y in X.

Proof. (1) $\Longrightarrow$ (2) : Assume that X is $\delta - R_0$. Let $x \in Cl_\delta(\{y\})$ and D be any $\delta - open$ set such that $y \in D$. Now by hypothesis, $x \in D$. Therefore, every $\delta - open$ set containing y contains x. Hence $y \in Cl_\delta(\{x\})$.

(2) $\Longrightarrow$ (1) : Let U be a $\delta - open$ set and $x \in U$. If $y \notin U$, then $x \notin Cl_\delta(\{y\})$ and hence $y \notin Cl_\delta(\{x\})$. This implies that $Cl_\delta(\{x\}) \subseteq U$. Hence (X,τ) is $\delta - R_0$.

Theorem 3.20. For a topological space (X,τ), the following properties are equivalent:

(1) (X,τ) is a $\delta - R_0$ space;

(2) If F is $\delta - closed$, then $F = Ker_\delta(F)$;

(3) If F is $\delta - closed$ and $x \in F$, then $Ker_\delta(X) \subseteq F$;

(4) If $x \in X$, then $Ker_\delta(\{x\}) \subseteq Cl_\delta(\{x\})$.

Proof. (1) $\Longrightarrow$ (2) : Let F be a $\delta - closed$ and $x \notin F$. Thus $(X - F)$ is a $\delta - open$ set containing x. Since (X,τ) $\delta - R_0$. $Cl_\delta(\{x\}) \subseteq (X - F)$. Thus $Cl_\delta(\{x\}) \cap F = \phi$ and by Lemma 3.11 $x \notin Ker_\delta(F)$. Therefore $Ker_\delta(F) = F$.

(2) $\Longrightarrow$ (3) : In general, $A \subseteq B$ implies $Ker_\delta(A) \subseteq Cl_\delta(B)$. Therefore, it follows from (2) that $Ker_\delta(\{x\}) \subseteq Ker_\delta(F) = F$.

(3) $\Longrightarrow$ (4) : Since $x \in Cl_\delta(\{x\})$ and $Cl_\delta(\{x\})$ is $\delta - closed$, by (3), $Ker_\delta(\{x\}) \subseteq Cl_\delta(\{x\})$.

(4) $\Longrightarrow$ (1) : We show the implication by using Theorem 3.19. Let $x \in Cl_\delta(\{y\})$. Then by Lemma 3.13, $y \in Ker_\delta(\{x\})$. Since $x \in Cl_\delta(\{x\})$ and $Cl_\delta(\{x\})$ is $\delta - closed$, by (4) we obtain $y \in Ker_\delta(\{x\}) \subseteq Cl_\delta(\{x\})$. Therefore $x \in Cl_\theta(\{y\})$ implies $y \in Cl_\theta(\{x\})$. The converse is obvious and (X,τ) is $\delta - R_0$.

Theorem 3.21. Let (X,τ) be a topological space. Then $\cap\{Cl_\delta(\{x\})\,|x\in X\}=\phi$ if and only if $Ker_\delta(\{x\})\neq X$ for every $x\in X$.

Proof. **Necessity**. Suppose that $\cap\{Cl_\delta(\{x\})\,|x\in X\}=\phi$. Assume that there is a point y in X such that $Ker_\delta(\{y\})=X$. Then $y\notin O$, where O is some proper $\delta-open$ subset of X. This implies that $y\in\cap\{Cl_\delta(\{x\})\,|x\in X\}$. But this is a contradiction.

Sufficiency. Assume that $Ker_\delta(\{x\})\neq X$ for every $x\in X$. If there exists a point $y\in X$ such that $y\in\cap\{Cl_\delta(\{x\})\,|x\in X\}$, then every $\delta-open$ set containing y must contain every point of X. This implies that the space X is the unique $\delta-open$ set containing y. Hence $Ker_\delta(\{x\})=X$ which is a contradiction. Therefore, $\cap\{Cl_\delta(\{x\})\,|x\in X\}=\phi$.

Definition 3.22. A filter base $\mathcal{F}$ is called $\delta-convergent$ to a point x in X, if for any $\delta-open$ set U of X containing x, there exists B in $\mathcal{F}$ such that B is a subset of U.

Lemma 3.23. Let (X,τ) be a topological space and x and y be any two points in X such that every net in X $\delta-converging$ to y $\delta-converges$ to x. Then $x\in Cl_\delta(\{y\})$.

Proof. Suppose that $x_\alpha=y$ for $\alpha\in I$. Then $\{x_\alpha:\alpha\in I\}$ is a net in $Cl_\delta(\{y\})$. Since $\{x_\alpha:\alpha\in I\}$ $\delta-converges$ to y, so $\{x_\alpha:\alpha\in I\}$ $\delta-converges$ to x and this implies that $x\in Cl_\delta(\{y\})$.

Theorem 3.24. For a topological space (X,τ), the following statements are equivalent:

(1) (X,τ) is $\delta-R_0$ space;

(2) If $x,y\in X$, then $y\in Cl_\delta(\{x\})$ if and only if every net in X $\delta-converging$ to y $\delta-converges$ to x.

Proof. (1) $\Longrightarrow$ (2) : Let $x,y\in X$ such that $y\in Cl_\delta(\{x\})$. Suppose that $\{x_\alpha:\alpha\in I\}$ is a net in X such that this net $\delta-converges$ to y. Since $y\in$

$Cl_\delta(\{x\})$ so by Theorem 3.15 we have $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$. Therefore $x \in Cl_\delta(\{y\})$. This means that the net $\{x_\alpha : \alpha \in I\}$ $\delta - converges$ to x.

Conversely, let $x, y \in X$ such that every net in X $\delta - converging$ to y $\delta - converges$ to x. Then $x \in Cl_\delta(\{y\})$ by Lemma 3.23. By Theorem 3.15, we have $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$. Therefore $y \in Cl_\delta(\{x\})$.

(2) $\Longrightarrow$ (1) : Assume that x and y are any two points of X such that $Cl_\delta(\{x\}) \cap Cl_\delta(\{y\}) \neq \phi$. Let $z \in Cl_\delta(\{x\}) \cap Cl_\delta(\{y\})$. So there exists a net $\{x_\alpha : \alpha \in I\}$ in $Cl_\delta(\{x\})$ such that $\{x_\alpha : \alpha \in I\}$ $\delta - converges$ to z. Since $z \in Cl_\delta(\{y\})$. So by hypothesis $\{x_\alpha : \alpha \in I\}$ $\delta - converges$ to y. It follows that $y \in Cl_\delta(\{x\})$. Similarly we obtain $x \in Cl_\delta(\{y\})$. Therefore $Cl_\delta(\{x\}) = Cl_\delta(\{y\})$ and by Theorem 3.15, (X, τ) is $\delta - R_0$.

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