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Polynomial solutions of certain differential equations arising in physics

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Abstract

Linear differential equations of arbitrary order with polynomial coefficients are considered. Specifically, necessary and sufficient conditions for the existence of polynomial solutions of a given degree are obtained for these equations. An algorithm to determine these conditions and to construct the polynomial solutions is given. The effectiveness of this algorithmic approach is illustrated by applying it to several differential equations that arise in mathematical physics.

Key words: Differential equations, polynomial solutions, Schrödinger equation, Heun's equation, Davidson potential, Asymptotic iteration method (AIM), Maple

1 Introduction

Differential equations of the form $\sum_{k=0}^{N} a_k y^{(k)} = 0$ where a_k is a polynomial of degree $\leq k$ $(1 \leq k \leq N)$ have been studied by many authors, notably Bochner [1] and Brenke [3] for $N = 2$, and Krall [9] and Littlejohn [11] (see also [12]) for orthogonal polynomial solutions. However, the case when the polynomials a_k have arbitrary degree has not been investigated as extensively. In their recent paper [6], Ciftci et al. considered certain types of such equations that arise in mathematical physics. They specifically gave conditions for the existence of polynomial solutions using, in particular, the asymptotic iteration method (AIM) they introduced in their earlier work [5].

In this paper we consider linear differential equations of arbitrary order with polynomial coefficients of arbitrary degree. Our approach is based on linear algebra and provides not only a necessary and sufficient condition for the existence of polynomial solutions of such equations, but also an algorithmic procedure for the verification of this condition as well as for constructing these solutions. This is discussed in detail in Section 2. In Section 3, we include a Maple program that can be implemented to determine the conditions that guarantee the existence of polynomial solutions of any degree and to find them, depending of course on the available computational power. We hope that the computer program given here will be useful for researchers in solving linear differential equations with polynomial coefficients. To illustrate the efficiency of our algorithmic procedure, the Maple code is implemented in several examples, namely the one-dimensional Schrödinger equation, planar Coulomb diamagnetic problem, Bohr Hamiltonian with Davidson potential, and radial Schrödinger equation with shifted potential. We point out that in [6] the authors stated that finding polynomial solutions is a problem that needs to be investigated and suggested AIM for that.

2 Polynomial Solutions

Throughout, $\mathbb{P}$ is the space of all real polynomials and $\mathbb{P}_n$ is the subspace of polynomials with degree at most n. Let $L : \mathbb{P} \to \mathbb{P}$ be the linear operator given $Ly = \sum_{k=0}^{N} p_k(x)D^k y$, where D is the usual differential operator and $p_k(x) = \sum_{h\geq 0} p_{kh}x^h$ is a polynomial of degree d_k (with the convention that the zero polynomial has degree $-\infty$ and that $D^0 y = y$). Our objective is to find a necessary and sufficient condition for the equation $Ly = 0$ to have non-trivial polynomial solutions. Although this can be achieved, for each specific case, by comparing coefficients (see for example the determinantal necessary condition in the recent interesting paper [6] on Heun's equations), or by using the Asymptotic Iteration Method in the case of second-order equations [5], we feel that a systematic approach that works for differential equations of all orders and that can easily be implemented in a computer algebra system is more desirable.

Assume first that for some i $(0 \leq i \leq N), d_i > i$. Let $m = \max_{0\leq i\leq N} (d_i - i)$ and put $y = D^m z$. In this way, the equation $Ly = 0$ is equivalent to $Hz = 0$ where H is the linear operator $\sum_{k=1}^{m+N} a_k(x)D^K$, and $Ly = 0$ has a polynomial solution of degree $n \geq 0$ if and only if $Hz = 0$ has a polynomial solution of degree $n + m$. Clearly, for each nonnegative integer $n, \mathbb{P}_n$ is H-invariant, and H has thus the advantage over L of being directly amenable to an eigenvalue analysis as demonstrated below. We note here that the easier case when $d_i \leq i$ for all i can be discussed almost verbatim, with obvious modifications.

Let a_k $(k \geq 1)$ be the sequence of polynomials defined by $a_k = 0$ if $k < m$ and $a_k = p_{k-m}$ if $k \geq m$. Put $a_k(x) = \sum_{h\geq 0} a_{kh}x^h$, where $a_{kh} = 0$ if $k < h$. Since, for each nonnegative integer n, $H(x^n)$ is a scalar multiple of x^n plus lower order terms, we see that the matrix representation of H, with respect to the standard basis $B_n = \{1, x, ..., x^n\}$ of $\mathbb{P}_n$ is upper triangular and its eigenvalues are the coefficients of x^n in $H(x^n)$. More specifically, the $(n+1)\times(n+1)$ matrix A_n of H operating on $\mathbb{P}_n$ has (i,j)-th entry $\sum_{k\geq 1} a_{k,\,k+i-j}(j-k)_k$, i.e.

$$A_n = \left[\sum_{k\geq 1} a_{k,\,k+i-j}(j-k)_k\right]_{1\leq i,\,j\leq n+1}$$

where $(j-k)_k = (j-1)(j-2)\cdots(j-k)$, and where each row and column has at most $(N+m+1)$ nonzero entries. Clearly, the first m columns of A_n are zero and A_{n+1} is obtained by A_n by adding one row and one column at the end. As diagonal entries of A_n, all the eigenvalues of the operator H are real and are given by $\lambda_n = n!\sum_{k=1}^{n}\frac{a_{kk}}{(n-k)!}$ for $n\geq 1$ (note that $\lambda_0=\lambda_1=\cdots=\lambda_{m-1}=0$). Each eigenvalue λ_n has an eigenpolynomial $y_n(x) = y_{n0}+y_{n1}x+\cdots+y_{nn}x^n$ of degree at most n and whose vector representation $(y_{n0},...,y_{nn})^T$ in the standard basis B_n can be directly computed from the homogeneous upper triangular system $(A_n-\lambda_n I)(y_{n0},...,y_{nn})^T=0$. Our problem is to find necessary and sufficient conditions for which the operator H has an eigenpolynomial of degree $n+m$ corresponding to $\lambda_{n+m}=0$, that is necessary and sufficient conditions for the homogeneous system $A_{n+m}(y_{n+m,0},...,y_{n+m,n+m})^T=0$ to have a solution $(y_{n+m,0},...,y_{n+m,n+m})^T$ with $y_{n+m,n+m}=1$. This will follow from

Lemma 1. Let A be an $m\times n$ matrix. Then the homogeneous system $AX=0$ has a solution $X=(x_1,x_2,...,x_n)^T$ with $x_k\neq 0$ for some k if and only if $rank(A)=rank(A_k)$ *where* (A_k) *is the matrix obtained from* A *by deleting the* k^{th} *column.*

Proof. Put $A=[c_{ij}]_{1\leq i,j\leq n}$ and let c_k be the k^{th} column of A. Clearly, A and the augmented matrix $[A_k \vdots c_k]$ have the same rank. Hence,

$$rank(A)=rank(A_k)\Leftrightarrow rank[A_k\vdots c_k]=rank(A_k)\Leftrightarrow \text{ the system} A_kX=-c_k \text{ is consistent}$$
$$\Leftrightarrow \text{there exists a solution } X=(x_1,\cdots,x_{k-1},1,x_{k+1},\cdots,x_n)^T \text{ to the system } AX=0. \quad \square$$

Since A_{n+m-1} is obtained from A_{n+m} by deleting the last column, the above lemma immediately yields that the differential equation $Ly=0$ has a polynomial solution of degree $n\geq 0$ if and only if $rank(A_{n+m})=rank(A_{n+m-1})$. In this case, since A_{n+m} is upper triangular, the last entry of A_{n+m} is zero i.e. $\lambda_{n+m}=\sum_{k\geq 1}a_{kk}(m+n-k)_k=0$, and therefore the last row of A_{n+m} is zero.

Now let M_n and M'_n be, respectively, the matrices obtained from A_{n+m} and A_{n+m-1} by deleting the first m zero columns. Clearly $rank(A_{n+m}) = rank(A_{n+m-1})$ if and only if $rank(M_n) = rank(M'_n)$. It is easy to see that the $(i,j)^{th}$ entry of the $(n+m+1)\times(n+1)$ matrix M_n is $\sum_{t=0}^{j-1} a_{t+m,\,t+i-j}(j-t)_{t+m} = \sum_{t=0}^{j-1} p_{t,\,t+i-j}(j-t)_{t+m}$. This proves the main result of this note:

Proposition 2. Let M_n be the $(n+m+1)\times(n+1)$ matrix with $(i,j)^{th}$ entry $\sum_{t=0}^{j-1} p_{t,\,t+i-j}(j-t)_{t+m}$ and let M'_n be the matrix obtained from M_n by deleting the last column. Then the differential equation $Ly = 0$ has a polynomial solution of degree $n \geq 0$ if and only if $rank(M_n) = rank(M'_n)$.□

It thus follows that if the equation $Ly = 0$ has a polynomial solution of degree $n \geq 0$, then $\lambda_{n+m} = \sum_{t\geq 1} p_{t,\,t+m}(n-t)_{(t+m)} = 0$, and since M'_n has n columns, $rank(M_n) = rank(M'_n)$ implies that $rank(M_n) \leq n$ and so every $(n+1)\times(n+1)$ submatrix of M_n has zero determinant. This generalizes Theorems 5 and 6 of [6].

3 Maple Code and Examples

In this section we illustrate the effectiveness of the algorithmic approach of Section 2 by applying it to four differential equations that appear in [6]. These arise in mathematical physics and, more precisely, in the study of solutions to Schrödinger equation [10], planar Coulomb diamagnetic problem [4], Bohr Hamiltonian with Davidson potential [2] and radial Schrödinger equation with shifted potential [6, 7, 8]. The examples show how to implement the method algorithmically to determine the conditions for the existence of polynomial solutions and also to calculate the corresponding polynomial solutions.

Example 1

As a first example, we consider the linear second order ODE arising in the study of one dimensional Schrödinger problems [10]. The investigation of Krylov and Robnik [10] about polynomial solutions of one dimensional Schrödinger problems leads to investigation of polynomial solutions of the following differential equation. The conditions for existence of polynomial solutions of this ODE have also been discussed by Ciftci et al. in the recent paper [6], by a different approach.

$$x^3 \frac{d^2}{dx^2} y(x) + \alpha \left(x^2 - 1\right) \frac{d}{dx} y(x) + (\beta x + g) y(x) = 0 \tag{3.1}$$

Here, we apply our method to determine existence conditions as well as to compute the corresponding polynomial solutions of the above differential equation. A Maple code is also provided to show how the method can be implemented algorithmically using software.

For $g = 0$, the algorithmic procedure of Section 2 can be implemented to generate a sequence of even degree polynomial solutions of ODE (3.1). A further analysis of these solutions yields the following result.

- For $g = 0$, ODE (3.1) admits polynomial solutions of degree $n = 2m$ ($m \geq 1$), with $\beta = -(\alpha n + n^2 - n)$, given by

$$y = x^{2m} + \sum_{i=1}^{m} \frac{(-1)^i \binom{m}{i} \alpha^i x^{2m-2i}}{(\alpha + 2n - 3)(\alpha + 2n - 5) \cdots (\alpha + 2n - 3 - 2(i-1))}$$

  It should be noted that, for $g = 0$, no lower odd degree polynomial solutions of ODE (3.1) were found.

For $g \neq 0$ some examples of polynomial solutions, found using the construction procedure of Section 2, are given in Table 1 below.

n	β	g	Polynomial solution of ODE (3.1) of degree n
1	$-\alpha$	$\pm\alpha$	$x \pm 1$
2	$-2\alpha - 2$	$\pm\sqrt{4\alpha^2 + 6\alpha}$	$x^2 \pm \frac{\sqrt{2\alpha(2\alpha+3)}}{\alpha+2} + \frac{\alpha}{\alpha+2}$
3	$-3\alpha - 6$	$\pm\sqrt{15\alpha + 5\alpha^2 + \alpha A}$	$4x^3 \pm \frac{12\alpha(9+2\alpha+A)}{\alpha\sqrt{15+5\alpha+A}(-3-2\alpha+A)} x^2 + \frac{24\alpha}{-3-2\alpha+A} x \pm \frac{24\alpha^2}{\alpha(-3-2\alpha+A)\sqrt{15+5\alpha+A}}$
		$\pm\sqrt{15\alpha + 5\alpha^2 - \alpha A}$	$4x^3 \pm \frac{12\alpha(-9-2\alpha+A)}{\alpha\sqrt{15+5\alpha-A}(3+2\alpha+A)} x^2 - \frac{24\alpha}{3+2\alpha+A} x \mp \frac{24\alpha^2}{\alpha(3+2\alpha+A)\sqrt{15+5\alpha-A}}$ where $A = \sqrt{153 + 96\alpha + 16\alpha^2}$

Table 1:

In general, for any given n and α, the algorithmic procedure of Section 2 can easily be implemented to determine β, g for which ODE (3.1) admits polynomial solutions of degree n as well as to compute the corresponding polynomial solution. We provide below a set of

Maple commands that can be used as template to compute polynomial solutions of given degree n of ODE (3.1) for any given value of α. For illustration we take $\alpha = \frac{-15}{2}$ and look for a solution of degree $n = 6$. The following Maple code determines $\beta = 15$, $g = 3(750)^{1/4}$ and computes the corresponding polynomial solution of degree 6 of ODE (3.1) as

$$\begin{aligned} y(x) &= 7\,x^6 + \frac{7}{30}\,1080^{3/4}x^5 + 126\,\frac{\sqrt{30}\left(-65+12\sqrt{30}\right)}{66\sqrt{30}-360}x^4 + 420\,\frac{\sqrt[4]{1080}\left(6\sqrt{30}-30\right)}{66\sqrt{30}-360}x^3 \\ &+ 1575\,\frac{\left(-72+6\sqrt{30}\right)}{66\sqrt{30}-360}x^2 + 630\,\frac{\left(-72+6\sqrt{30}\right)750^{3/4}}{\left(66\sqrt{30}-360\right)\left(-60+5\sqrt{30}\right)}x \\ &- 7875\,\frac{\left(-72+6\sqrt{30}\right)\sqrt{30}}{\left(66\sqrt{30}-360\right)\left(-60+5\sqrt{30}\right)} \end{aligned} \tag{3.2}$$

The Maple code with brief explanations is presented below.

```
restart:
with(LinearAlgebra):
α := −15/2 :
N := 2 :
pcoeff := Array(0 .. N):
pcoeff[0] := βx + g :
pcoeff[1] := α(x^2 − 1) :
pcoeff[2] := x^3 :
```

These commands define the order N of the ODE, the value of α and the coefficients $p_k(x)$ of the operator $\sum_{k=0}^{N} D^k y$. Next we determine the value of m and the coefficients $a_k(x)$ of the operator $\sum_{k=1}^{m+N} D^k z$ using the following commands.

```
Vm := Array(0..N) :
dk := Array(0..N) :
for i from 0 to N do
   dk[i] := degree(pcoeff[i], x)
   Vm[i]:= degree(pcoeff[i],x)-i end do:
m := max(Vm) :
dkmax := max(dk) :
acoeff := Array(1 .. m+N):
for i from m to (m+N) do
   acoeff[i]:= pcoeff[i-m] end do:
```

Matrices A_{n+m}, A_{n+m-1} of the procedure of Section 2 are computed by the following set of commands in which "*soldegree*", *An* and *Anprime* respectively denote the degree of the sought polynomial solution, the matrix A_{n+m} and the matrix A_{n+m-1}.

```
soldegree := 6:
```

```
n := soldegree+m:
cAM := max(dkmax, n+m+N):
rAM := m+N:
AM := Array(1 .. rAM, 0 .. cAM):
for i from m to rAM do
  AM[i,0]:= coeff(x acoeff[i],x^1) end do:
for i from m to rAM do
  for j from 1 to cAM do
    AM[i, j] := coeff(acoeff[i], x^j)
  end do
end do
An := Matrix(n+1,n+1):
for i from 1 to n+1 do
  for j from 1 to n+1 do
    for k from 1 to rAM do
      if (k+i-j) ≥ 0 and (k-j+1) ≤ 0
        then An[i,j]:= An[i,j]+AM[k,k+i-j] (j−1)!/(j−k−1)!
      end if
    end do
  end do
end do
Anprime := Matrix(n+1,n)
for i from 1 to n+1 do
  for j from 1 to n do
    Anprime[i, j] := An[i, j]
  end do
end do
```

At this stage we have $Rank(An) \neq Rank(Anprime)$. Next we determine the values of parameters so that $Rank(An)$ equals $Rank(Anprime)$. The first condition employed is the vanishing of the last diagonal entry of the upper triangular matrix An which determines β by the following commands.

```
ansbeta := solve(An[n+1, n+1] = 0, β):
β := ansbeta
```

Implementing the fact that the 7×7 submatrix, obtained by deleting the last zero row of An, must have zero determinant provides the value of g via the commands below.

```
Andet := Matrix(n,n):
for i from 1 to n do
```

for j from 1 to n do

Andet[i, j] := An[i, j+1]

end do

end do

Determinant(Andet):

ansdet := solve(Determinant(Andet) = 0, g)

This leads to seven roots. At this stage a root needs to be chosen before checking the rank condition. For illustration we choose $g = 3(750)^{1/4}$.

g:= $3(750)^{1/4}$:

Rank(An)

Rank(Anprime)

As the rank condition is satisfied so the desired polynomial solution can be obtained by the following set of commands.

kern := NullSpace(An)

The output of the above command contains two vectors $kern_1$ and $kern_2$ with $kern_2 = (1,0,0,0,0,0,0)^T$ so the vector $kern_1$ is used as follows to find the solution.

Vkern := $kern_1$:

solm := 0:

for i from 1 to (n+1) do

solm:= solm + Vkern[i] x^{i-1} *end do:*

soln := diff(solm, x)

The output *soln* provides the solution given in Equation (3.2).

Example 2

Consider the ODE

$$\frac{d^2}{dx^2}y(x) + \left(p - 2x^2\right)\frac{d}{dx}y(x) + (\delta\, x + \alpha)\, y(x) = 0 \tag{3.3}$$

The question of investigating the polynomial solutions of ODE (3.3) arises from the study of polynomial solutions of Coulomb diamagnetic problem by Chhajlany and Malnev [4]. Ciftci et al. [6, Eq.18,19] provide conditions for the existence of polynomial solutions of ODE (3.3). Here we implement our procedure to demonstrate how to generate polynomial solutions of ODE (3.3) in a straightforward algorithmic manner.

For general $\alpha \neq 0$ some examples of polynomial solutions listed in Table 2 below are obtained by adapting the Maple code presented in Example 1. The conditions on the parameters δ and p, for having these solutions, are also determined.

n	δ	p	Polynomial solution of ODE (3.3) of degree n
1	2	$\frac{\alpha^2}{2}$	$2x-\alpha$
2	4	$\frac{\alpha^3+16}{8\alpha}$	$x^2-\frac{\alpha}{2}x+\frac{\alpha^3-16}{16\alpha}$
3	6	$\frac{5\alpha^2}{18}\pm\frac{2\sqrt{\alpha^4-54\,\alpha}}{9}$	$4x^3-2\alpha x^2-\frac{1}{3}\frac{\left(\pm 5\,\alpha^3+4\,\sqrt{\alpha\,(\alpha^3-54)}\alpha\mp 216\right)\alpha\,x}{\pm\alpha^2+2\,\sqrt{\alpha\,(\alpha^3-54)}}$ $+\frac{1}{54}\frac{\pm 41\,\alpha^5+40\,\alpha^3\sqrt{\alpha\,(\alpha^3-54)}\mp 1728\,\alpha^2-432\,\sqrt{\alpha\,(\alpha^3-54)}}{\pm\alpha^2+2\,\sqrt{\alpha\,(\alpha^3-54)}}$
4	8	$\frac{1}{64}\,\frac{5\,\alpha^3+192\pm 3\,A}{\alpha}$	$5x^4-\frac{5\alpha}{2}x^3+\frac{15}{32}\,\frac{\left(\mp\alpha^6\pm 768\,\alpha^3+\alpha^3A\mp 4096-64\,A\right)x^2}{\alpha\,(\pm 3\,\alpha^3\mp 192+5\,A)}$ $\pm\frac{5}{64}\,\frac{\left(\mp\alpha^9\mp 128\,\alpha^6+\alpha^6A\pm 8192\,\alpha^3-1728\,\alpha^3A\pm 262144+4096\,A\right)x}{(\pm 3\,\alpha^3\mp 192+5\,A)(\mp\alpha^3\pm 64+A)}$ $\mp\frac{5}{2048}\,\frac{B}{\alpha^2(\pm 3\,\alpha^3\mp 192+5\,A)(\mp\alpha^3\pm 64+A)}$ where $A=\sqrt{\alpha^6-384\,\alpha^3+4096}$ and $B=\mp\alpha^{12}\mp 2816\,\alpha^9+\alpha^9A\pm 524288\,\alpha^6-5184\,\alpha^6A$ $\pm 19922944\,\alpha^3+200704\,\alpha^3A\mp 150994944-2359296\,A.$

Table 2:

Depending on the available computational power, for a given value of α, the algorithmic procedure of Section 2 can be implemented, as in Example 1, to compute polynomial solutions of ODE (3.3) of any given degree n. As example the following solutions of degree 9 and 25 are found.

- $\alpha=0$, $n=9$ implies $\delta=18$, $p=\frac{4}{35}(144830)^{1/3}$ and the polynomial solution of degree 9 of ODE (3.3) given by

$$y(x)=10x^9-\frac{18}{7}\sqrt[3]{144830}x^7-120x^6+\frac{9}{35}\,144830^{2/3}x^5+\frac{666}{35}\sqrt[3]{144830}x^4 -\frac{10314}{7}x^3-\frac{8478}{8575}\,144830^{2/3}x^2+\frac{4239}{245}\sqrt[3]{144830}x+\frac{19803534}{8575}$$

- For $\alpha=0$, $n=25$ with $\delta=50$ and $p=0$ the polynomial solution of degree 25 of ODE (3.3) is given by

$$y(x)=26\,x^{25}-2600\,x^{22}+100100\,x^{19}-1901900\,x^{16}+19019000\,x^{13} -98898800\,x^{10}+247247000\,x^7-247247000\,x^4+61811750\,x$$

Example 3

The analysis of solutions of the Bohr Hamiltonian for Davidson potential leads to investigation of exact solutions of a differential equation [2, Eq.49] which can be rewritten as

[6, Eq.22]

$$x\frac{d^2}{dx^2}y(x) - \left(2x^2 - 2\mu - 2\right)\frac{d}{dx}y(x) - (2\mu + 3 - \epsilon)\,xy(x) = 0 \tag{3.4}$$

Adapting the Maple code of Example 1 for ODE (3.4) readily generates polynomial solutions of a given degree n. Some examples for solutions of even as well as odd degrees are provided below in Tables 3 and 4 respectively.

n	ϵ	Polynomial solution of ODE (3.4) of degree $n = 2m$
0	$2\mu + 3$	1
2	$2\mu + 7$	$2x^2 - (2\mu + 3)$
4	$2\mu + 11$	$4x^4 - 4(2\mu + 5)x^2 + (2\mu + 3)(2\mu + 5)$
6	$2\mu + 15$	$8x^6 - 12(2\mu + 7)x^4$ $+6(2\mu + 5)(2\mu + 7)x^2 - (2\mu + 3)(2\mu + 5)(2\mu + 7)$
8	$2\mu + 19$	$16x^8 - 32(2\mu + 9)x^6 + 24(2\mu + 7)(2\mu + 9)x^4$ $-8(2\mu + 5)(2\mu + 7)(2\mu + 9)x^2 + (2\mu + 3)(2\mu + 5)(2\mu + 7)(2\mu + 9)$
10	$2\mu + 23$	$32x^{10} - 80(2\mu + 11)x^8 + 80(2\mu + 9)(2\mu + 11)x^6$ $-40(2\mu + 7)(2\mu + 9)(2\mu + 11)x^4$ $+10(2\mu + 5)(2\mu + 7)(2\mu + 9)(2\mu + 11)x^2$ $-(2\mu + 3)(2\mu + 5)(2\mu + 7)(2\mu + 9)(2\mu + 11)$

Table 3:

It should be noted that for this case polynomial solutions can easily be computed without much computational cost. For instance, Maple could compute polynomial solution of degree $n = 100$ in computational time of 1.8 seconds.

n	ϵ	μ	Polynomial solution of ODE (3.4) of degree $n = 2m + 1$
1	$2\mu + 5$	-1	x
3	$2\mu + 9$	-1	$x^3 - \frac{3}{2}x$
		-2	x^3
5	$2\mu + 13$	-1	$x^5 - 5x^3 + \frac{15}{4}x$
		-2	$x^5 - \frac{5}{2}x^3$
		-3	x^5
7	$2\mu + 17$	-1	$x^7 - \frac{21}{2}x^5 + \frac{105}{4}x^3 - \frac{105}{8}x$
		-2	$x^7 - 7x^5 + \frac{35}{4}x^3$
		-3	$x^7 - \frac{7}{2}x^5$
		-4	x^7

Table 4:

A further analysis of the odd lower degree polynomial solutions of ODE (3.4), given in Table 4, yields the following family of solutions of degree $n = 2m + 1$.

For $k = 0, 1, 2, 3$ ODE (3.4) admits the following class of polynomial solutions of degree $n = 2m + 1$ $(m \geq k)$ with $\mu = -(m + 1 - k)$ and $\epsilon = 2m + 2k + 3$.

- If $k = 0$ $(m \geq 0)$

$$y = x^{2m+1}$$

- if $k = 1$ $(m \geq 1)$

$$y = x^{2m+1} - \frac{2m+1}{2} x^{2m-1}$$

- if $k = 2, 3$ $(m \geq k)$

$$y = x^{2m+1} + k \sum_{i=1}^{k-1} \frac{(-1)^i (2m+1)(2m-1) \cdots (2m+1-2(i-1))}{2^i} x^{2m+1-2i} + \frac{(-1)^k (2m+1)(2m-1) \cdots (2m+1-2(k-1))}{2^k} x^{2m+1-2k}$$

Example 4

As a final example, we consider a question related to the investigation of the radial Schrödinger equation with shifted Coulomb potential which has been discussed recently in [6, 7, 8]. The anstaz of [6, Eq.35] that the radial Schrödinger equation admits a solution which vanishes at the origin and at infinity leads to the question of obtaining solutions of the following differential equation; the reader is referred to [6] for details.

$$x(x+\beta)\frac{d^2}{dx^2}y(x) + \left(-2\alpha x^2 + 2(K+1-\alpha\beta)x + 2\beta(K+1)\right)\frac{d}{dx}y(x) + \left((-2\alpha(K+1) + 2Z)x - 2\alpha\beta(K+1)\right)y(x) = 0 \tag{3.5}$$

This is a particular case of the confluent Heun equation whose polynomial solutions can be studied algorithmically using our procedure. While discussing the question of polynomial solutions of ODE (3.5), Ciftci et al. [6] provide conditions on parameters α, β to have polynomial solutions. In particular a table was provided which listed conditions on parameters for the existence of polynomial solutions for $n = 1, 2, 3, 4$. However, it was pointed out in [6] that finding the corresponding polynomial solutions is an open problem that remains to be solved. For a given value K and the given degree n of the required polynomial solution, adapting the Maple code of example 1 for ODE (3.5) can determine conditions on parameters α, β as well as generate corresponding polynomial solutions of ODE (3.5). In Table 5 below, we demonstrate this by providing some examples of polynomial solutions of ODE (3.5) of degree $n = 1, 2, 3, 4, 5$.

n	α	K	β	Polynomial solution of ODE (3.5) of degree n
1	$\frac{Z}{K+2}$	any	$\frac{K+2}{Z}$	$x+\frac{K+2}{Z}$
2	$\frac{Z}{K+3}$	any	$\frac{\left(3\,K+6\pm\sqrt{K^2+8\,K+12}\right)(K+3)}{2(K+2)Z}$	$3\,x^2+3\,\frac{(K+3)\left(3\,K+6\pm\sqrt{K^2+8\,K+12}\right)(2\,K+3)x}{(K+2)Z\left(K\pm\sqrt{K^2+8\,K+12}\right)}$ $+3\,\frac{\left(3\,K+6\pm\sqrt{K^2+8\,K+12}\right)(K+3)^2(2\,K+3)}{(K+2)Z^2\left(K\pm\sqrt{K^2+8\,K+12}\right)}$
3	$\frac{Z}{K+4}$	$-\frac{3}{2}$	$\frac{5}{2Z}$	$4x^3+\frac{10}{Z}x^2$
			$\frac{25}{2Z}$	$4x^3+\frac{110}{Z}x^2+\frac{1875}{2Z^2}x+\frac{9375}{4Z^3}$
4	$\frac{Z}{K+5}$	$-\frac{3}{2}$	$\frac{49}{2Z}$	$5x^4+\frac{1435}{4}\,\frac{x^3}{Z}+\frac{18375}{2}\,\frac{x^2}{Z^2}+\frac{3109295}{32}\,\frac{x}{Z^3}+\frac{21765065}{64Z^4}$
			$\frac{7(\pm15+\sqrt{65})}{20Z}$	$5x^4+28\,\frac{\left(\pm15+\sqrt{65}\right)x^3}{Z\left(\pm1+\sqrt{65}\right)}+98\,\frac{\left(\pm15+\sqrt{65}\right)x^2}{\left(\pm1+\sqrt{65}\right)Z^2}$
5	$\frac{Z}{K+6}$	$-\frac{3}{2}$	$\frac{81}{2Z}$	$6x^5+891\,\frac{x^4}{Z}+51030\,\frac{x^3}{Z^2}+1390932\,\frac{x^2}{Z^3}$ $+\frac{282195171}{16}\,\frac{x}{Z^4}+\frac{2539756539}{32Z^5}$

Table 5:

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