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Characterizations of Mappings in $\theta-$Open Sets*

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Abstract

Velicko [1968] introduced the concepts of $\theta - closure$ and $\theta - interior$ operations. The collection of all $\theta - open$ sets in a topological soace X forms a topology on X. In this paper we introduce $\theta - irresolute$, $\theta - closed$, $pre - \theta - open$ and $pre - \theta -$ $closed$ mappings and investigate properties and characterizations of these new types of mappings and also explore further properties of the well-known notions of $\theta - continuous$ and $\theta - open$ mappings.

1 Introduction

The notions of $\theta - open$ subsets, $\theta - closed$ subsets and $\theta - closure$ were introduced by Velicko [16] for the purpose of studying the important class of $H - closed$ spaces in terms of arbitrary filterbases. Dickman and Porter [2], [3], Joseph[9] and [14] continued the work of Velicko. Recently Noiri and Jafari [15] and Jafari [6] have also obtained several new and interesting results related to these sets. In what follows (X, τ) $(or\ X)$ denotes topological spaces on which no separation axioms are assumed unless explicitly stated. We denote the interior and the closure of a subset A of X by $Int\,(A)$ and $Cl\,(A)$, respectively. A

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point $x \in X$ is called a $\theta - adherent$ point of A [4], if $A \cap Cl(A) \neq \phi$ for every open set V containing x. The set of all $\theta - adherent$ points of A is called the $\theta - closure$ of A and is denoted by $Cl_\theta(A)$. A subset A of X is called $\theta - closed$ if $A = Cl_\theta(A)$. Dontchev and Maki [[4], Lemma 3.9] have shown that if A and B are subsets of a space (X, τ), then $Cl_\theta(A \cup B) = Cl_\theta(A) \cup Cl_\theta(B)$ and $Cl_\theta(A \cap B) = Cl_\theta(A) \cap Cl_\theta(B)$. Note also that the $\theta - closure$ of a given set need not be a $\theta - closed$ set. But it is always closed. The complement of a $\theta - closed$ set is called a $\theta - open$ set. The $\theta - interior$ of set A in X, written $Int_\theta(A)$, consists of those points x of A such that for some open set U containing x, $Cl(U) \subseteq A$. A set A is $\theta - open$ if and only if $A = Int_\theta(A)$, or equivalently, $X - A$ is $\theta - closed$. The collection of all $\theta - open$ sets in a topological space (X, τ) forms a topology τ_θ on X, weaker than τ. In this paper we continue to explore further properties and characterizations of $\theta - continuous$ and $\theta - open$ mappings. We also introduce $\theta - irresolute$, $\theta - closed$, $pre - \theta - open$ and $pre - \theta - closed$ mappings and investigate properties and characterizations of these new types of mappings.

2 Characterizations of Mappings

The purpose of this section is to explore properties and characterizations of $\theta - continuous$, $\theta - irresolute$, $\theta - open$, $\theta - closed$, $pre - \theta - open$ and $pre - \theta - closed$ functions.

A. θ–Continuous Functions

The purpose of this section is to investigate further properties and characterizations of $\theta - continuous$ functions.

Definition 2.1. A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be $\theta - continuous$ if $f^{-1}(V) \in \tau_\theta$ for every $V \in \sigma$.

Theorem 2.2. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function. Then the following are equivalent:

(1) f is $\theta - continuous$;

(2) The inverse image of each closed set in Y is a $\theta - closed$ set in X;

(3) $Cl_\theta\left[f^{-1}(V)\right] \subseteq f^{-1}[Cl(V)]$, for every $V \subseteq Y$;

(4) $f[Cl_\theta(U)] \subseteq Cl[f(U)]$, for every $U \subseteq X$;

(5) For any point $x \in X$ and any open set V of Y containing $f(x)$, there exists $U \in \tau_\theta$ such that $x \in U$ and $f(U) \subseteq V$;

(6) $Bd_\theta\left[f^{-1}(V)\right] \subseteq f^{-1}[Bd(V)]$, for every $V \subseteq Y$;

(7) $f[D_\theta(U)] \subseteq Cl[f(U)]$, for every $U \subseteq X$;

(8) $f^{-1}[Int(V)] \subseteq Int_\theta\left[f^{-1}(V)\right]$, for every $V \subseteq Y$.

Proof. (1) $\Longrightarrow$ (2) : Let $F \subseteq Y$ be closed. Since f is $\theta-$*continuous*, $f^{-1}(Y-F) = X - f^{-1}(F)$ is $\theta-$*open*. Therefore, $f^{-1}(F)$ is $\theta-$*closed* in X.

(2) $\Longrightarrow$ (3) : Since $Cl(V)$ is closed for every $V \subseteq Y$, then $f^{-1}[Cl(V)]$ is $\theta-$*closed*. Therefore $f^{-1}[Cl(V)] = Cl_\theta\left[f^{-1}(Cl(V))\right] \supseteq Cl_\theta\left[f^{-1}(V)\right]$.

(3) $\Longrightarrow$ (4) : Let $U \subseteq X$ and $f(U) = V$. Then $f^{-1}[Cl(V)] \supseteq Cl_\theta\left[f^{-1}(V)\right]$. Thus $f^{-1}[Cl(f(U))] \supseteq Cl_\theta\left[f^{-1}(f(U))\right] \supseteq Cl_\theta(U)$ and $Cl[f(U)] \supseteq f[Cl_\theta(U)]$.

(4) $\Longrightarrow$ (2) : Let $W \subseteq Y$ be a closed set, and $U = f^{-1}(W)$, then $f[Cl_\theta(U)] \subseteq Cl[f(U)] = Cl\left[f\left(f^{-1}(W)\right)\right] \subseteq Cl(W) = W$. Thus $Cl_\theta(U) \subseteq f^{-1}[f(Cl_\theta(U))] \subseteq f^{-1}(W) = U$. So U is $\theta-$*closed*.

(2) $\Longrightarrow$ (1) : Let $V \subseteq Y$ be an open set, then $Y-V$ is *closed*. Then $f^{-1}(Y-V) = X - f^{-1}(V)$ is $\theta-$*closed* in X and hence $f^{-1}(V)$ is $\theta-$*open* in X.

(1) $\Longrightarrow$ (5) : Let $f : X \longrightarrow Y$ be $\theta-$*continuous*. For any $x \in X$ and any open set V of Y containing $f(x)$, $U = f^{-1}(V) \in \tau_\theta$, and $f(U) = f\left[f^{-1}(V)\right] \subseteq V$.

(5) $\Longrightarrow$ (1) : Let $V \in \sigma$. We prove $f^{-1}(V) \in \tau_\theta$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists $U \in \tau_\theta$ such that $x \in U$ and $f(x) \in f(U) \subseteq V$. Hence $x \in U \subseteq f^{-1}[f(U)] \subseteq f^{-1}(V)$. It shows that $f^{-1}(V)$ is a $\theta-$*neighborhood* of each of its points. Therefore $f^{-1}(V) \in \tau_\theta$.

(6) $\Longrightarrow$ (8) : Let $V \subseteq Y$. Then by hypothesis, $Bd_\theta\left[f^{-1}(V)\right] \subseteq f^{-1}[Bd(V)]$
$\Longrightarrow f^{-1}(V) - Int_\theta\left[f^{-1}(V)\right] \subseteq f^{-1}[V - Int(V)] = f^{-1}(V) - f^{-1}[Int(V)]$
$\Longrightarrow f^{-1}[Int(V)] \subseteq Int_\theta\left[f^{-1}(V)\right]$.

(8) $\Longrightarrow$ (6) : Let $V \subseteq Y$. Then by hypothesis, $f^{-1}[Int(V)] \subseteq Int_\theta\left[f^{-1}(V)\right]$
$\Longrightarrow f^{-1}(V) - Int_\theta\left[f^{-1}(V)\right] \subseteq f^{-1}(V) - f^{-1}[Int(V)] = f^{-1}[V - Int(V)]$

$\Longrightarrow Bd_{\theta}\left[f^{-1}(V)\right] \subseteq f^{-1}[Bd(V)]$.

(1) $\Longrightarrow$ (7) : It is obvious, since f is $\theta-continuous$ and by (4) $f[Cl_{\theta}(U)] \subseteq Cl[f(U)]$ for each $U \subseteq X$. So $f[D_{\theta}(U)] \subseteq Cl[f(U)]$.

(7) $\Longrightarrow$ (1) : Let $U \subseteq Y$ be an open set, $V = Y - U$ and $f^{-1}(V) = W$. Then by hypothesis $f[D_{\theta}(W)] \subseteq Cl[f(W)]$. Thus $f\left[D_{\theta}\left(f^{-1}(V)\right)\right] \subseteq Cl\left[f\left(f^{-1}(V)\right)\right] \subseteq Cl(V) = V$. Then $D_{\theta}\left[f^{-1}(V)\right] \subseteq f^{-1}(V)$ and $f^{-1}(V)$ is $\theta - closed$. Therefore, f is $\theta - continuous$.

(1) $\Longrightarrow$ (8) : Let $V \subseteq Y$. Then $f^{-1}[Int(V)]$ is $\theta - open$ in X. Thus $f^{-1}[Int(V)] = Int_{\theta}\left[f^{-1}(Int(V))\right] \subseteq Int_{\theta}\left[f^{-1}(V)\right]$. Therefore, $f^{-1}[Int(V)] \subseteq Int_{\theta}\left[f^{-1}(V)\right]$.

(8) $\Longrightarrow$ (1) : Let $V \subseteq Y$ be an open set. Then $f^{-1}(V) = f^{-1}[Int(V)] \subseteq Int_{\theta}\left[f^{-1}(V)\right]$. Therefore, $f^{-1}(V)$ is $\theta - open$. Hence f is $\theta - continuous$.

In the next Theorem, $\#\theta - c.$ denotes the set of points x of X for which a function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is not $\theta - continuous$.

Theorem 2.3. $\#\theta - c.$ is identical with the union of the $\theta - frontiers$ of the inverse images of $\theta - open$ sets containing $f(x)$.

Proof. Suppose that f is not $\theta - continuous$ at a point x of X. Then there exists an open set $V \subseteq Y$ containing $f(x)$ such that $f(U)$ is not a subset of V for every $U \in \tau_{\theta}$ containing x. Hence, we have $U \cap \left[X - f^{-1}(V)\right] \neq \phi$ for every $U \in \tau_{\theta}$ containing x. It follows that $x \in Cl_{\theta}\left[X - f^{-1}(V)\right]$. We also have $x \in f^{-1}(V) \subseteq Cl_{\theta}\left[f^{-1}(V)\right]$. This means that $x \in Fr_{\theta}\left(f^{-1}(V)\right)$.

Now, let f be $\theta - continuous$ at $x \in X$ and $V \subseteq Y$ any open set containing $f(x)$. Then, $x \in f^{-1}(V)$ is a $\theta - open$ set of X. Thus, $x \in Int_{\theta}\left[f^{-1}(V)\right]$ and therefore $x \notin Fr_{\theta}\left[f^{-1}(V)\right]$ for every *open* set V containing $f(x)$.

Remarks. 2.4. (1) Every $\theta - continuous$ function is continuous but the converse may not be true.

(2) If a function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is $\theta - continuous$ and a function $g : (Y, \sigma) \longrightarrow (Z, \vartheta)$ is $\theta - continuous$, then $g \circ f : (X, \tau) \longrightarrow (Z, \vartheta)$ is $\theta - continuous$.

(3) If a function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is $\theta - continuous$ and a function $g : (Y, \sigma) \longrightarrow (Z, \vartheta)$ is *continuous*, then $g \circ f : (X, \tau) \longrightarrow (Z, \vartheta)$ is $\theta - continuous$.

(4) Let (X, τ) and (Y, σ) be topological spaces. If $f : X \longrightarrow Y$ is a function, and one of the following

(a) $f^{-1}[Int(B)] \subseteq Int_{\theta}\left[f^{-1}(B)\right]$ for each $B \subseteq Y$,

(b) $Cl_{\theta}\left[f^{-1}(B)\right] \subseteq f^{-1}[Cl(B)]$ for each $B \subseteq Y$,

(c) $f[Cl_{\theta}(A)] \subseteq Cl[f(A)]$ for each $A \subseteq X$.

holds, then f is continuous.

Lemma. 2.5. Let $A \subseteq Y \subseteq X$, Y is $\theta-open$ in X and A is $\theta-open$ in Y. Then A is $\theta-open$ in X.

Proof. Since A is $\theta-open$ in Y, there exists a $\theta-open$ set $U \subseteq X$ such that $A = Y \cap U$. Thus A being the intersection of two $\theta-open$ sets in X, is $\theta-open$ in X.

Theorem. 2.6. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a mapping and $\{U_i : i \in I\}$ be a cover of X such that $U_i \in \tau_{\theta}$ for each $i \in I$. Suppose that $f\,|U_i : U_i \longrightarrow Y$ is $\theta-continuous$ for each $i \in I$. Then prove that f is $\theta-continuous$.

Proof. Let $V \subseteq Y$ be an open set, then $(f\,|U_i)^{-1}(V)$ is $\theta-open$ in U_i for each $i \in I$. Since U_i is $\theta-open$ in X for each $i \in I$. So by Lemma 2.5, $(f\,|U_i)^{-1}(V)$ is $\theta-open$ in X for each $i \in I$. But, $f^{-1}(V) = \cup\left\{(f\,|U_i)^{-1}(V) : i \in I\right\}$, then $f^{-1}(V) \in \tau_{\theta}$ because τ_{θ} is a topology on X. This implies that f is $\theta-continuous$.

B. θ–Irresolute Functions

In this section, the functions to be considered are those for which inverses of $\theta-open$ sets are $\theta-open$. We investigate some properties and characterizations of such functions.

Definition. 2.7. Let (X, τ) and (Y, σ) be topological spaces. A function $f : X \longrightarrow Y$ is called $\theta-irresolute$ if the inverse image of each $\theta-open$ set of Y is a $\theta-open$ set in X.

Theorem 2.8. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function between topological spaces. Then the following are equivalent:

(1) f is $\theta-irresolute$;

(2) the inverse image of each $\theta-closed$ set in Y is a $\theta-closed$ set in X;

(3) $Cl_\theta\left[f^{-1}(V)\right] \subseteq f^{-1}\left[Cl_\theta(V)\right]$ for every $V \subseteq Y$;

(4) $f\left[Cl_\theta(U)\right] \subseteq Cl_\theta\left[f(U)\right]$ for every $U \subseteq X$;

(5) $f^{-1}\left[Int_\theta(B)\right] \subseteq Int_\theta\left[f^{-1}(B)\right]$ for every $B \subseteq Y$.

Theorem 2.9. Prove that a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is $\theta-irresolute$ if and only if for each point p in X and each $\theta-open$ set B in Y with $f(p) \in B$, there is a $\theta-open$ set A in X such that $p \in A$, $f(A) \subseteq B$.

Proof. **Necessity**. Let $p \in X$ and $B \in \sigma_\theta$ such that $f(p) \in B$. Let $A = f^{-1}(B)$. Since f is $\theta-irresolute$, A is $\theta-open$ in X. Also $p \in f^{-1}(B) = A$ as $f(p) \in B$. Thus we have $f(A) = f\left[f^{-1}(B)\right] \subseteq B$.

Sufficiency. Let $B \in \sigma_\theta$, let $A = f^{-1}(B)$. We show that A is $\theta-open$ in X. For this let $x \in A$. It implies that $f(x) \in B$. Then by hypothesis, there exists $A_x \in \tau_\theta$ such that $x \in A_x$ and $f(A_x) \subseteq B$. Then $A_x \subseteq f^{-1}\left[f(A_x)\right] \subseteq f^{-1}(B) = A$. Thus $A = \cup\{A_x : x \in A\}$. It follows that A is $\theta-open$ in X. Hence f is $\theta-irresolute$.

Definition. 2.10. Let (X,τ) be a topological space. Let $x \in X$ and $N \subseteq X$. We say that N is a $\theta-neighborhood$ of x if there exists a $\theta-open$ set M of X such that $x \in M \subseteq N$.

Theorem 2.11. Prove that a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is $\theta-irresolute$ if and only if for each x in X, the inverse image of every $\theta-neighborhood$ of $f(x)$, is a $\theta-neighborhood$ of x.

Proof. **Necessity**. Let $x \in X$ and let B be a $\theta-neighborhood$ of $f(x)$. Then there exists $U \in \sigma_\theta$ such that $f(x) \in U \subseteq B$. This implies that $x \in f^{-1}(U) \subseteq f^{-1}(B)$. Since f is $\theta-irresolute$, so $f^{-1}(U) \in \tau_\theta$. Hence $f^{-1}(B)$ is a $\theta-neighborhood$ of x.

Sufficiency. Let $B \in \sigma_\theta$. Put $A = f^{-1}(B)$. Let $x \in A$. Then $f(x) \in B$. But then, B being $\theta-open$ set, is a $\theta-neighborhood$ of $f(x)$. So by hypothesis, $A = f^{-1}(B)$ is a $\theta-neighborhood$ of x. Hence by definition, there exists $A_x \in \tau_\theta$ such that $x \in A_x \subseteq A$. Thus $A = \cup\{A_x : x \in A\}$. It follows that A is a $\theta-open$ set in X. Therefore f is $\theta-irresolute$.

Theorem 2.12. Prove that a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is $\theta-irresolute$

if and only if for each x in X, and each $\theta-neighborhood$ U of $f(x)$, there is a $\theta-neighborhood$ V of x such that $f(V)\subseteq U$.

Proof. **Necessity**. Let $x\in X$ and let U be a $\theta-neighborhood$ of $f(x)$. Then there exists $O_{f(x)}\in\sigma_\theta$ such that $f(x)\in O_{f(x)}\subseteq U$. It follows that $x\in f^{-1}\left[O_{f(x)}\right]\subseteq f^{-1}(U)$. By hypothesis, $f^{-1}\left[O_{f(x)}\right]\in\tau_\theta$. Let $V=f^{-1}(U)$. Then it follows that V is a $\theta-neighborhood$ of x and $f(V)=f\left[f^{-1}(U)\right]\subseteq U$.

Sufficiency. Let $B\in\sigma_\theta$. Put $O=f^{-1}(B)$. Let $x\in O$. Then $f(x)\in B$. Thus B is a $\theta-neighborhood$ of $f(x)$. So by hypothesis, there exists a $\theta-neighborhood$ V_x of x such that $f(V_x)\subseteq B$. Thus it follows that $x\in V_x\subseteq f^{-1}[f(V_x)]\subseteq f^{-1}(B)=O$. Since V_x is a $\theta-neighborhood$ of x, so there exists an $O_x\in\tau_\theta$ such that $x\in O_x\subseteq V_x$. Hence $x\in O_x\subseteq O$, $O_x\in\tau_\theta$. Thus $O=\cup\{O_x:x\in O\}$. It follows that O is $\theta-open$ in X. Therefore, f is $\theta-irresolute$.

Theorem 2.13. Prove that a function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is $\theta-irresolute$ if and only if $f[D_\theta(A)]\subseteq f(A)\cup D_\theta[f(A)]$, for all $A\subseteq X$.

Proof. **Necessity**. Let $f:X\longrightarrow Y$ be $\theta-irresolute$. Let $A\subseteq X$, and $a_0\in D_\theta(A)$. Assume that $f(a_0)\notin f(A)$ and let V denote a $\theta-neighborhood$ of $f(a_0)$. Since f is $\theta-irresolute$, so by Theorem 2.12, there exists a $\theta-neighborhood$ U of a_0 such that $f(U)\subseteq V$. From $a_0\in D_\theta(A)$, it follows that $U\cap A\neq\phi$; there exists, therefore, at least one element $a\in U\cap A$ such that $f(a)\in f(A)$ and $f(a)\in V$. Since $f(a_0)\notin f(A)$, we have $f(a)\neq f(a_0)$. Thus every $\theta-neighborhood$ of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$, consequently, $f(a_0)\in D_\theta[f(A)]$. This proves necessity of the condition.

Sufficiency. Assume that f is not $\theta-irresolute$. Then by Theorem 2.12, there exists $a_0\in X$ and a $\theta-neighborhood$ V of $f(a_0)$ such that every $\theta-neighborhood$ U of a_0 contains at least one element $a\in U$ for which $f(a)\notin V$. Put $A=\{a\in X:f(a)\notin V\}$. Then $a_0\notin A$ since $f(a_0)\in V$, and therefore $f(a_0)\notin f(A)$; also $f(a_0)\notin D_\theta[f(A)]$ since $f(A)\cap(V-\{f(a_0)\})=\phi$. It follows that $f(a_0)\in f[D_\theta(A)]-[f(A)\cup D_\theta(f(A))]\neq\phi$, which is a contradiction to the given condition. The condition of the Theorem is therefore sufficient and the theorem is proved.

Theorem 2.14. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be a $one-to-one$ function. Then f is $\theta-irresolute$ if and only if $f\left[D_\theta(A)\right]\subseteq D_\theta\left[f(A)\right]$, for all $A\subseteq X$.

Proof. **Necessity**. Let f be $\theta-$ $irresolute$. Let $A\subseteq X$, $a_0\in D_\theta(A)$ and V be a $\theta-neighborhood$ of $f(a_0)$. Since f is $\theta-irresolute$, so by Theorem 2.12, there exists a $\theta-neighborhood$ U of a_0 such that $f(U)\subseteq V$. But $a_0\in D_\theta(A)$; hence there exists an element $a\in U\cap A$ such that $a\neq a_0$; then $f(a)\in f(A)$ and, since f is $1-1$, $f(a)\neq f(a_0)$. Thus every $\theta-neighborhood$ V of $f(a_0)$ contains an element of $f(A)$ different from $f(a_0)$; consequently $f(a_0)\in D_\theta\left[f(A)\right]$. We have therefore $f\left[D_\theta(A)\right]\subseteq D_\theta\left[f(A)\right]$.

Sufficiency. Follows from Theorem 2.13.

C. $\theta-$Open Functions

The purpose of this section is to investigate some characterizations of $\theta-open$ mappings.

Definition 2.15. Let (X,τ) and (Y,σ) be topological spaces. A function $f:X\longrightarrow Y$ is called $\theta-open$ if for every open set G in X, $f(G)$ is a $\theta-open$ set in Y.

Theorem 2.16. Prove that a mapping $f:(X,\tau)\longrightarrow(Y,\sigma)$ is $\theta-open$ if and only if for each $x\in X$, and $U\in\tau$ such that $x\in U$, there exists a $\theta-open$ set $W\subseteq Y$ containing $f(x)$ such that $W\subseteq f(U)$.

Proof. Follows immediately from Definition 2.15.

Theorem. 2.17. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be $\theta-open$. If $W\subseteq Y$ and $F\subseteq X$ is a closed set containing $f^{-1}(W)$, then there exists a $\theta-closed$ $H\subseteq Y$ containing W such that $f^{-1}(H)\subseteq F$.

Proof. Let $H=Y-f(X-F)$. Since $f^{-1}(W)\subseteq F$, we have $f(X-F)\subseteq(Y-W)$. Since f is $\theta-open$, then H is $\theta-closed$ and $f^{-1}(H)=X-f^{-1}\left[f(X-F)\right]\subseteq X-(X-F)=F$.

Theorem 2.18. Let $f:(X,\tau)\longrightarrow(Y,\sigma)$ be $\theta-open$ and let $B\subseteq Y$. Then $f^{-1}\left[Cl_\theta\left(Int_\theta\left(Cl_\theta(B)\right)\right)\right]\subseteq Cl\left[f^{-1}(B)\right]$.

Proof. $Cl\left[f^{-1}(B)\right]$ is closed in X containing $f^{-1}(B)$. By Theorem 2.17, there exists a $\theta-closed$ set $B \subseteq H \subseteq Y$, such that $f^{-1}(H) \subseteq Cl\left[f^{-1}(B)\right]$. Thus, $f^{-1}[Cl_\theta(Int_\theta(Cl_\theta(B)))] \subseteq f^{-1}[Cl_\theta(Int_\theta(Cl_\theta(H)))]$

$\subseteq f^{-1}(H) \subseteq Cl\left[f^{-1}(B)\right]$.

Theorem 2.19. Prove that a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is $\theta-open$ if and only if $f[Int(A)] \subseteq Int_\theta[f(A)]$, for all $A \subseteq X$.

Proof. **Necessity**. Let $A \subseteq X$. Let $x \in Int(A)$. Then there exists $U_x \in \tau$ such that $x \in U_x \subseteq A$. So $f(x) \in f(U_x) \subseteq f(A)$ and by hypothesis, $f(U_x) \in \sigma_\theta$. Hence $f(x) \in Int_\theta[f(A)]$. Thus $f[Int(A)] \subseteq Int_\theta[f(A)]$.

Sufficiency. Let $U \in \tau$. Then by hypothesis, $f[Int(U)] \subseteq Int_\theta[f(U)]$. Since $Int(U) = U$ as U is open. Also $Int_\theta[f(U)] \subseteq f(U)$. Hence $f(U) = Int_\theta[f(U)]$. Thus $f(U)$ is $\theta-open$ in Y. So f is $\theta-open$.

Remark 2.20. The equality may not hold in the preceding Theorem.

Theorem 2.21. Prove that a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is $\theta-open$ if and only if $Int\left[f^{-1}(B)\right] \subseteq f^{-1}[Int_\theta(B)]$, for all $B \subseteq Y$.

Proof. **Necessity**. Let $B \subseteq Y$. Since $Int\left[f^{-1}(B)\right]$ is open in X and f is $\theta-open$, $f\left[Int\left(f^{-1}(B)\right)\right]$ is $\theta-open$ in Y. Also we have $f\left[Int\left(f^{-1}(B)\right)\right] \subseteq f\left[f^{-1}(B)\right] \subseteq B$. Hence, $f\left[Int\left(f^{-1}(B)\right)\right] \subseteq Int_\theta(B)$. Therefore $Int\left[f^{-1}(B)\right] \subseteq f^{-1}[Int_\theta(B)]$.

Sufficiency. Let $A \subseteq X$. Then $f(A) \subseteq Y$. Hence by hypothesis, we obtain $Int(A) \subseteq Int\left[f^{-1}(f(A))\right] \subseteq f^{-1}[Int_\theta(f(A))]$. Thus $f[Int(A)] \subseteq Int_\theta[f(A)]$, for all $A \subseteq X$. Hence, by Theorem 2.19, f is $\theta-open$.

Theorem 2.22. Let $f:(X,\tau) \longrightarrow (Y,\sigma)$ be a mapping. Then a necessary and sufficient condition for f to be $\theta-open$ is that $f^{-1}[Cl_\theta(B)] \subseteq Cl\left[f^{-1}(B)\right]$ for every subset B of Y.

Proof. **Necessity**. Assume f is $\theta-open$. Let $B \subseteq Y$. Let $x \in f^{-1}[Cl_\theta(B)]$. Then $f(x) \in Cl_\theta(B)$. Let $U \in \tau$ such that $x \in U$. Since f is $\theta-open$, then $f(U)$ is a $\theta-open$ set in Y. Therefore, $B \cap f(U) \neq \phi$. Then $U \cap f^{-1}(B) \neq \phi$. Hence $x \in Cl\left[f^{-1}(B)\right]$. We conclude that $f^{-1}[Cl_\theta(B)] \subseteq Cl\left[f^{-1}(B)\right]$.

Sufficiency. Let $B \subseteq Y$. Then $(Y-B) \subseteq Y$. By hypothesis, $f^{-1}[Cl_\theta(Y-B)] \subseteq Cl\left[f^{-1}(Y-B)\right]$. This implies $X - Cl\left[f^{-1}(Y-B)\right] \subseteq X - f^{-1}[Cl_\theta(Y-B)]$.

Hence $X - Cl\left[X - f^{-1}(B)\right] \subseteq f^{-1}\left[Y - Cl_{\theta}(Y - B)\right]$. By applying Theorem 10 [12], $Int\left[f^{-1}(B)\right] \subseteq f^{-1}\left[Int_{\theta}(B)\right]$. Now form Theorem 2.21, it follows that f is $\theta - open$.

D. $\theta-$ Closed Functions

In this section we introduce $\theta - closed$ functions and study certain properties and characterizations of this type of functions.

Definition 2.23. A mapping $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called $\theta - closed$ if the image of each closed set in X is a $\theta - closed$ set in Y.

Theorem 2.24. Prove that a mapping $f : (X, \tau) \longrightarrow (Y, \sigma)$ is $\theta - closed$ if and only $Cl_{\theta}[f(A)] \subseteq f[Cl(A)]$ for each $A \subseteq X$.

Proof. **Necessity**. Let f be $\theta - closed$ and let $A \subseteq X$. Then $f(A) \subseteq f[Cl(A)]$ and $f[Cl(A)]$ is a $\theta - closed$ set in Y. Thus $Cl_{\theta}[f(A)] \subseteq f[Cl(A)]$.

Sufficiency. suppose that $Cl_{\theta}[f(A)] \subseteq f[Cl(A)]$, for each $A \subseteq X$. Let $A \subseteq X$ be a closed set. Then $Cl_{\theta}[f(A)] \subseteq f[Cl(A)] = f(A)$. This shows that $f(A)$ is a $\theta - closed$ set. Hence f is $\theta - closed$.

Theorem 2.25. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be $\theta - closed$. If $V \subseteq Y$ and $E \subseteq X$ is an open set containing $f^{-1}(V)$, then there exists a $\theta - open$ set $G \subseteq Y$ containing V such that $f^{-1}(G) \subseteq E$.

Proof. Let $G = Y - f(X - E)$. Since $f^{-1}(V) \subseteq E$, we have $f(X - E) \subseteq Y - V$. Since f is $\theta - closed$, then G is a $\theta - open$ set and $f^{-1}(G) = X - f^{-1}[f(X - E)] \subseteq X - (X - E) = E$.

Theorem 2.26. Suppose that $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a $\theta - closed$ mapping. Then $Int_{\theta}[Cl_{\theta}(f(A))] \subseteq f[Cl(A)]$ for every subset A of X.

Proof. Suppose f is a $\theta - closed$ mapping and A is an arbitrary subset of X. Then $f[Cl(A)]$ is $\theta - closed$ in Y. Then $Int_{\theta}[Cl_{\theta}(f(Cl(A)))] \subseteq f[Cl(A)]$. But also $Int_{\theta}[Cl_{\theta}(f(A))] \subseteq Int_{\theta}[Cl_{\theta}(f(Cl(A)))]$. Hence $Int_{\theta}[Cl_{\theta}(f(A))] \subseteq f[Cl(A)]$.

Theorem 2.27. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a $\theta - closed$ function, and $B, C \subseteq Y$.

Proof. (1) If U is an open neighborhood of $f^{-1}(B)$, then there exists a

$\theta - open$ neighborhood V of B such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

(2) If f is also onto, then if $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint open neighborhoods, so have B and C.

Proof. (1) Let $V = Y - f(X - U)$. Then $V^c = Y - V = f(U^c)$. Since f is $\theta - closed$, so V is a $\theta - open$ set. Since $f^{-1}(B) \subseteq U$, we have $V^c = f(U^c) \subseteq f\left[f^{-1}(B^c)\right] \subseteq B^c$. Hence, $B \subseteq V$, and thus V is a $\theta - open$ neighborhood of B. Further $U^c \subseteq f^{-1}[f(U^c)] = f^{-1}(V^c) = \left[f^{-1}(V)\right]^c$. This proves that $f^{-1}(V) \subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint open neighborhoods M and N, then by (1), we have $\theta - open$ neighborhoods U and V of B and C respectively such that $f^{-1}(B) \subseteq f^{-1}(U) \subseteq Int_\theta(M)$ and $f^{-1}(C) \subseteq f^{-1}(V) \subseteq Int_\theta(N)$. Since M and N are disjoint, so are $Int_\theta(M)$ and $Int_\theta(N)$, and hence so $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint as well. It follows that U and V are disjoint too as f is onto.

Theorem 2.28. Prove that a surjective mapping $f : (X, \tau) \longrightarrow (Y, \sigma)$ is $\theta - closed$ if and only if for each subset B of Y and each open set U in X containing $f^{-1}(B)$, there exists a $\theta - open$ set V in Y containing B such that $f^{-1}(V) \subseteq U$.

Proof. **Necessity**. This follows from (1) of Theorem 2.27. **Sufficiency**. Suppose F is an arbitrary closed set in X. Let y be an arbitrary point in $Y - f(F)$. Then $f^{-1}(y) \subseteq X - f^{-1}[f(F)] \subseteq (X - F)$ and $(X - F)$ is open in X. Hence by hypothesis, there exists a $\theta - open$ set V_y containing y such that $f^{-1}(V_y) \subseteq (X - F)$. This implies that $y \in V_y \subseteq [Y - f(F)]$. Thus $Y - f(F) = \cup \{V_y : y \in Y - f(F)\}$. Hence $Y - f(F)$, being a union of $\theta - open$ sets, is $\theta - open$. Thus its complement $f(F)$ is $\theta - closed$. This shows that f is $\theta - closed$.

Theorem 2.29. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a bijection. Then the following are equivalent:

(a) f is $\theta - closed$.

(b) f is $\theta - open$.

(c) f^{-1} is $\theta - continuous$.

Proof. $(a) \Longrightarrow (b)$: Let $U \in \tau$. Then $X - U$ is closed in X. By (a), $f(X-U)$ is $\theta - closed$ in Y. But $f(X-U) = f(X) - f(U) = Y - f(U)$. Thus $f(U)$ is $\theta - open$ in Y. This shows that f is $\theta - open$.

$(b) \Longrightarrow (c)$: Let $U \subseteq X$ be an open set. Since f is $\theta - open$. So $f(U) = \left(f^{-1}\right)^{-1}(U)$ is $\theta - open$ in Y. Hence $f^{-1} is\ \theta - continuous$.

$(c) \Longrightarrow (a)$: Let A be an arbitrary closed set in X. Then $X - A$ is open in X. Since f^{-1} is $\theta - continuous$, $\left(f^{-1}\right)^{-1}(X-A)$ is $\theta - open$ in Y. But $\left(f^{-1}\right)^{-1}(X-A) = f(X-A) = Y - f(A)$. Thus $f(A)$ is $\theta - closed$ in Y. This shows that f is $\theta - closed$.

Remark 2.30. A bijection $f : (X,\tau) \longrightarrow (Y,\sigma)$ may be open and closed but neither $\theta - open$ nor $\theta - closed$.

E. Pre $-\theta-$ Open Functions

The purpose of this section is to introduce and discuss certain properties and characterizations of $pre - \theta - open$ functions.

Definition 2.31. Let (X,τ) and (Y,σ) be topological spaces. Then a function $f : X \longrightarrow Y$ is said to be $pre - \theta - open$ if and only if for each $A \in \tau_\theta$, $f(A) \in \sigma_\theta$.

Theorem 2.32. Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ and $g : (Y,\sigma) \longrightarrow (Z,\mu)$ be any two $pre - \theta - open$ functions. Then the composition function $g \circ f : X \longrightarrow Z$ is a $pre - \theta - open$ function.

Proof. Let $U \in \tau_\theta$. Then $f(U) \in \sigma_\theta$ since f is $pre - \theta - open$. But then $g(f(U)) \in \mu_\theta$ as g is $pre - \theta - open$. Hence, $g \circ f$ is $pre - \theta - open$.

Theorem 2.33. Prove that a mapping $f : (X,\tau) \longrightarrow (Y,\sigma)$ is $pre - \theta - open$ if and only if for each $x \in X$ and for any $U \in \tau_\theta$ such that $x \in U$, there exists $V \in \sigma_\theta$ such that $f(x) \in V$ and $V \subseteq f(U)$.

Proof. Routine.

Theorem 2.34. Prove that a mapping $f : (X,\tau) \longrightarrow (Y,\sigma)$ is $pre - \theta - open$ if and only if for each $x \in X$ and for any $\theta-$ neighborhood U of x in X, there exists a $\theta - neighborhood$ V of $f(x)$ in Y such that $V \subseteq f(U)$.

Proof. **Necessity**. Let $x \in X$ and let U be a $\theta - neighborhood$ of x. Then there exists $W \in \tau_\theta$ such that $x \in W \subseteq U$. Then $f(x) \in f(W) \subseteq f(U)$. But

$f(W) \in \sigma_\theta$ as f is $pre-\theta-open$. Hence $V = f(W)$ is a $\theta-neighborhood$ of $f(x)$ and $V \subseteq f(U)$.

Sufficiency. Let $U \in \tau_\theta$. Let $x \in U$. Then U is a $\theta-neighborhood$ of x. So by hypothesis, there exists a $\theta-neighborhood$ $V_{f(x)}$ of $f(x)$ such that $f(x) \in V_{f(x)} \subseteq f(U)$. It follows at once that $f(U)$ is a $\theta-neighborhood$ of each of its points. Therefore $f(U)$ is $\theta-open$. Hence f is $pre-\theta-open$.

Theorem 2.35. Prove that a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is $pre-\theta-open$ if and only if $f[Int_\theta(A)] \subseteq Int_\theta[f(A)]$, for all $A \subseteq X$.

Proof. **Necessity**. Let $A \subseteq X$. Let $x \in Int_\theta(A)$. Then there exists $U_x \in \tau_\theta$ such that $x \in U_x \subseteq A$. So $f(x) \in f(U_x) \subseteq f(A)$ and by hypothesis, $f(U_x) \in \sigma_\theta$. Hence $f(x) \in Int_\theta[f(A)]$. Thus $f[Int_\theta(A)] \subseteq Int_\theta[f(A)]$.

Sufficiency. Let $U \in \tau_\theta$. Then by hypothesis, $f[Int_\theta(U)] \subseteq Int_\theta[f(U)]$. Since $Int_\theta(U) = U$ as U is $\theta-open$. Also $Int_\theta[f(U)] \subseteq f(U)$. Hence $f(U) = Int_\theta[f(U)]$. Thus $f(U)$ is $\theta-open$ in Y. So f is $pre-\theta-open$.

We remark that the equality does not hold in Theorem 2.35 as the following example shows.

Example 2.36. *Let* $X = Y = \{1,2\}$. suppose X is antidiscrete and Y is discrete. Let $f = Id.$, $A = \{1\}$. Then $\phi = f[Int_\theta(A)] \neq Int_\theta[f(A)] = \{1\}$.

Theorem 2.37. Prove that a function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is $pre-\theta-open$ if and only if $Int_\theta\left[f^{-1}(B)\right] \subseteq f^{-1}[Int_\theta(B)]$, for all $B \subseteq Y$.

Proof. **Necessity**. Let $B \subseteq Y$. Since $Int_\theta\left[f^{-1}(B)\right]$ is $\theta-open$ in X and f is $pre-\theta-open$, $f\left[Int_\theta\left(f^{-1}(B)\right)\right]$ is $\theta-open$ in Y. Also we have $f\left[Int_\theta\left(f^{-1}(B)\right)\right] \subseteq f\left[f^{-1}(B)\right] \subseteq B$. Hence, $f\left[Int_\theta\left(f^{-1}(B)\right)\right] \subseteq Int_\theta(B)$. Therefore $Int_\theta\left[f^{-1}(B)\right] \subseteq f^{-1}[Int_\theta(B)]$.

Sufficiency. Let $A \subseteq X$. Then $f(A) \subseteq Y$. Hence by hypothesis, we obtain $Int_\theta(A) \subseteq Int_\theta\left[f^{-1}(f(A))\right] \subseteq f^{-1}[Int_\theta(f(A))]$. This implies that $f[Int_\theta(A)] \subseteq f\left[f^{-1}(Int_\theta(f(A)))\right] \subseteq Int_\theta[f(A)]$. Thus $f[Int_\theta(A)] \subseteq Int_\theta[f(A)]$, for all $A \subseteq X$. Hence, by Theorem 2.35, f is $pre-\theta-open$.

Theorem 2.38. Prove that a mapping $f:(X,\tau) \longrightarrow (Y,\sigma)$ is $pre-\theta-open$ if and only if $f^{-1}[Cl_\theta(B)] \subseteq Cl_\theta\left[f^{-1}(B)\right]$, for every subset B of Y.

Proof. **Necessity**. Let $B \subseteq Y$. Let $x \in f^{-1}\left[Cl_\theta(B)\right]$. Then $f(x) \in Cl_\theta(B)$. Let $U \in \tau_\theta$ such that $x \in U$. By hypothesis, $f(U) \in \sigma_\theta$ and $f(x) \in f(U)$. Thus $f(U) \cap B \neq \phi$. Hence $U \cap f^{-1}(B) \neq \phi$. Therefore, $x \in Cl_\theta\left[f^{-1}(B)\right]$. So we obtain $f^{-1}\left[Cl_\theta(B)\right] \subseteq Cl_\theta\left[f^{-1}(B)\right]$.

Sufficiency. Let $B \subseteq Y$. Then $(Y-B) \subseteq Y$. By hypothesis, $f^{-1}\left[Cl_\theta(Y-B)\right] \subseteq Cl_\theta\left[f^{-1}(Y-B)\right]$. This implies that $X - Cl_\theta\left[f^{-1}(Y-B)\right] \subseteq X - f^{-1}\left[Cl_\theta(Y-B)\right]$. Hence $X - Cl_\theta\left[X - f^{-1}(B)\right] \subseteq f^{-1}\left[Y - Cl_\theta(Y-B)\right]$. By Theorem 2.7 (6) [13], $Int_\theta\left[f^{-1}(B)\right] \subseteq f^{-1}\left[Int_\theta(B)\right]$. Now by Theorem 2.37, it follows that f is $pre-\theta-open$.

Theorem 2.39. Let $f:(X,\tau) \longrightarrow (Y,\sigma)$ and $g:(Y,\sigma) \longrightarrow (Z,\mu)$ be two mappings such that $g \circ f:(X,\tau) \longrightarrow (Z,\mu)$ is $\theta-irresolute$. Then

(1) If g is a $pre-\theta-open$ injection, then f is $\theta-irresolute$.

(2) If f is a $pre-\theta-open$ surjection, then g is $\theta-irresolute$.

Proof. (1) Let $U \in \sigma_\theta$. Then $g(U) \in \mu_\theta$ since g is $pre-\theta-open$. Also $g \circ f$ is $\theta-irresolute$. Therefore, we have $(g \circ f)^{-1}\left[g(U)\right] \in \tau_\theta$. Since g is an injection, so we have : $(g \circ f)^{-1}\left[g(U)\right] = \left(f^{-1} \circ g^{-1}\right)\left[g(U)\right] = f^{-1}\left[g^{-1}(g(U))\right] = f^{-1}(U)$. Consequently $f^{-1}(U)$ is $\theta-open$ in X. This proves that f is $\theta-irresolute$.

(2) Let $V \in \mu_\theta$. Then $(g \circ f)^{-1}(V) \in \tau_\theta$ since $g \circ f$ is $\theta-irresolute$. Also f is $pre-\theta-open$, $f\left[(g \circ f)^{-1}(V)\right]$ is $\theta-open$ in Y. Since f is surjective, we note that $f\left[(g \circ f)^{-1}(V)\right] = \left[f \circ (g \circ f)^{-1}\right](V) = \left[f \circ \left(f^{-1} \circ g^{-1}\right)\right](V) = \left[\left(f \circ f^{-1}\right) \circ g^{-1}\right](V) = g^{-1}(V)$. Hence g is $\theta-irresolute$.

F. Pre $-\theta-$ Closed Functions

In this last section, we introduce and explore several properties and characterizations of $pre-\theta-closed$ functions.

Definition 2.40. A function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is said to be $pre-\theta-closed$ if and only if the image set $f(A)$ is $\theta-closed$ for each $\theta-closed$ subset A of X.

Theorem 2.41. The composition of two $pre-\theta-closed$ mappings is a $pre-\theta-closed$ mapping.

Proof. The straight forward proof is omitted.

Theorem 2.42. Prove that a mapping $f : (X, \tau) \longrightarrow (Y, \sigma)$ is $pre-\theta-closed$ if and only if $Cl_\theta [f (A)] \subseteq f [Cl_\theta (A)]$ for every subset A of X.

Proof. **Necessity**. Suppose f is a $pre - \theta - closed$ mapping and A is an arbitrary subset of X. Then $f [Cl_\theta (A)]$ is $\theta - closed$ in Y. Since $f (A) \subseteq f [Cl_\theta (A)]$, we obtain $Cl_\theta [f (A)] \subseteq f [Cl_\theta (A)]$.

Sufficiency. Suppose F is an arbitrary $\theta - closed$ set in X. By hypothesis, we obtain $f (F) \subseteq Cl_\theta [f (F)] \subseteq f [Cl_\theta (F)] = f (F)$. Hence $f (F) = Cl_\theta [f (F)]$. Thus $f (F)$ is $\theta - closed$ in Y. It follows that f is $pre - \theta - closed$.

Theorem 2.43. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a $pre - \theta - closed$ function, and $B, C \subseteq Y$.

(1) If U is a $\theta - open$ neighborhood of $f^{-1} (B)$, then there exists a $\theta - open$ neighborhood V of B such that $f^{-1} (B) \subseteq f^{-1} (V) \subseteq U$.

(2) If f is also onto, then if $f^{-1} (B)$ and $f^{-1} (C)$ have disjoint $\theta - open$ neighborhoods, so have B and C.

Proof. (1) Let $V = Y - f (X - U)$. Then $V^c = Y - V = f (U^c)$. Since f is $pre - \theta - closed$, so V is $\theta - open$. Since $f^{-1} (B) \subseteq U$, we have $V^c = f (U^c) \subseteq f \left[f^{-1} (B^c)\right] \subseteq B^c$. Hence, $B \subseteq V$, and thus V is a $\theta - open$ neighborhood of B. Further $U^c \subseteq f^{-1} [f (U^c)] = f^{-1} (V^c) = \left[f^{-1} (V)\right]^c$. This proves that $f^{-1} (V) \subseteq U$.

(2) If $f^{-1} (B)$ and $f^{-1} (C)$ have disjoint $\theta - open$ neighborhoods M and N, then by (1), we have $\theta - open$ neighborhoods U and V of B and C respectively such that $f^{-1} (B) \subseteq f^{-1} (U) \subseteq Int_\theta (M)$ and $f^{-1} (C) \subseteq f^{-1} (V) \subseteq Int_\theta (N)$. Since M and N are disjoint, so are $Int_\theta (M)$ and $Int_\theta (N)$, and hence so $f^{-1} (U)$ and $f^{-1} (V)$ are disjoint as well. It follows that U and V are disjoint too as f is onto.

Theorem 2.44. Prove that a surjective mapping $f : (X, \tau) \longrightarrow (Y, \sigma)$ is $pre - \theta - closed$ if and only if for each subset B of Y and each $\theta - open$ set U in X containing $f^{-1} (B)$, there exists a $\theta - open$ set V in Y containing B, such that $f^{-1} (V) \subseteq U$.

Proof. **Necessity**. This follows from (1) of Theorem 2.43. **Sufficiency**. Suppose F is an arbitrary $\theta - closed$ set in X. Let y be an arbitrary point in $Y - f(F)$. Then $f^{-1}(y) \subseteq X - f^{-1}[f(F)] \subseteq (X - F)$ and $(X - F)$ is $\theta - open$ in X. Hence by hypothesis, there exists a $\theta - open$ set V_y containing y such that $f^{-1}(V_y) \subseteq (X - F)$. This implies that $y \in V_y \subseteq [Y - f(F)]$. Thus $Y - f(F) = \cup \{V_y : y \in Y - f(F)\}$. Hence $Y - f(F)$, being a union of $\theta - open$ sets is $\theta - open$. Thus its complement $f(F)$ is $\theta - closed$. This shows that f is $\theta - closed$.

Theorem 2.45. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a bijection. Then the following are equivalent:

(1) f is $pre - \theta - closed$.

(2) f is $pre - \theta - open$.

(3) f^{-1} is $\theta - irresolute$.

Proof. (1) $\Longrightarrow$ (2) : Let $U \in \tau_\theta$. Then $X - U$ is $\theta - closed$ in X. By (1), $f(X - U)$ is $\theta - closed$ in Y. But $f(X - U) = f(X) - f(U) = Y - f(U)$. Thus $f(U)$ is $\theta - open$ in Y. This shows that f is $pre - \theta - open$.

(2) $\Longrightarrow$ (3) : Let $A \subseteq X$. Since f is $pre - \theta - open$, so by Theorem 2.38, $f^{-1}[Cl_\theta(f(A))] \subseteq Cl_\theta\left[f^{-1}(f(A))\right]$. It implies that $Cl_\theta[f(A)] \subseteq f[Cl_\theta(A)]$. Thus $Cl_\theta\left[\left(f^{-1}\right)^{-1}(A)\right] \subseteq \left(f^{-1}\right)^{-1}[Cl_\theta(A)]$, for all $A \subseteq X$. Then by Theorem 2.8, it follows that f^{-1} is $\theta - irresolute$.

(3) $\Longrightarrow$ (1) : Let A be an arbitrary $\theta - closed$ set in X. Then $X - A$ is $\theta - open$ in X. Since f^{-1} is a $\theta - irresolute$, $\left(f^{-1}\right)^{-1}(X - A)$ is $\theta - open$ in Y. But $\left(f^{-1}\right)^{-1}(X - A) = f(X - A) = Y - f(A)$. Thus $f(A)$ is $\theta - closed$ in Y. This shows that f is $pre - \theta - closed$.

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