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# On the Product of Two Correlated Chi-Square Variables

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We derive the distribution of the product of two chi-squure variables when they are correlated through a bivariate chi-square distribution. The result is well known if the variables are independent. The cumulative distribution function of the distribution is also derived. Closed form expressions for raw moments and centered moments are obtained. The density function is graphed. The results are simply extended to the distribution of sample variances of bivariate normal distribution. Results match with the independent case in case coefficient of correlation vanishes.

## 1. Introduction

Let $X_1, X_2, \cdots X_N \; (N > 2)$ be two-dimensional independent normal random vectors with mean vector $\bar{X} = (\bar{X}_1, \bar{X}_2)'$ so that the sums of squares and cross product matrix is given by $\sum_{j=1}^{N}(X_j - \bar{X})(X_j - \bar{X})' = A$ which can be denoted by $A = (a_{ik}), i = 1,2; k = 1,2$ where $a_{ii} = ms_i^2, m = N-1, (i = 1,2)$ and $a_{12} = mrs_1 s_2$. Also let $\Sigma = (\sigma_{ik})$, $i = 1,2;\; k = 1,2$ where $\sigma_{11} = \sigma_1^2,\; \sigma_{22} = \sigma_2^2,\; \sigma_{12} = \rho\sigma_1\sigma_2$ with $\sigma_1 > 0,\; \sigma_2 > 0$. The quantity $\rho \;\; (-1 < \rho < 1)$ is the product moment correlation coefficient between $X_1$ and $X_2$.

The joint density function of $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$, called the bivariate chi-square distribution, was derived by Joarder (2009). The product moment correlation coefficient between $U$ and $V$ can be calculated to be $\rho^2$. For the estimation of correlation coefficient by modern techniques, we refer to Ahmed (1992). In case the correlation coefficient $\rho = 0$, the density function of $U$ and $V$ becomes that of the product of two independent chi-square variables each with $m$ degrees of freedom.

The distribution of the product of two variables arises in many contexts. Certain cases in traditional portfolio selection models involve the distribution of the product of two variables. The best examples of this are in the case of investment in a number of different overseas markets. In portfolio diversification models (see, e.g., Grubel, 1968) not only are the prices of shares in local markets uncertain but also the exchange rates, and so the value of the portfolio in domestic currency is related to a product of random variables. Similarly, in models of diversified production by multinationals (see e.g. Rugman 1979), there are local production uncertainty and exchange rate uncertainty, and so profits in home currency are again related to a product of random variables..

An entirely different example can be drawn from the econometric literature. While forecasting from an estimated equation, Feldstein (1971), pointed out that both the parameter and the value of the exogenous variable in the forecast period could be considered random variables. Hence the forecast was proportional to a product of random variables.

Wells, Anderson and Cell (1962) derived the distribution of the product of two independent chi-square variables with degrees of freedom $m_1$ and $m_2$. Note that Springer (1979, 365) also derived the same but with some misprints.

We derive the distribution of $W = UV$ in Theorem 3.1 when $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$ have a bivariate chi-square distribution with common degrees of freedom $m$. Our contribution is more general than Wells, Anderson and Cell (1962) or Springer (1979) in the sense of accommodating correlated chi-square variables $U$ and $V$. In case the variables are uncorrelated, Theorem 3.1 matches exactly with Wells, Anderson and Cell for $m_1 = m_2$. The cumulative distribution function of $W$ is derived in Theorem 3.2. Higher order raw moments, centered moments, and coefficient of skewness and kurtosis of $W$ are derived in Section 4. The results are often simply extended to the distribution of sample variances of bivariate normal distribution.

## 2. The Bivariate Chi-Square Distribution

**Theorem 2.2** Let $S_1^2$ and $S_2^2$ be two sample variances based on a bivariate normal distribution as discussed in the introduction. Then $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$ have the following joint density function:

$$f(u,v) = \frac{(uv)^{(m-2)/2}}{2^m \, \Gamma^2(m/2)(1-\rho^2)^{m/2}} \exp\left(-\frac{u+v}{2-2\rho^2}\right) {}_0F_1\left(\frac{m}{2}; \frac{\rho^2 uv}{(2-2\rho^2)^2}\right), \tag{2.1}$$

where $-1 < \rho < 1$, $m > 2$ and ${}_0F_1(;b;z)$ is a generalized hypergeometric function.

The random variables $U$ and $V$ are said to have a correlated bivariate chi-square distribution each with $m$ degrees of freedom, if its density function is given by (2.1).

In case $\rho = 0$, the density function of the joint probability distribution in (2.1), would be that of the product of two independent chi-square random variables.

By a simple transformation in (2.1) we have the following corollary.

**Corollary 2.1** The joint density of $S_1^2$ and $S_2^2$ is given by

$$f(s_1^2, s_2^2) = \left(\frac{m}{2\sigma_1\sigma_2}\right)^m \frac{(s_1^2 s_2^2)^{(m-2)/2}}{(1-\rho^2)^{m/2}\Gamma^2(m/2)} \exp\left[\frac{-m}{2-2\rho^2}\left(\frac{s_1^2}{\sigma_1^2}+\frac{s_2^2}{\sigma_2^2}\right)\right]$$
$$\times {}_0F_1\left(\tfrac{m}{2}; \frac{(m\rho)^2 s_1^2 s_2^2}{(2-2\rho^2)^2\sigma_1^2\sigma_2^2}\right), \tag{2.2}$$

where $m = N-1 > 2$, $-1 < \rho < 1$ and ${}_0F_1(a;z)$ is a generalized hypergeometric function.

By integrating out $v$ in (2.1), it can be easily checked that $U = mS_1^2/\sigma_1^2 \sim \chi_m^2$. Similarly, it can be proved that $V = mS_2^2/\sigma_2^2 \sim \chi_m^2$. Then by a simple transformation we have the following corollary.

**Corollary 2.2** Let $S_1^2$ and $S_2^2$ be two correlated chi-square variables with density function given by (3.2). Then $S_i^2 (i = 1,2)$ has the density function

$$g_i(s_i^2) = \left(\frac{m}{2\sigma_i^2}\right)^{m/2} \frac{(s_i^2)^{(m-2)/2}}{\Gamma(m/2)} \exp\left(\frac{-ms_i^2}{2\sigma_i^2}\right), \quad (i = 1,2), \tag{2.3}$$

where $m = N-1 > 2$.

In case $S_1^2$ and $S_2^2$ are independent, then the density function of $S_1^2$ and $S_2^2$ would be

$$f(s_1^2, s_2^2) = \left(\frac{m}{2\sigma_1\sigma_2}\right)^m \frac{(s_1^2 s_2^2)^{(m-2)/2}}{\Gamma^2(m/2)} \exp\left[\frac{-m}{2}\left(\frac{s_1^2}{\sigma_1^2}+\frac{s_2^2}{\sigma_2^2}\right)\right], \tag{2.4}$$

where $m = N-1 > 2$.

## 3. Main Results

**Theorem 3.1** Let $U$ and $V$ be two correlated chi-square variables with density function given by Theorem 2.1. Then the density function of $W = UV$ is given by

$$f_W(w) = \frac{(1-\rho^2)^{-m/2} w^{(m-2)/2}}{2^{m-1}\Gamma^2(m/2)} K_0\left(\frac{\sqrt{w}}{1-\rho^2}\right) {}_0F_1\left(\frac{m}{2}; \frac{\rho^2 w}{(2-2\rho^2)^2}\right), \qquad w > 0 \tag{3.1}$$

where $m > 2$, $-1 < \rho < 1$, ${}_0F_1(x)$ is a generalized hypergeometric function and $K_0(x)$ is a modified Bessel function of the second kind (Gradshteyn and Ryzhik, 1994).

**Proof.** Let $y = h_1(u,v) = u + v$, $w = h_2(u,v) = u\,v$, $u = h_1^{-1}(y,w)$, $v = h_2^{-1}(y,w)$ in (2.1) so that the joint density function of $Y$ and $W$ is given by

$$f(y,w)= f(h_1^{-1}(y,w),h_2^{-1}(y,w))\,|J_1| + f(h_1^{-1}(y,w),h_2^{-1}(y,w))\,|J_2| \tag{3.2}$$

where $|J_i|,(i=1,2)$ is the Jacobian of transformation in the domain $D_1=\{(u,v),u>v\}$ and $D_2=\{(u,v):u<v\}$ respectively. In $D_1=\{(u,v):u>v\}$ we have $2u=y+\sqrt{y^2-4w}$, $2v=y-\sqrt{y^2-4w}$ so that $\frac{\partial u}{\partial y}\frac{\partial v}{\partial w}-\frac{\partial u}{\partial w}\frac{\partial v}{\partial y}=(y^2-4w)^{-1/2}$

yielding $J(u,v\to y,w)=(y^2-4w)^{-1/2},\ y>2\sqrt{w}$. In $D_2=\{(u,v):u<v\}$, we have $2u=y-\sqrt{y^2-4w}, 2v=y+\sqrt{y^2-4w}$ and as above, it can be proved that

$$\frac{\partial u}{\partial y}\frac{\partial v}{\partial w}-\frac{\partial u}{\partial w}\frac{\partial v}{\partial y}=-(y^2-4w)^{-1/2},$$

so that the Jacobian of the transformation is $|J(u,v\to y,w)|=\left(y^2-4w\right)^{-1/2}, y>2\sqrt{w}$.
Then the joint probability density function of $Y$ and $V$ is given by

$$f(y,w)=\frac{w^{(m-2)/2}\ (y^2-4w)^{-1/2}\exp\left(\frac{-y}{2-2\rho^2}\right)}{2^{m-1}\ \Gamma^2(m/2)(1-\rho^2)^{m/2}}\ {}_0F_1\left(\frac{m}{2};\frac{\rho^2 w}{(2-2\rho^2)^2}\right), \tag{3.3}$$

where $w>0,\ y>2\sqrt{w},\ m>2$ and $-1<\rho<1$.

By integrating out $y$, it follows from (3.3) that

$$f_W(w)=\frac{(1-\rho^2)^{-m/2}w^{(m-2)/2}}{2^{m-1}\ \Gamma^2(m/2)}\ {}_0F_1\left(\frac{m}{2};\frac{\rho^2 w}{(2-2\rho^2)^2}\right)\ I(w,\rho),$$

where

$$I(w,\rho)=\int_{2\sqrt{w}}^{\infty}(y^2-4w)^{-1/2}\exp\left(\frac{-y}{2-2\rho^2}\right)dy\ .$$

Substituting $\frac{y}{2\sqrt{w}}=t,\ dy=2\sqrt{w}\,dt$, in the integral, we have

$$\begin{aligned}I(w,\rho)&=\int_1^{\infty}(t^2-1)^{-1/2}\exp\left(\frac{t\sqrt{w}}{1-\rho^2}\right)dt\\&=K_0\left(\frac{\sqrt{w}}{1-\rho^2}\right),\end{aligned}$$

where $K_\alpha(x)$ is the modified Bessel function of the second kind (Gradshteyn and Ryzhik, 1994).

Figure 1 in the Appendix 1 shows the graph of the density function (3.1) of the product of two chi-square variables for various values of $\rho$ for $m=5$. If $W$ is the product of two independent chi-square variables with degrees of freedom $m_1$ and $m_2$, then

$$f_W(w)=\frac{w^{[(m_1+m_2)/4]-1}}{2^{[(m_1+m_2)/2]-1}\Gamma(m_1/2)\Gamma(m_2/2)}K_{(m_1-m_2)/2}(\sqrt{w}), \qquad w>0 \tag{3.4}$$

(Wells, Anderson and Cell, 1962) where $K_\alpha(x)$ is the modified Bessel function of the second kind (Gradshteyn and Ryzhik, 1994). Note that Springer (1979, 365) derived the above but there is a misprint in the density function. If $\rho=0$ and $m_1=m_2=m$ in (4.4), it reduces to the joint density of the product of two independent chi-square variables each having $m$ degrees of freedom.

**Corollary 3.1** Let $S_1^2$ and $S_2^2$ be two correlated variables with density function given by (2.2). Then the density function of $W=S_1^2S_2^2$ is given by

$$f_W(w)=\frac{m^m w^{(m/2)-1}}{2^{m-1}(1-\rho^2)^{m/2}\Gamma^2(m/2)(\sigma_1\sigma_2)^m}\,{}_0F_1\left(\frac{m}{2};\frac{(m\rho)^2 w}{(2-2\rho^2)^2\sigma_1^2\sigma_2^2}\right)K_0\left(\frac{m\sqrt{w}}{(1-\rho^2)\sigma_1\sigma_2}\right), \tag{3.5}$$

where $w>0$, $-1<\rho<1$, $m>2$, ${}_0F_1(b;z)$ is a generalized hypergeometric function and $K_\alpha(x)$ is the modified Bessel function of the second kind (Gradshteyn and Ryzhik, 1994).

**Corollary 3.2** Let $S_1^2$ and $S_2^2$ be independent chi-square random variable with density functions given by (3.3). Then the density function of $W=S_1^2S_2^2$ is given by

$$f_W(w)=\frac{m^m}{2^{m-1}\,\Gamma^2(m/2)(\sigma_1\sigma_2)^m}w^{(m-2)/2}K_0\left(\frac{m}{\sigma_1\sigma_2}\sqrt{w}\right), \tag{3.6}$$

where $w>0$, and $m>2$.

**Theorem 3.2** Let $U$ and $V$ have the joint chi-square distribution with density function given by (2.1). Then the cumulative distribution function (CDF) of $W=UV$ is given by

$$F_W(w)=\frac{(1-\rho^2)^{-m/2}}{2^{m-1}\,\Gamma^2(m/2)}\sum_{k=0}^{\infty}\frac{\Gamma(m/2)}{\Gamma((k+(m/2))}\frac{\rho^{2k}}{(2-2\rho^2)^{2k}k!}J(k;m,\rho), \tag{3.7}$$

where

$$J(k;m,\rho)=\int_0^w y^{(m/2)+k-1}K_0\left(\frac{\sqrt{y}}{1-\rho^2}\right)dy. \tag{3.8}$$

with $m>2$, $-1<\rho<1$, ${}_0F_1(b;x)$ is a generalied hypergeometric function and $K_0(x)$ is a Bessel function of the second kind (Gradshteyn and Ryzhik, 1994).

**Proof.** The CDF of the distribution is given by

$$F_W(w) = \int_0^w f_W(y)dy$$

which can be written as

$$F_W(w) = \frac{(1-\rho^2)^{-m/2}}{2^{m-1}\ \Gamma^2(m/2)} I(w;\rho,m),$$

where

$$I(w;\rho,m) = \int_0^w y^{(m/2)-1} K_0\left(\frac{\sqrt{y}}{1-\rho^2}\right) {}_0F_1\left(\frac{m}{2};\frac{\rho^2 y}{(2-2\rho^2)^2}\right) dy.$$

By expanding the hypergeometric function, the above integral can be written as

$$I(w;\rho,m) = \sum_{k=0}^{\infty} \frac{\Gamma(m/2)}{\Gamma((k+(m/2))} \frac{\rho^{2k}}{(2-2\rho^2)^{2k} k!} J(k;m,\rho) \qquad (3.9)$$

where

$$J(k;m,\rho) = \int_0^w y^{(m/2)+k-1} K_0\left(\frac{\sqrt{y}}{1-\rho^2}\right) dy. \qquad (3.10)$$

The integral is evaluated in Appendix 2.

**Corollary 3.3** Let $U$ and $V$ be distributed as independent chi-square variables each having $m$ degrees of freedom. Then the cumulative distribution function of $W$ $=UV$ is given by

$$F_W(w) = \frac{1}{2^{m-1}\ \Gamma^2(m/2)} I(w;m), \quad w > 0, \qquad (3.11)$$

where $m > 2$,

$$I(w;m) = \int_0^w y^{(m/2)-1} K_0\left(\sqrt{y}\right) dy, \qquad (3.12)$$

and $K_0(x)$ is a modified Bessel function of the second kind (Gradshteyn and Ryzhik, 1994).

In case $S_1^2$ and $S_2^2$ are independent with density function in (2.4), then the CDF of $S_1^2 S_2^2$ follows from the above theorem by virtue of $m^2 S_1^2 S_2^2 = \sigma_1^2 \sigma_2^2 W$.

## 4. Higher Order Moments

**Theorem 4.1** Let $W$ have the density function given by Theorem 4.2. Then for $m > 2$ and $-1 < \rho < 1$, the a-th moment of $W$ is given by $\mu_a' = E(W^a)$, is given by

$$E(W^a) = 4^a \frac{\Gamma^2\left(a + (m/2)\right)}{\Gamma^2(m/2)} {}_2F_1(-a, -a; m/2; \rho^2) \tag{4.1}$$

where ${}_2F_1(a_1, a_2; b_1; z)$ is a generalized hypergeometric function.

**Proof.** The $a$-th moment $E(W^a)$ is given by

$$E(W^a) = \frac{4^a (1-\rho^2)^{2a+(m/2)}}{\sqrt{\pi}\ \Gamma(m/2)} \sum_{k=0}^{\infty} \frac{(2\rho)^{2k}}{(2k)!\Gamma\left(k + (m/2)\right)} \Gamma^2\left(k + a + (m/2)\right)\Gamma\left(k + (1/2)\right)$$

which, by virtue of duplication formula of gamma function, can be written as

$$E(W^a) = 4^a (1-\rho^2)^{2a+(m/2)} \frac{\Gamma^2\left(a + (m/2)\right)}{\Gamma^2(m/2)} {}_2F_1(a + (m/2), a + (m/2); m/2; \rho^2)$$

which can be transformed to (4.1).

**Corollary 4.1** Let $U$ and $V$ have the bivariate chi-square distribution with density function given by Theorem 3.1. Then for any integer $a$, $E(W^a)$ is given by

$$E(W^a) = 4^a (m/2)_{\{a\}} \sum_{k=0}^{a} \binom{a}{k} (1 + a - k)_{\{k\}} \left((m/2) + k\right)_{\{a-k\}} \rho^{2k}, \tag{4.2}$$

where $k_{\{a\}} = k(k+1)\cdots(k+a-1)$, $m > 2$ and $-1 < \rho < 1$.

**Proof.** Since $a$ and $b$ are integers, we have

$${}_2F_1(-a, -b; m/2; \rho^2) = \sum_{k=0}^{a} \frac{(-a)_{\{k\}} (-b)_{\{k\}}}{(m/2)_{\{k\}}} \frac{\rho^k}{k!},$$

and hence, from Theorem 4.1, we have

$$E(W^a) = 4^a \frac{\Gamma^2\left(a + (m/2)\right)}{\Gamma^2(m/2)} \sum_{k=0}^{a} \frac{[(-a)_{\{k\}}]^2}{(m/2)_{\{k\}}} \frac{\rho^k}{k!}.$$

Further by virtue of $(-a)_{\{k\}}=(-1)^k(a-k+1)_{\{k\}}$ we have

$$E(W^a)=4^a(m/2)_{\{a\}}\sum_{k=0}^{a}\binom{a}{k}\frac{\Gamma\big(a+(m/2)\big)(a-k+1)_{\{k\}}}{\Gamma\big((m/2)+k\big)}\rho^{2k},$$

since $(a)_{\{k\}}\Gamma(a)=\Gamma(a+k)$. The above is equivalent to what we have in (4.2).

The above moments are represented by Jacobi's Polynomials in the following corollary:

**Corollary 4.2** Let $U$ and $V$ have the bivariate chi-square distribution with density function given by Theorem 2.1. Then for nonnegative integers $a$ and $b$, we have the following:

$$E(W^a)=\ 4^a(m/2)_{\{a\}}(1-\rho^2)^{-2a}a!P_a^{((m/2)-1,-2a-(m/2))}(1-2\rho^2),\ m>2.$$

**Proof.** The proof is obvious from Theorem 4.1 by virtue of

$${}_2F_1(-a,-b;m/2;\rho^2)=\frac{a!}{(m/2)_{\{a\}}}P_a^{((m/2)-1,-a-b-(m/2))}(1-2\rho^2).$$

The corollary below follows from Corollary 4.1 (cf. Joarder , 2009).

**Corollary 4.3** For $m>2,-1<\rho<1$, then the first raw four moments $E(W^a)$, $a=1,2,3,4$ of $W=UV$ are given by

$$\begin{aligned}
E(W)&=m(m+2\rho^2),\\
E(W^2)&=m(m+2)[8\rho^4+8(m+2)\rho^2+m(m+2)],\\
E(W^3)&=m(m+2)(m+4)[48\rho^6+72(m+4)\rho^4+18(m+2)(m+4)\rho^2)+m(m+2)(m+4)],\\
E(W^4)&=m(m+2)(m+4)(m+6)\big[m(m+2)(m+4)(m+6)+32(m+2)(m+4)(m+6)\rho^2\\
&\quad+288(m+4)(m+6)\rho^4+768(m+6)\rho^6+384\rho^8\ \big].
\end{aligned}$$

Since $m^2S_1^2S_2^2=\sigma_1^2\sigma_2^2W$ , moments, coefficient of skewness and kurtosis of $S_1^2S_2^2$ can be simply derived from the results in this section. In particular in the following corollary we report the mean and variance.

**Corollary 4.4** The mean and variance of $S_1^2S_2^2$ are respectively given by

$$E(S_1^2S_2^2)=\frac{1}{m}(m+2\rho^2)\sigma_1^2\sigma_2^2,\ \text{and}$$

$$Var(S_1^2S_2^2)=\frac{1}{m^3}\ [4(1+\rho^2)m^2+4(1+8\rho^2-\rho^4)m+16\rho^4]\ \sigma_1^4\sigma_2^4,$$

where $m > 2$ and $-1 < \rho < 1$. The centered moments of $W$ are given by $\mu_a = E(E - E(W))^a$. In particular, the second, third and fourth order mean corrected moments are respectively given by

$$\mu_2 = E(W^2) - \mu^2,$$
$$\mu_3 = E(W^3) - 3E(W^2)\mu + 2\mu^3,$$
$$\mu_4 = E(W^4) - 4E(W^3)\mu + 6E(W^2)\mu^2 - 3\mu^4.$$

**Corollary 4.5** The first four centered moments of $W$ are given respectively by

$$\mu_2 = 4m[(m+4)\rho^4 + (m^2+8m+8)\rho^2 + m(m+1)],$$

$$\mu_3 = 8m[m(5m^2+12m+8) + 6(2m^3+17m^2+36m+24)\rho^2 + 3(m+4)(m^2+16m+24)\rho^4$$
$$+ 2(m^2+12m+24)\rho^6],$$

$$\mu_4 = 48m[m(m^4+16m^3+59m^2+88m+48) + 2(m^5+37m^4+304m^3+976m^2+1408m+768)\rho^2$$
$$+ (m^5+56m^4+798m^3+4176m^2+9024m+6912)\rho^4 + 2(3m^4+84m^3+728m^2+2304m+2304)\rho^6$$
$$+ (3m^3+56m^2+288m+384\rho^8],$$

where $m > 2$ and $-1 < \rho < 1$.

**Corollary 4.6** The coefficient of skewness and kurtosis of $W$ are given by the moment ratios

$$\alpha_3(W) = \frac{\mu_3}{\mu_2^{3/2}}, \text{ and } \alpha_4(W) = \frac{\mu_4}{\mu_2^2}$$

respectively where $\mu_2$, $\mu_3$ and $\mu_4$ are given by Corollary 4.4.

The asymptotic behaviour of coefficient of skewness and kurtosis can be graphed. In case $\rho = 0$, then $W$ will be the product of two independent chi-square random variables each with $m$ degrees of freedom and evidently the resulting moments are in agreement with that situation of independence. In particular, if $\rho = 0$, that is if $W$ is the product of two independent chi-square variables each having $m$ degrees of freedom, then the mean and variance of $W$ will be $E(W) = m$ and $Var(W) = 4m^2(m+1)$ respectively. In that case the coefficient of skewness and kurtosis of $W$ are respectively given by

$$\frac{5m^2+12+8}{m(m+1)^{3/2}}, \text{ and } \frac{3(m^4+16m^3+59m^2+88m+48)}{[m(m+1)]^2}.$$

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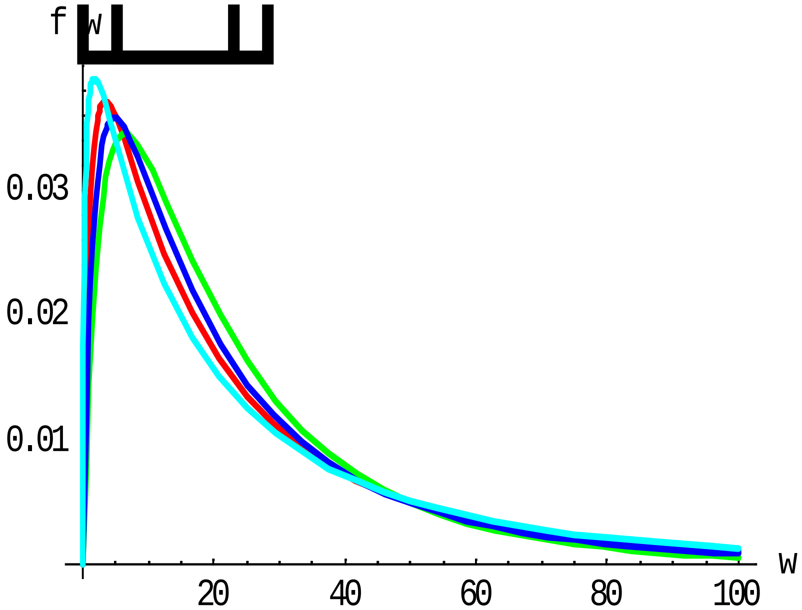
## Appendix 1

$\rho$ = -0.7 (red), 0(green), 0.5(blue), 0.9 (cyan)




$\rho$ = -0.7 (red), 0(green), 0.5(blue), 0.9 (cyan)

**Figure 1.** Density function of $W$ for $m$ = 5 and various $\rho$ values

**Appendix 2** The evaluation of the integral

$$J(k;m,\rho)=\int_0^w y^{(m/2)+k-1}K_0\left(\frac{\sqrt{y}}{1-\rho^2}\right)dy. \tag{A1}$$

From Gradshteyn and Ryzhik (1994), we have

$$K_0(z)=-\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{y}{4(1-\rho^2)^2}\right)^k\ln\left(\frac{y}{4(1-\rho^2)^2}\right)+\sum_{k=0}^{\infty}\frac{\psi(k+1)}{(k!)^2}\left(\frac{y}{4(1-\rho^2)^2}\right)^k, \tag{A2}$$

where the psi function is defined in Gradshteyn and Ryzhik (1994). Using (A2) in (A1), we have

$$J(k;m,\rho) = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{4^k(1-\rho^2)^{2k}(k!)^2}\int_{y=0}^{y=w} y^{2k+(m/2)-1}\ln y dy$$
$$+\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{4^k(1-\rho^2)^{2k}(k!)^2}\ln[4(1-\rho^2)^2]dy \qquad \text{(A3)}$$
$$+\sum_{k=0}^{\infty}\frac{\psi(k+1)}{4^k(1-\rho^2)^{2k}(k!)^2}\int_{y=0}^{y=w} y^{2k+(m/2)-1}dy.$$

Having evaluated the simple integrals in (A3), we have

$$J(k;m,\rho) = -\frac{1}{2}w^{m/2}\ln w\sum_{k=0}^{\infty}\frac{w^{2k}}{4^k(1-\rho^2)^{2k}(k!)^2}\times\frac{1}{2k+(m/2)}$$
$$+\frac{1}{2}w^{m/2}\sum_{k=0}^{\infty}\frac{1}{4^k(1-\rho^2)^{2k}(k!)^2}\frac{w^{2k}}{[2k+(m/2)]^2} \qquad \text{(A4)}$$
$$+\frac{1}{2}\sum_{k=0}^{\infty}\frac{\ln[4(1-\rho^2)^2]}{4^k(1-\rho^2)^{2k}(k!)^2}dy$$
$$+w^{m/2}\sum_{k=0}^{\infty}\frac{\psi(k+1)}{4^k(1-\rho^2)^{2k}(k!)^2}\frac{w^{2k}}{2k+(m/2)},$$

where the psi function is defined in Gradshteyn and Ryzhik (1994).