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Abstract. We study linear parabolic differential equations subjected to nonlocal initial conditions. We are concerned with the existence of solutions. Our technique is based on the Green's function, the maximum principle, integral representation of solutions.
Keywords. parabolic problems; integral representation of solutions; maximum principles; nonlocal initial conditions.
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1. Introduction

Let Ω be a an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. We denote the norm (usually the Euclidean norm) of $x \in \Omega$ by $\|x\|$. Let T be a positive real number, $D = \Omega \times (0, T)$ and $\Gamma = \partial\Omega \times [0, T]$. Then Γ is smooth and any point on Γ satisfies the inside (and outside) strong sphere property , i.e. for any $(x_0, t_0) \in \Gamma$ there is a closed ball $B \subset \Omega$ (and a closed ball $\widetilde{B}$ outside Ω) such that $\Gamma \cap (B \times [0, T]) = \{(x_0, t_0)\}$, (and $\Gamma \cap \left(\widetilde{B} \times [0, T]\right) = \{(x_0, t_0)\}$) (see [8]). For $u : D \to \mathbb{R}$ we denote its partial derivatives (when they exists) by $D_t u = \partial u/\partial t$, $D_i u = \partial u/\partial x_i$, $D_i D_j u = \partial^2 u/\partial x_i \partial x_j$, $i, j = 1, ..., N$.

$C(D)$ denotes the Banach space of continuous functions $u : D \to \mathbb{R}$, endowed with the norm

$$|u|_0 = \sup\{|u(x,t)| \,;\ (x,t) \in \overline{D}\}.$$

We say that $u \in C^{2,1}(D)$ if $u,\ D_i u,\ D_i D_j u$ and $D_t u$ exist and are continuous on D. In fact, we can write

$$C^{2,1}(D) = \{u \in C(D);\ u(.,t) \in C^2(\Omega), t \in (0,T)\,,\ u(x,.) \in C^1\,(0,T)\,, x \in \Omega\}.$$

$u \in C(D)$ is called Hölder continuous of order $\alpha \in (0,1]$ if

$$H_\alpha\,(u) = \sup\{\frac{|u(x,t) - u(\xi,\tau)|}{\left(\|x-\xi\|^2 + |t-\tau|\right)^{\alpha/2}}; (x,t)\,, (\xi,\tau) \in D\} < +\infty.$$

In this case we write $u \in C^\alpha(D)$ and we define its norm by

$$|u|_\alpha = |u|_0 + H_\alpha\,(u)\,.$$

If $\alpha = 1$, u is called Lipschitz continuous. Note that the natural injection $i : C^\alpha(D) \to C(D)$ is continuous. We say that that $C^\alpha(D)$ is continuously embedded in $C(D)$, and we write $C^\alpha(D) \hookrightarrow C(D)$.

Also, $u \in C^{2+\alpha,1+\alpha}(D)$ if $u(.,t) \in C^{2+\alpha}(\Omega)$ for all $t \in (0,T)$ and $u(x,.) \in C^{1+\alpha}(0,T)$ for all $x \in \Omega$. For $u \in C^{2+\alpha,1+\alpha}(D)$ we define its norm by

$$|u|_{2+\alpha,1+\alpha} = |u|_\alpha + \sum_{i=1}^{N} |D_i u|_\alpha + \sum_{i,j=1}^{N} |D_i D_j u|_\alpha + |D_t u|_\alpha\,.$$

We say that $\partial\Omega$ is in the class $C^{\ell+\alpha}, \ell \in \mathbb{N},\ \alpha \in [0,1)$ if in a neighborhood of each point of $\partial\Omega$ there is a local reprentation of $\partial\Omega$ having the form $x_i = \vartheta_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_N)$ with $\vartheta_i \in C^{\ell+\alpha}$.

Next, we introduce the Lebesgue spaces. For $1 \le p < +\infty$, we say that $u : D \to \mathbb{R}$ is in $L^p\,(D)$ if u is measurable and $\int_D |u(x,t)|^p\, dxdt < +\infty$, in which case we define its norm by

$$|u|_{L^p} = \left(\int_D |u(x,t)|^p\, dxdt\right)^{1/p}.$$

For $p = +\infty$, we write

$$|u|_\infty = ess\sup\{|u(x,t)|\,; (x,t) \in D\} = \inf_{N \subset D, \mu(N)=0}\ \sup_{(x,t)\in D\setminus N} |u(x)|\,,\ \mu = \text{Lebesgue measure.}$$

Given a nonnegative continuous function $\phi : [0, T] \to \mathbb{R}$, we are concerned with the existence of solutions of the following parabolic problem with a nonlocal initial condition

$$D_t u + Lu = f(x,t) \qquad (x,t) \in D, \tag{1}$$

$$u(x,t) = 0 \qquad (x,t) \in \Gamma, \tag{2}$$

$$u(x,0) = \int_0^T \phi(t)\, u(x,t)dt, \quad x \in \Omega, \tag{3}$$

where L is a strongly elliptic operator given by

$$Lu = -\sum_{i,j=1}^{N} D_i \left(a_{ij}(x,t) D_j u\right) + c(x,t)u.$$

We shall assume throughout this paper that the functions $a_{ij}, c : D \to \mathbb{R}$ are Hölder continuous, $a_{ij} = a_{ji}$ and moreover there exist positive numbers λ_0, λ_1 such that

$$\lambda_0 \|\xi\|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \leq \lambda_1 \ \|\xi\|^2, \ \forall \xi \in \mathbb{R}^N \text{ and } \forall (x,t) \in D.$$

The linear nonhomogeneous problem (1), (2) with the condition

$$u(x,0) + \sum_{i=1}^{m} \beta_i(x) u(x,t_i) = \varphi(x) \quad x \in \Omega. \tag{4}$$

has been investigated by several authors (see for instance [4], [10], [17] and the references therein). Problems with integral boundary conditions appear in the modeling of concrete problems, such as heat conduction [3], [5], [9], [13], [16] and in thermoelasticity [7]. The case of nonlocal problems for the Poisson equation is analyzed in [6]. Some general problems of independent interest are considered in [15].

Problem (1), (2) and the initial condition $u(x,0) = u_0(x)$, is well known and completely solved (see the books [8], [11], [12], [14]). The paper [2] is devoted to an interesting study of a linear problem with a nonlinear integral boundary condition.

2. Existence and uniqueness of solutions

Definition 1. A strong solution of problem (1), (2), (3) is a function $u \in C^{2,1}(D) \cap C(\overline{D})$ which satisfies (1), (2), (3).

We have the following version of the maximum principle.

Lemma 1. Let $u \in C^{2,1}(D) \cap C(\overline{D})$. Assume that $c(x,t) \geq c_0 > 0$ on D and $\int_0^T \phi(t)\,dt < 1$. If $D_t u + Lu \geq 0$ in D, $u(x,t) \geq 0$ on Γ, $u(x,0) - \int_0^T \phi(t)\,u(x,t)dt \geq 0$ on Ω. Then $u(x,t) \geq 0$ on $\overline{D}$. Moreover, either $u(x,t) = 0\ \forall\,(x,t) \in D$, or $u(x,t) > 0$ on D.

Proof. Suppose there exists $(\eta,\tau) \in \overline{D}$ such that $u(\eta,\tau) < 0$. It follows from the continuity of u that u achieves a negative minimum at some point $(x_0,t_0) \in \overline{D}$. By the strong maximum principle (see for instance [8]) we have either $(x_0,t_0) \in \Gamma$ or $(x_0,t_0) = (x_0,0)$ with $x_0 \in \Omega$. From the above assumptions we see that $\min_{\overline{D}} u(x,t) = u(x_0,0) < 0$. Since $u(x,t) \geq u(x_0,0)$ for all $(x,t) \in D$, it follows that if $\int_0^T \phi(t)\,dt < 1$,

$$0 \leq u(x_0,0) - \int_0^T \phi(t)\,u(x,t)dt \leq u(x_0,0)\left(1 - \int_0^T \phi(t)\,dt\right) < 0,$$

which is a contradiction. This completes the proof of the Lemma.

The investigation of the existence and uniqueness of solutions of problem (1), (2), (3) is based on the integral representation of solutions.

The homogeneous problem

$$\begin{aligned} &D_t u + Lu = 0, && (x,t) \in D \\ &u(x,t) = 0, && (x,t) \in \Gamma \\ &u(x,0) = 0, && x \in \Omega \end{aligned}$$

has only the trivial solution.

There exists a unique function, $G(x,t;y,s)$, called the Green's function corresponding to the linear homogeneous problem. This function satisfies the following (see [8], [14]),

(i) $D_tG + LG = \delta(t-s)\,\delta(x-y), \quad s < t,\ x, y \in \Omega$

(ii) $G(x,t;y,s) = 0, \quad s > t,\ \ x, y \in \Omega$

(iii) $G(x,t;y,s) = 0$, $(x,t)\,,(y,s) \in \Gamma$

(iv) $G(x,t;y,s) > 0$ for $(x,t) \in D$

(v) $G,\ G_t,\ G_x,\ G_{xx}$ are continuous functions of $(x,t),\ (y,s) \in D,\ t - s > 0,$

(vi) for any Hölder continuous function $f : D \to \mathbb{R}$ and any continuous function $u_0 : \Omega \to \mathbb{R}$, the function $u : D \to \mathbb{R}$, given by

$$u(x,t) = \int_\Omega G(x,t;y,0)\ u_0(y)\,dy + \int_0^t \int_\Omega G(x,t;y,s)\ f(y,s)\,dyds, \quad (x,t) \in D$$

is the unique solution of the nonhomogeneous problem

$$\begin{aligned} D_tu + Lu &= f, && (x,t) \in D, \\ u(x,t) &= 0, && (x,t) \in \Gamma, \\ u(x,0) &= u_0(x), && x \in \Omega. \end{aligned}$$

(vii) $G(x,t;y,s)$ satisfies the following important estimate (see [11, pages 412-413]). For some positive constants C and a

$$|G(x,t;y,s)| \le C\,(t-s)^{-N/2} \exp\left(\frac{-a\,\|x-y\|^2}{t-s}\right).$$

Remark. Since $u \in C^{2,1}(D) \cap C(\overline{D})$ it is clear that the functions $(x,t) \to \int_\Omega G(x,t;y,0)dy$ and $(x,t) \to \int_0^t \int_\Omega G(x,t;y,s)dyds$ are continuous. Let $\gamma_0 := \max_{(x,t)\in\overline{D}} \int_\Omega G(x,t;y,0)dy$ and let $\delta := \max_{(x,t)\in\overline{D}} \int_0^T \int_\Omega G(x,t;y,s)dyds$. Also, property (vii) above shows that $G \in L^2(D \times D)$.

Theorem 1. Assume that the function f is Hölder continuous and $\gamma_0 \int_0^T \phi(t)\,dt < 1$. Then, Problem (1), (2), (3) has a unique strong solution.

Proof. Consider the following representation (see property (vi) above),

$$u(x,t) = \int_\Omega G(x,t;y,0)\,u(y,0)\,dy + \int_0^t \int_\Omega G(x,t;y,s)\,f(y,s)\,dyds, \quad (x,t) \in D. \tag{5}$$

The condition $u(x,0)=\int_0^T \phi(t)\,u(x,t)dt$ implies

$$u(x,t)=\int_\Omega G(x,t;y,0)\int_0^T \phi(s)\,u(y,s)dsdy+\int_0^t\int_\Omega G(x,t;y,s)\,f(y,s)\,dyds,\quad (x,t)\in D$$

or

$$u(x,t)=\int_0^T\int_\Omega G(x,t;y,0)\phi(s)\,u(y,s)dyds+\int_0^t\int_\Omega G(x,t;y,s)\,f(y,s)\,dyds,\quad (x,t)\in D. \tag{6}$$

Define an operator $G:C^\alpha(D)\to C^{2,1}(D)\cap C(\overline{D})$ by

$$Gf(x,t)=\int_0^t\int_\Omega G(x,t;y,s)\;f(y,s)\,dyds,\quad (x,t)\in D. \tag{7}$$

Then G is a bounded linear operator with

$$|Gf|_0\le\delta\,|f|_0\,. \tag{8}$$

Since the Green's function has a weak singularity at $t=0$ it follows that the function $K:D\times D\to\mathbb{R}$, given by $K(x,t;y,s)=G(x,t;y,0)\phi(s)$, is continous. It is clear that (6) is equivalent to

$$u(x,t)=\int_0^T\int_\Omega K(x,t;y,s)u(y,s)dyds+Gf(x,t). \tag{9}$$

This is a Fredlhom integral equation of the second kind with a continuous kernel $K(x,t;y,s)$.

The condition $\gamma_0\int_0^T\phi(t)\,dt<1$ implies that Eq. (9) with $f=0$ has only the trivial solution. Hence there exists a unique solution $u(.,.)$ of (9), which is the unique solution of (1), (2), (3). Moreover, using the method of successive substitutions we obtain

$$u(x,t)=\int_0^T\int_\Omega R(x,t;y,s)f(y,s)dyds,$$

where $R(x,t;y,s)$ is the resolvent kernel.

.

Lemma 2.If f is Hölder continuous and $\gamma_0 \int_0^T \phi(t)\,dt < 1$ then solutions of (1), (2), (3) satisfy the estimate

$$|u|_0 \leq \frac{\delta}{1-\gamma_0 \int_0^T \phi(t)\,dt} |f|_0 .$$

Proof. It follows from (6) that

$$|u(x,t)| \leq \int_0^T \int_\Omega G(x,t;y,0)\phi(s)\,|u(y,s)|\,dyds + \int_0^t \int_\Omega G(x,t;y,s)\ |f(y,s)|\,dyds, \quad (x,t)\in D,$$

which implies that

$$|u(x,t)| \leq \gamma_0 |u|_0 \int_0^T \phi(t)\,dt + \delta |f|_0 .$$

Hence

$$\left(1-\gamma_0 \int_0^T \phi(t)\,dt\right) |u|_0 \leq \delta |f|_0 .$$

This completes the proof of the lemma.

If the source function f is not continuous, but is only square integrable then (1), (2), (3) cannot have a continuous solution. But, the linear operator G defined by (7) maps $L^2(D)$ into itself and is bounded. We have the following

Lemma 3.Assume $|\phi|_0 |G|_{L^2(D\times D)} < 1$ and $f \in L^2(D)$.Then (1), (2), (3) has a unique solution $u \in L^2(D)$ and the solution satisfies the estimate

$$|u|_{L^2(D)} \leq \frac{|G|_{L^2(D\times D)} |f|_{L^2(D)}}{1-|\phi|_0 |G|_{L^2(D\times D)}}.$$

We now prove a comparison result. Consider the following problems

$$\begin{cases} D_t u + Lu = f, & (x,t)\in D, \\ u(x,t) = 0, & (x,t)\in \Gamma, \\ u(x,0) = \int_0^T \phi(t)\,u(x,t)dt, & x\in\Omega, \end{cases} \tag{10}$$

and

$$\begin{cases} D_t v + Lv = g, & (x,t)\in D, \\ v(x,t) = 0, & (x,t)\in \Gamma, \\ v(x,0) = \int_0^T \phi(t)\,v(x,t)dt, & x\in\Omega. \end{cases} \tag{11}$$

Theorem 2.Assume that the functions f and g are Hölder continuous and $\gamma_0 \int_0^T \phi(t)\, dt < 1$. If $g(x,t) \geq f(x,t)$ for all $(x,t) \in D$ and $\int_0^T \phi(t)\, [v(x,t) - u(x,t)] dt \geq 0$ for all $x \in \Omega$, then the unique solution v of (11) dominates the unique solution u of (10), in the sense that $v(x,t) \geq u(x,t)$ for all $(x,t) \in \overline{D}$.

Proof.Let $z(x,t) = v(x,t) - u(x,t)$ for all $(x,t) \in D$. Then z satisfies

$$D_t z + Lz = g - f \geq 0 \qquad (x,t) \in D,$$

$$z(x,t) \geq 0, \qquad (x,t) \in \Gamma,$$

$$z(x,0) - \int_0^T \phi(t)\, z(x,t) dt \geq 0, \quad x \in \Omega.$$

By the maximum principle we have $z(x,t) \geq 0$ for all $(x,t) \in \overline{D}$.

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