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Some Properties of Bivariate Chi-Square Distribution and Their Applications

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Abstract The cumulative distribution function, characteristic function and reliability function of a chi-square probability function of two correlated variables have been derived in closed forms. The characteristic function has been used to derive raw product moments and mean centered product moments of general order. Results match with the independent case if the coefficient of correlation vanishes.

*Keywords and Phrases: B*ivariate chi-square distribution; correlated chi-square variables; characteristic function; cumulative distribution function; reliability function

AMS Mathematics Subject Classification (2000): 62E15, 60E05, 60E10

1. Introduction

Let two dimensional vectors $X_1, X_2, \cdots X_N \ (N > 2)$ be a random sample from a bivariate normal distribution $N_2(\mu, \Sigma)$ where the mean vector μ and the variance-covariance matrix Σ are unknown. Then the sample mean vector is $\bar{X} = (\bar{X}_1, \bar{X}_2)'$ and the sums of squares and cross product matrix is $\sum_{j=1}^{N}(X_j - \bar{X})(X_j - \bar{X})'$ which can be denoted by $A = (a_{ik}), i = 1,2; k = 1,2$ where $a_{ii} = ms_i^2,\ m = N-1,\ (i = 1,2)$ and $a_{12} = mrs_1s_2$. Also let $\Sigma = (\sigma_{ik})$, $i = 1,2;\ k = 1,2$ where $\sigma_{11} = \sigma_1^2,\ \sigma_{22} = \sigma_2^2,\ \sigma_{12} = \rho\sigma_1\sigma_2$ with $\sigma_1 > 0,\ \sigma_2 > 0$. The quantity $\rho\ \ (-1 < \rho < 1)$ is the product moment correlation coefficient between X_1 and X_2.

The joint density function of $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$, called the bivariate chi-square distribution, was derived by Joarder (2009) in a similar spirit to Krishnaiah, Hagis and Steinberg (1963) who studied the bivariate chi-distribution. The product moment correlation coefficient between U and V can be calculated to be ρ^2. For the estimation of correlation coefficient by modern techniques, we refer to Ahmed (1992). In case the correlation coefficient $\rho = 0$, the density function of U and V becomes that of the product of two independent chi-square variables each with m degrees of freedom.

The distribution plays an important role in radar systems, the detection of signals in noise etc. (Lawson and Uhlenbeck, 1950). For example when a radar signal passes through fog, cloud or rain, the water droplets produce a scattering effect. The scattered signal consists of waves, each having a different amplitude and phase. The bivariate chi-square distribution is useful in determining the probability that the resultant of these scattered waves will have a given amplitude and phase. It is also useful in determining the probability of missing a target by a specified distance when firing projectiles or missiles.

The cumulative distribution function has been derived in Section 2. The characteristic function of the bivariate chi-square function has been derived in closed form in Section 3. Raw product moments and mean centered product moments of general order has been derived by differentiating the characteristic function in Sections 4 and 5. The Reliability Function has been derived in Section 6.

2. The Cumulative Distribution Function

The following theorem is due to Joarder (2009) who derived it in a similar spirit to Krishnaiah, Hagis and Steinberg (1963).

Theorem 2.1 Let S_1^2 and S_2^2 be variances of a sample of size N from a bivariate normal distribution with unknown mean vector μ and variance covariance matrix Σ. Let the true variances be σ_1^2 and σ_2^2 while the covariance be $\rho\sigma_1\sigma_2$. Then the joint density function of $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$ is given by

$$f^*(u,v) = \frac{(uv)^{(m-2)/2}}{2^m\ \Gamma^2(m/2)(1-\rho^2)^{m/2}} \exp\left(-\frac{u+v}{2-2\rho^2}\right)\ {}_0F_1\left(\frac{m}{2};\frac{\rho^2 uv}{(2-2\rho^2)^2}\right), \tag{2.1}$$

where $m = N-1 > 2$, $-1 < \rho < 1$, and ${}_0F_1(;b;z)$ is defined in (1.2).

Theorem 2.2 Let U and V have the bivariate chi-square distribution with density function given by (2.1). Then for $m > 2$ and $-1 < \rho < 1$, the cumulative distribution function (CDF) of the distribution is given by

$$F(u,v) = \frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)} \sum_{k=0}^{\infty} \frac{\rho^{2k}}{\Gamma\left(k+(m/2)\right)k!}\gamma\left(k+\frac{m}{2},\frac{u}{2-2\rho^2}\right)\gamma\left(k+\frac{m}{2},\frac{v}{2-2\rho^2}\right), \tag{2.2}$$

where $\gamma\left(\alpha,x\right)$ the well known incomplete gamma function.

Proof. From (2.1), the CDF of U and V is given by

$$F(u,v) = \frac{(1-\rho^2)^{-m/2}}{2^m\ \Gamma(m/2)} \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k+(m/2)\right)} \frac{\rho^{2k}}{(2-2\rho^2)^{2k}k!} I(u)I(v), \tag{2.3}$$

where

$$I(u)=\int_{x=0}^{u} x^{k+(m-2)/2}\exp\left(-x/(2-2\rho^2)\right)dx.$$

Since $I(u)=(2-2\rho^2)^{k+(m/2)}\gamma(k+(m/2),u/(2-2\rho^2))$, $F(u,v)$ equivalent to what we have in the theorem.

3. The Characteristic Function

Theorem 3.1 Let U and V have the density function given by (2.1). Then the characteristic function of the density function at t_1 and t_2 is given by

$$\phi(t_1,t_2)=[(1-2it_1)(1-2it_2)+4t_1t_2\rho^2]^{-m/2}, \tag{3.1}$$

where $i=\sqrt{-1}$, $m>2,-1<\rho<1$.

Proof. By expanding the hypergeometric function in (2.1) and a bit of algebraic simplification, we have

$$\phi(t_1,t_2)=\frac{(1-\rho^2)^{-m/2}}{2^m\ \Gamma(m/2)}\sum_{k=0}^{\infty}\frac{1}{\Gamma\left((m/2)+k\right)}\frac{\rho^{2k}}{(2-2\rho^2)^{2k}k!}I(t_1)I(t_2),$$

where

$$I(t_1)=\int_{u=0}^{\infty} u^{k+(m-2)/2}\exp\left(it_1u-\frac{u}{2-2\rho^2}\right)du,\ \text{and}$$

$$I(t_2)=\int_{v=0}^{\infty} v^{k+(m-2)/2}\exp\left(it_2v-\frac{v}{2-2\rho^2}\right)dv.$$

Since

$$I(t)=\frac{[(1-2it)+2it\rho^2)]^{-k-(m/2)}}{(2-2\rho^2)^{-k-(m/2)}}\Gamma\left(k+(m/2)\right),$$

it can be checked that

$$I(t_1)I(t_2)=(2-2\rho^2)^{m+2k}\Gamma^2(k+(m/2))[J(t_1,t_2)]^{-k-(m/2)},$$

where $J(t_1,t_2)=(1-2it_1)(1-2it_2)+2i\rho^2(t_1+t_2)+8t_1t_2\rho^2-4t_1t_2\rho^4$. Then

$$\phi(t_1,t_2)=\frac{(1-\rho^2)^{m/2}}{\Gamma(m/2)}[J(t_1,t_2)]^{-m/2}\sum_{k=0}^{\infty}\Gamma\left((m/2)+k\right)\frac{\rho^{2k}}{k!}[J(t_1,t_2)]^{-k},$$

which simplifies to

$$\phi(t_1,t_2)=(1-\rho^2)^{m/2}[J(t_1,t_2)]^{-m/2}\left[1-\rho^2\{J(t_1,t_2)\}^{-1}\right]^{-m/2},$$

or ,

$$\phi(t_1,t_2)=(1-\rho^2)^{m/2}\left[J(t_1,t_2)-\rho^2\right]^{-m/2}.$$

The above can then be reduced to what we have in (3.1).

4. Raw Product Moments

The raw product moment of U and V of order a and b respectively denoted by $\mu'(a,b;\rho)=E(U^aV^b)$, can be derived by differentiating the characteristic function as demonstrated below. For brevity, we use the symbol $\phi_{ab}^{a+b}(t_1,t_2)=\dfrac{\partial^{a+b}\phi(t_1,t_2)}{\partial t_1^a \partial t_2^b}$.

Theorem 4.1 Let U and V have the characteristic function given by (3.1). Then the (a,b)-th raw moment of U is given by

$$\mu'(a,b;\rho)=(-i)^{a+b}\phi_{ab}^{a+b}(0,0) \tag{4.1}$$

where $i=\sqrt{-1}$, $\phi(t_1,t_2)=[(1-2it_1)(1-2it_2)+4t_1t_2\rho^2]^{-m/2}$, $m>2$, and $-1<\rho<1$.

Corollary 4.1 Let U and V have the characteristic function given by (3.1). Then the a-th raw product moment of the density function is given by

$$\mu'(a)=(-i)^a\phi_{a0}^{a+0}(t_1,0),\ \text{evaluated at } t_1=0 \tag{4.2}$$

where $i=\sqrt{-1}$, $\phi(t_1,0)=(1-2it_1)^{-m/2}$, $m>2, -1<\rho<1$.

Since $\phi(t_1,0)=(1-2it_1)^{-m/2}$, is known to be the characteristic function of a chi-square variable with m degrees of freedom, it also implies that U has a univariate chi-square distribution with m degrees of freedom.

Example 4.1 To calculate $\mu'(1,1;\rho)$ and $\mu'(2,1;\rho)$ by (4.1) we proceed as follows:
Letting $\psi(t_1,t_2)=(1-2it_1)(1-2it_2)+4t_1t_2\rho^2$, the characteristic function can be written as $\phi(t_1,t_2)=[\psi(t_1,t_2)]^{-m/2}$. It can be checked that

$$\phi_{11}^{1+1}(t_1,t_2) = m(m+2)[\psi(t_1,t_2)]^{-(m/2)-2}[i+2t_2(1-\rho^2)]^2$$
$$+2m[\psi(t_1,t_2)]^{-(m/2)-1}(1-\rho^2),$$

which is $-m(m+2\rho^2)$ at $t_1 = t_2 = 0$, so that $\mu'(1,1;\rho) = m(m+2\rho^2)$. Similarly, it can be checked that

$$\phi_{21}^{2+1}(t_1,t_2) = m(m+2)(m+4)[\psi(t_1,t_2)]^{-(m/2)-3}[i+2t_1(1-\rho^2)][i+2t_2(1-\rho^2)]^2$$
$$+4m(m+2)[\psi(t_1,t_2)]^{-(m/2)-2}\left[i+2t_2(1-\rho^2)\right](1-\rho^2),$$

which is $-im(m+2)(m+4\rho^2)$, at $t_1 = t_2 = 0$, so that $\mu'(2,1;\rho) = m(m+2)(m+4\rho^2)$.

5. Centered Product Moments

In the following theorem, we present how the mean centered raw product moments $\mu(a,b;\rho) = E\left([U-E(U)]^a[V-E(V)]^b\right)$ can be obtained by differentiating the joint characteristic function.

Theorem 5.1 Let U and V have the bivariate chi-square distribution with joint density function in (2.1). Then the mean centered product moment of order (a,b) of the density function is given by the following:

$$\mu(a,b) = \sum_{j=0}^{a}\sum_{k=0}^{b}\binom{a}{j}\binom{b}{k}(-m)^{a+b-(j+k)}(-i)^{j+k}\phi_{jk}^{j+k}(0,0), \tag{5.1}$$

where $i=\sqrt{-1}$, $\phi(t_1,t_2) = [(1-2it_1)(1-2it_2)+4t_1t_2\rho^2]^{-m/2}$, $m>2, -1<\rho<1$.

Proof. By expanding the binomial terms $[U-E(U)]^a$ and $[V-E(V)]^b$, in $\mu(a,b;\rho) = E\left([U-E(U)]^a[V-E(V)]^b\right)$, we have

$$\mu(a,b) = E\sum_{j=0}^{a}\binom{a}{j}U^j(-m)^{a-j}\sum_{k=0}^{b}\binom{b}{k}V^k)(-m)^{b-k},$$

which can be written as

$$\mu(a,b) = \sum_{j=0}^{a}\sum_{k=0}^{b}(-m)^{a+b-(j+k)}\binom{a}{j}\binom{b}{k}\mu'(j,k;\rho), \tag{5.2}$$

where $\mu'(j,k;\rho) = E(U^jV^k)$. Since by (4.1)

$$\mu'(j,k;\rho) = (-i)^{j+k}\phi_{jk}^{j+k}(0,0),$$

the identity in (5.1) is immediate.

Example 5.1 To derive $\mu(2,1)$, let us put $a = 2, b = 1$ in (5.2) so that

$$\mu(2,1) = \sum_{j=0}^{2}\sum_{k=0}^{1}(-m)^{2+1-(j+k)}\binom{2}{j}\binom{1}{k}\mu'(j,k;\rho),$$

which simplifies to

$$\mu(2,1) = -m^3\mu'(0,0;\rho) + m^2\mu'(0,1;\rho) + 2m^2\mu'(1,0;\rho) - 2m\mu'(1,1;\rho) - m\mu'(2,0;\rho) + \mu'(2,1;\rho).$$

Obviously $\mu'(0,0;\rho) = 1$. By (4.2), we have $\mu'(a,0;\rho) = \mu'(0,a;\rho) = E(U^a)$. From Example 4.1, we have $\mu'(1,1;\rho) = m(m+2\rho^2)$, and $\mu'(2,1;\rho) = m(m+2)(m+4\rho^2)$. Then it follows that $\mu(2,1) = 8m\rho^2$.

6. Reliability Function

In the context of reliability, the stress-strength model describes the life of a component which has a random strength Y and is subjected to random stress X. The component fails if the stress (X) applied to it exceeds the strength (Y) and the component will function satisfactorily whenever $Y > X$. Thus $P(X < Y)$ is a measure of component reliability.

Theorem 6.1 Let U and V have the bivariate chi-square distribution with density function given by (2.1). Then the reliability function $P(U < V)$ is given by

$$P(U < V) = \frac{(1-\rho^2)^{m/2}}{2\Gamma(m/2)\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma\left[k+((m+1)/2)\right]}{[k+(m/2)]}\frac{\rho^{2k}}{k!}\,{}_2F_1\left(1,\ 2k+m;\ k+1+(m/2);\ 1/2\right), \tag{6.1}$$

where $-1 < \rho < 1$ and $m > 2$.

Proof. From (2.1), the reliability function can be written as

$$P(U < V) = \frac{(1-\rho^2)^{-m/2}}{2^m\Gamma(m/2)}\sum_{k=0}^{\infty}\frac{1}{\Gamma[k+(m/2)]}\frac{\rho^{2k}J(k;m,\rho)}{(2-2\rho^2)^{2k}k!}, \tag{6.2}$$

where

$$J(k;m,\rho) = \int_0^{\infty} v^{k-1+(m/2)}\exp\left(-\frac{v}{2-2\rho^2}\right)I(v,k;m,\rho)dv$$

with

$$I(v,k;m,\rho)=\int_{u=0}^{v} u^{k-1+(m/2)}\exp\left(-\frac{u}{2-2\rho^2}\right)du.$$

Evaluating the gamma integral, we have

$$I(v,k;m,\rho)=(2-2\rho^2)^{k+(m/2)}\gamma\left(k+(m/2),\frac{v}{2-2\rho^2}\right),$$

so that

$$J(k;m,\rho^2)=(2-2\rho^2)^{k+(m/2)}\int_0^{\infty} v^{k-1+(m/2)}\exp\left(-\frac{v}{2-2\rho^2}\right)\gamma\left(k+(m/2),\frac{v}{2-2\rho^2}\right)dv,$$

Which can be evaluated to be

$$J(k;m,\rho^2)\ =\frac{\Gamma(2k+m)}{[k+(m/2)](1-\rho^2)^{-(2k+m)}}\ {}_2F_1\left(1,2k+m;k+1+(m/2);1/2\right).$$

Then from (6.2), the reliability function is given by

$$P(U<V)=\frac{(1-\rho^2)^{-m/2}}{2^m\ \Gamma(m/2)}\sum_{k=0}^{\infty}\frac{1}{\Gamma[k+(m/2)]}\frac{\rho^{2k}}{(2-2\rho^2)^{2k}k!}$$
$$\times\frac{\Gamma(2k+m)}{[k+(m/2)](1-\rho^2)^{-(2k+m)}}\ {}_2F_1\left(1,2k+m;k+1+(m/2);1/2\right),$$

which after some algebraic simplification leads to what we have in (6.1).

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