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The Distribution of the Sum of Two Correlated Chi-Square Variables

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The exact distribution of the sum of two chi-square variables is well known if the variables are independent. We derive the exact distribution of the sum of two correlated chi-square variables when they are correlated through a bivariate chi-square distribution. The graph of the density function is presented. Some properties of the distribution, namely, the characteristic function, cumulative distribution function, raw moments, mean centered moments, coefficient of skewness and kurtosis are derived. If the results of the paper are specialized to the uncorrelated case, then they, as expected, match with that of the independent case.

1. Introduction

Let S_1^2 and S_2^2 be sample variances based on a sample of size $N = m+1$ from a bivariate normal distribution with unknown means and unknown variances σ_1^2 and σ_2^2, and correlation coefficient ρ $(-1 < \rho < 1)$. The joint density function of $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$, called the bivariate chi-square distribution follows from Krishnaiah, Hagis and Steinberg (1963). Gunst and Webster (1973) derived it through Canonical Correlation Analysis. Some moments of the distribution are derived by Joarder (2009). The product moment correlation coefficient between U and V can be calculated to be ρ^2. For the estimation of correlation coefficient by modern techniques, we refer to Ahmed (1992). In case the correlation coefficient $\rho = 0$, the density function of U and V becomes that of the product of two independent chi-square variables each with m degrees of freedom.

The distribution plays an important role in radar systems, the detection of signals in noise etc. (Lawson and Uhlenbeck, 1950). It is also useful in determining the probability of missing a target by specified distance when firing projectiles or missiles.

Gerkmann and Martin (2009) has recently considered correlated bivariate chi-square model of Joarder (2009) to derive explicit expression for the variance and covariance of correlated spectral amplitudes and the resulting cepstral coefficients. The results in their work allow for cepstral smoothing of spectral quantities without affecting their signal power. Interested readers may go through the paper and the references therein.

The exact density function of the linear combination of independent gamma variables was derived by Provost (1988). A special case of his paper is the linear combination of independent chi-square variables. However results in the correlated case is unknown, so we consider the sum of two chi-square variables.

The distribution of the sum of correlated chi-square variables arises in the construction of confidence intervals for the common variance of multivariate normal populations with correlation structure having unknown correlation coefficients. By application of the inversion formula to the characteristic function of the sum of correlated chi-squares, Gordon and Ramig (1983) derived an integral form of the cumulative distribution function of the sum and used trapezoidal rule to evaluate it.

In Section 3, we report the exact distribution of the sum $(U+V)$ of two correlated chi-square variables when they are governed through a bivariate chi-square density function. We also report the exact cumulative distribution function (CDF) of the sum of two correlated chi-squares. The graph of the density function of the sum is presented at the end of the paper. In Section 4, we report the characteristic function of the sum of two correlated chi-square variables. In Section 5, we report some properties of the distribution of the sum, namely, raw moments, mean centered moments, coefficient of skewness and kurtosis. If the results of the paper are specialized to the uncorrelated case $(\rho = 0)$, then they, as expected, match with that of the independent case.

2. The Bivariate Chi-Square Distribution

The joint density function of the bivariate chi-square distribution in Gunst and Webster (1973) is represented below by the generalized hypergeometric function.

Theorem 2.1 Let S_1^2 and S_2^2 be variances of a sample drawn from a bivariate normal distribution as discussed in the introduction. Then the joint density function of $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$ is given by the following function:

$$f_{U,V}(u,v) = \frac{(uv)^{(m/2)-1}}{2^m\ \Gamma^2(m/2)(1-\rho^2)^{m/2}} \exp\left(-\frac{u+v}{2-2\rho^2}\right)\ {}_0F_1\left(\frac{m}{2};\frac{\rho^2 uv}{(2-2\rho^2)^2}\right), \tag{2.1}$$

where $m > 2$, $-1 < \rho < 1$ and ${}_0F_1(;b;z)$ is the generalized hypergeometric function.

The random variables U and V are said to have a correlated bivariate chi-square distribution each with m degrees of freedom, if its density function is given by (2.1). In case $\rho = 0$, it can be verified that the density function in (2.1), would be $f_U(u)\, f_V(v)$, the product of the density functions of two independent chi-square random variables each having m degrees of freedom.

3. The Density Function and the CDF of the Sum

In the following theorem, we report the density function of the sum of two correlated chi-square variables.

Theorem 3.1 Let U and V be two correlated chi-square variables with joint density function given by (2.1). Then the density function of $Z = U + V$ is given by

$$f_Z(z) = \frac{(1-\rho^2)^{-m/2}}{2^m\ \Gamma(m)} z^{m-1} \exp\left(\frac{-z}{2-2\rho^2}\right)\ {}_0F_1\left(\frac{m+1}{2};\frac{\rho^2 z^2}{(4-4\rho^2)^2}\right), \quad 0 \le z < \infty, \tag{3.1}$$

where $m > 2$ and $-1 < \rho < 1$.

Figure 1 in the Appendix shows the graph of the density function (3.1) of the sum of two correlated chi-square variables for various values of ρ, and for $m = 5$. If $\rho = 0$, then the density function in (3.1) simplifies to that of a chi-square variable with $2m$ degrees of freedom.

Theorem 3.2 Let U and V be two correlated chi-square variables governed by the joint density function (2.1). Then the Cumulative Density Function of $Z = U + V$ is given by

$$F_Z(z) = \frac{(1-\rho^2)^{m/2}}{\Gamma(m)} \sum_{k=0}^{\infty} \frac{\Gamma\big((m+1)/2\big)}{\Gamma\big(k+(m+1)/2\big)(k!)} \left(\frac{\rho^2}{4}\right)^k \gamma\left(2k+m, \frac{z}{2-2\rho^2}\right), \tag{3.2}$$

$m > 2, -1 < \rho < 1,$ and $\gamma(\alpha, x)$ is the incomplete gamma function (Gradshteyn and Ryzhik, 1995).

Percentage points of the distribution of Z can be calculated by (3.2) for particular values of the correlation coefficient ρ. By putting $\rho = 0$ in Theorem 3.2, we have the following corollary:

Corollary 3.1 Let U and V be two independent chi-square variables. Then the Cumulative Density Function of $Z=U+V$ is given by

$$F_Z(z)=\frac{\gamma(m,z/2)}{\Gamma(m)}, \qquad m>2,$$

where $\gamma(\alpha,x)$ is the incomplete gamma function.

4. The Characteristic Function of the Sum

Theorem 4.1 Let U and V have two correlated chi-square variables governed by the density (2.1). Then the characteristic function of $Z=U+V$ is given by

$$\phi_Z(t)=\left((1-2it)^2+4t^2\rho^2\right)^{-m/2}, \quad m>2 \text{ and } -1<\rho<1. \tag{4.1}$$

Note that in case $\rho=0$, the characteristic function in (4.1) reduces to $\phi_Z(t)=(1-2it)^{-m}$ which is known to be the characteristic function of χ^2_{2m}, a chi-square random variable with $2m$ degrees of freedom.

5. Moments, Coefficient of Skewness and Kurtosis for Sum

In this section, we provide some moment characteristics of the sum of two correlated chi-square variables.

Theorem 5.1 Let U and V have the bivariate chi-square distribution with joint density function in (2.1). Then for any number a, the a-th moment of $Z=U+V$ denoted by $\mu'_a=E(Z^a)$ is given by

$$\mu'_a=\frac{2^a(1-\rho^2)^{a+(m/2)}\Gamma(m+a)}{\Gamma(m)}\,{}_2F_1\left(\frac{m+a}{2},\frac{m+a+1}{2};\frac{m+1}{2};\rho^2\right), \tag{5.1}$$

where $m>2,\ a>-2$ and $-1<\rho<1$.

If $\rho=0$, then prove that (5.1) simplifies to a-th raw moment of $Z\sim\chi^2_{2m}$.

Corollary 5.1 Let U and V have the bivariate chi-square distribution with joint density function in (2.1). Then the first four raw moments of $Z=U+V$ are respectively given by

$$(i)\ E(Z)=2m, \tag{5.2}$$

$$(ii)\ E(Z^2)=4m(m+1+\rho^2), \tag{5.3}$$

$(iii)\ E(Z^3) = 8m(m+2)(m+1+3\rho^2),$ (5.4)

$(iv)\ E(Z^4) = 16m(m+2)\big((m+1)(m+3)+3\rho^2(\rho^2+2m+6)\big),$ (5.5)

where $m > 2$ and $-1 < \rho < 1.$

Theorem 5.2 Let U and V have the bivariate chi-square distribution with joint density function in (2.1). Then for any integer a, the a-th centered moment of $Z = U + V$, denoted by $\mu_a = E(Z-2m)^a$, is given by

$$\mu_a = (-2m)^a \sum_{k=0}^{\infty} \frac{(-1)^k a^{\{k\}}}{k!} (2m)^k \mu_k' \tag{5.6}$$

where $m > 2$, $-1 < \rho < 1$, and $a^{\{k\}}$ is the Pochammer polynomial defined by $a^{\{k\}} = a(a-1)\cdots(a-k+1).$

Corollary 5.2 Let U and V have the bivariate chi-square distribution with joint density function in (2.1). The first 3 centered moments of $Z = U + V$ of order 2, 3 and 4 are respectively given by

$(i)\ \mu_2 = 4m(1+\rho^2),$ (5.7)

$(ii)\ \mu_3 = 16m(1+3\rho^2),$ (5.8)

$(iii)\ \mu_4 = 48m[m(1+\rho^2)^2 + 2(1+6\rho^2+\rho^4)],$ (5.9)

where $m > 2$ and $-1 < \rho < 1.$

Corollary 5.3 Let U and V have the bivariate chi-square distribution with joint density function in (2.1). The coefficient of skewness and kurtosis of $Z = U + V$ are respectively given by

$$\alpha_3(Z) = \frac{2(1+3\rho^2)}{(1+\rho^2)^{3/2}\sqrt{m}}, \tag{5.10}$$

and

$$\alpha_4(Z) = 3 + \frac{6(1+6\rho^2+\rho^4)]}{m(1+\rho^2)^2}, \tag{5.11}$$

where $m > 2$ and $-1 < \rho < 1$.

If $\rho = 0$, then it follows from Corollary 5.3 that $\alpha_3(Z) = 2/(\sqrt{m})$, and $\alpha_4(Z) = 3 + (6/m)$, which matches with the coefficient of skewness and kurtosis of $Z \sim \chi^2_{2m}$. Further if the degrees of freedom m increases indefinitely, the coefficient of skewness and kurtosis converges, as expected, to 0 and 3 of the univariate normal distribution.

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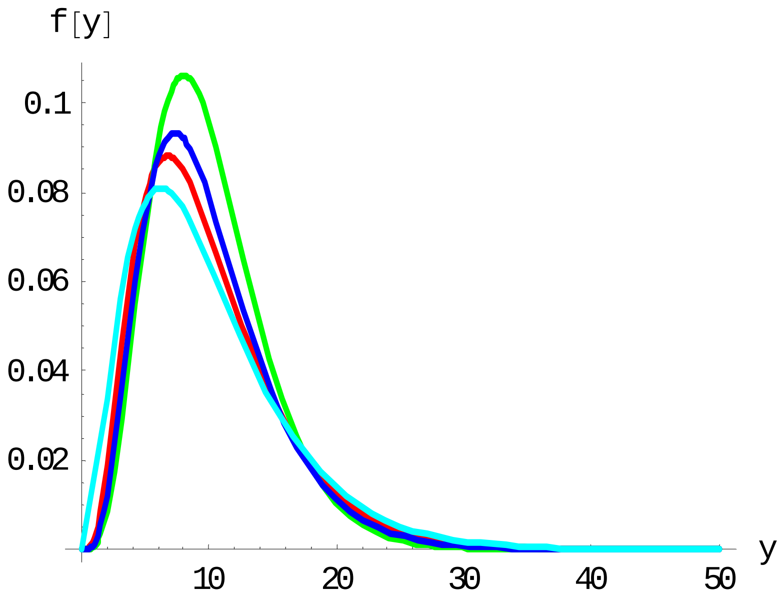

ρ = -0.7 (red), 0(green), 0.5(blue), 0.9 (cyan)

Figure 1. Sum of Chi-square variables for m = 5 and various ρ values