King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 428

Dec 2012

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A Bivariate Probability Model Based on Type II Gumbel Random Variable and its Properties

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Abstract

A bivariate Type II Gumbel Probability Model belonging to Farlie-Gumbel-Morgenstern family has been proposed and studied. We obtain the distribution of the product of the components, distribution of the ratio of the components and the reliability of the model. We also obtain different types of moments.

Keywords and phrases: Bivariate Type-II Gumbel Probability Model; distribution of the product; distribution of the ratio; moments; reliability function.

1 Introduction

Univariate Gumbel Type II Probability Model is used in life time data analysis. Distributions with copulas accommodate a variety of situations and have got applications in areas including flood prediction, risk management, portfolio analysis, epidemiology etc. See Yan, J. (2007) and the references therein. Proposed by Morgenstern (1956) and extended by Farlie (1960), this class of distributions is nowadays known as the Farlie-Gumbel-Morgenstern (FGM) class of distributions. See, for example, Kotz, Balakrsihnan and Johnson (2000, pp 51-55).

In this paper, we introduce a new of Bivariate Type-II Gumbel distribution which belongs to Farlie-Gumbel-Morgenstern (FGM) family satisfying the following property:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)\left[1+\gamma(1-2F_X(x))(1-2F_Y(y))\right], \tag{1.1}$$

where $f\left(x,y\right)$ is the joint density function of a bivariate probability model, $f_X(x)$ and $f_Y(y)$ are the marginal density functions of their components X and Y respectively, $F(x)$ and $F_Y(y)$ are the corresponding cumulative distribution functions and $|\rho|<1$. The model in (1.1) appears in Gumbel (1960).

Many works have been done on the product and ratio in case of independence of the components. We refer to Nakagami and Ota (1957) and Malik and Trudel (1986) among others for the distribution of the product and that of the ratio of components of correlated bivariate distributions. Since life distributions of coherent dependent systems involve components that are correlated, it is important to develop foundation results for these correlated probability models. See for example, Mukherjee and Sasmal (1977).

We develop the FGM family of Bivariate Type-II Gumbel Probability Model in Section 2. We derive the product moments of the model in Section 3, the distribution of the product of the components in Section 4, the distribution of the ratio of the components in Section 5. Finally we derive the reliability function of the model in Section 6.

The following mathematical formula will be used in the paper.

$$a.\ \int_0^\infty e^{-cx}x^{\alpha-1}dx = c^{-\alpha}\ \Gamma(\alpha), \quad \alpha>0. \tag{1.1}$$

$$b.\ B(p,q)=\int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy,\ Re\left(p\right)>0,\ Re\left(q\right)>0, \tag{1.2}$$

(eq. 8.380.3 of Gradshteyn and Ryzhik, 2007, 908).

$$c.\ B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{1.3}$$

(eq. 8.384.1 of Gradshteyn and Ryzhik, 2007, 909).

d. The modified Bessel function of second kind of order p is given by

$$K_p(x)=\left\{\sqrt{\pi}(x/2)^p\big/\Gamma\left(p+(1/2)\right)\right\}\int_1^\infty e^{-xt}(t^2-1)^{p-(1/2)}dt, \tag{1.4}$$

for $Re\left(p+(1/2)\right)>0$, $|arg(x)|<\pi/2$, or, for $Re\left(x\right)=0$, and $p=0$ (eq. 8.432.3 of Gradshteyn and Ryzhik, 2007, 917) .

$$e.\ I_p(\beta,\gamma)=\int_0^\infty x^{p-1}\exp\left(-(\beta/x)-\gamma x\right)dx = 2(\beta/\gamma)^{p/2}K_p(2\sqrt{\beta\gamma}), \tag{1.5}$$

$Re(\beta) > 0,\ Re(\gamma) > 0$ (eq. 3.471.9 of Gradshteyn and Ryzhik, 2007, 368), where $K_p(.)$ is the modified Bessel function of second kind given in equation (1.4). A special form of the modified Bessel function of second kind is given below:

$$f.\ K_0(x) = \int_0^\infty \frac{\cos(xt)}{\sqrt{1+t^2}} dt. \tag{1.6}$$

$$g.\ \int_0^\infty x^{\mu-1} e^{-\beta x} \Gamma(\nu, \xi x) dx = \frac{\xi^\nu \Gamma(\mu+\nu)}{\mu(\xi+\beta)^{\mu+\nu}} {}_2F_1\left(1, \mu+\nu; \mu+1; \frac{\beta}{\xi+\beta}\right), \tag{1.7}$$

where ${}_2F_1(a,b;c;z)$ is the generalized hypergeometric function, $\xi+\beta > 0,\ \mu > 0,\ \mu+\nu > 0$, (Gradshteyn and Ryzhik, 2007, 368),

2 The Bivariate Type-II Gumbel Probability Model

Let X and Y be two independent Type II Gumbel random variables with probability density functions (pdf) given by

$$f_X(x;\alpha,\beta) = \alpha\beta x^{-(\alpha+1)} \exp\left(-\beta x^{-\alpha}\right); x > 0,\ \alpha > 0,\ \beta > 0, \tag{2.1}$$

and

$$f_Y(y;\alpha,\beta) = \alpha\beta y^{-(\alpha+1)} \exp\left(-\beta y^{-\alpha}\right);\ \ y > 0, \alpha > 0, \beta > 0, \tag{2.2}$$

respectively.

The cumulative distribution functions (cdf) of these distributions are known to be

$$F_X(x;\alpha,\beta) = 1 - \exp\left(-\beta x^{-\alpha}\right),\ x > 0,\ \alpha > 0,\ \beta > 0, \tag{2.3}$$

and

$$F_Y(y;\alpha,\beta) = 1 - \exp\left(-\beta y^{-\alpha}\right),\ \ y > 0,\ \alpha > 0,\ \beta > 0, \tag{2.4}$$

respectively. The following theorem is obvious from (1.1).

Theorem 2.1 Let X and Y be two Type II Gumbel random variables with probability density functions $f_X(x;\alpha,\beta)$ and $f_Y(y;\alpha,\beta)$ given by the equations (2.1) and (2.2) respectively, with cumulative distribution functions $F_X(x;\alpha,\beta)$ and $F_Y(y;\alpha,\beta)$ given by the equations (2.3) and (2.4) respectively. Then the joint density function is given by the following:

$$f_{X,Y}(x,y;\alpha,\beta;\rho)=(\alpha\beta)^2(xy)^{-(\alpha+1)}\Big[\big(1+\rho\big)\exp\big(-\beta(x^{-\alpha}+y^{-\alpha})\big)+4\rho\exp\big(-2\beta(x^{-\alpha}+y^{-\alpha})\big)$$
$$-2\rho\exp\big(-\beta(2x^{-\alpha}+y^{-\alpha})\big)-2\rho\exp\big(-\beta(x^{-\alpha}+2y^{-\alpha})\big)\Big],$$

where $x>0,\ y>0,\ \alpha>0,\ \beta>0$ and $-1\le\rho\le 1$. (2.5)

Theorem 2.2 Let X and Y be two Type II Gumbel random variables with probability density functions $f_X(x;\alpha,\beta)$ and $f_Y(y;\alpha,\beta)$ given by the equations (2.1) and (2.2) respectively, with cumulative distribution functions $F_X(x;\alpha,\beta)$ and $F_Y(y;\alpha,\beta)$ given by the equations (2.3) and (2.4) respectively. Then the Cumulative Probability function $F_{X,Y}(x,y)$ is given by the following:

$$F_{X,Y}(x,y)=\big(1-\exp(-\beta x^{-\alpha})\big)\big(1-\exp(-\beta y^{-\alpha})\big)\Big[1+\rho\exp\big(-\beta(x^{-\alpha}+y^{-\alpha})\big)\Big], \tag{2.6}$$

where $\alpha>0,\beta>0$ and $|\rho|<1$.

Proof. It follows from (1.1) that $F_{X,Y}(x,y)=F_X(x)F_Y(y)[1+\rho(1-F_X(x))(1-F_Y(y))],\ |\rho|<1$. Then by using the equations (2.3) and (2.4), we have

$$F_{X,Y}(x,y)=\big(1-\exp(-\beta x^{-\alpha})\big)\big(1-\exp(-\beta y^{-\alpha})\big)\Big[1+\rho\exp(-\beta x^{-\alpha})\exp(-\beta y^{-\alpha})\Big],$$

which simplifies to the equation (2.6).

Let X and Y be two uncorrelated Type II Gumbel random variables with probability density functions $f_X(x;\alpha,\beta)$ and $f_Y(y;\alpha,\beta)$ given by the equations (2.1) and (2.2) respectively. Then the joint density function of X and Y, and their Cumulative Probability Function are given by

$$f_{X,Y}(x,y;\alpha,\beta;0)=(\alpha\beta)^2(xy)^{-(\alpha+1)}\exp\big(-\beta(x^{-\alpha}+y^{-\alpha})\big), \tag{2.7}$$

and

$$F_{X,Y}(x,y)=\big(1-\exp(-\beta x^{-\alpha})\big)\big(1-\exp(-\beta y^{-\alpha})\big), \tag{2.8}$$

respectively where $\alpha>0$ and $\beta>0$.

3. Product Moments

Theorem 3.1. Let X and Y have a Bivariate Type II Gumbel Probability Model with density function given by equation (2.5). Then the (a,b)-th product moment of Bivariate Type II Gumbel density function denoted by $\mu'_{a,b;\rho}(X,Y)$ is given by

$$\mu'_{a,b;\rho}(X,Y) = \beta^{(a+b)/\alpha}\Gamma\left(1-(a/\alpha)\right)\Gamma\left(1-(b/\alpha)\right)\left[1+\rho(2^{a/\alpha}-1)(2^{b/\alpha}-1)\right], \qquad (3.1)$$

where the real numbers a and b are such that $\max\{a,b\} < \alpha$ and $-1 < \rho < 1$.

Proof. Using the equation (2.5), the (a,b)-th product moment $\mu'_{a,b;\rho}(X,Y)$ is given by

$$E(X^aY^b) = \int_0^\infty \int_0^\infty x^a y^b f_{X,Y}(x,y)dxdy,$$

which can be simplified to

$$E(X^aY^b) = (\alpha\beta)^2\left[(1+\rho)I_a(1)I_b(1) + 4\rho I_a(2)I_b(2) - 2\rho I_a(2)I_b(1) - 2\rho I_a(1)I_b(2)\right], \quad (3.2)$$

where

$$I_k(c) = \int_0^\infty x^{k-\alpha-1}\exp\left(-\beta c x^{-\alpha}\right)dx \qquad (3.3)$$

By making a transformation $\beta x^{-\alpha} = s$ we have

$$I_k(c) = \int_0^\infty s^{-k/\alpha}\exp\left(-\beta c s\right)ds.$$

which, by (1.1), is evaluated to be

$$I_k(c) = \frac{1}{\alpha}(\beta c)^{(k/\alpha)-1}\,\Gamma\left(1-(k/\alpha)\right), \quad k < \alpha.$$

Having evaluated the other integrals similarly, we have

$$I_a(1)\,I_b(1) = \frac{1}{\alpha^2}\beta^{((a+b)/\alpha)-2}\,\Gamma\left(1-(a/\alpha)\right)\Gamma\left(1-(b/\alpha)\right),$$

$$I_a(2)\,I_b(2) = \frac{1}{\alpha^2}(2\beta)^{((a+b)/\alpha)-2}\,\Gamma\left(1-(a/\alpha)\right)\Gamma\left(1-(b/\alpha)\right),$$

$$I_a(2)\,I_b(1) = \frac{1}{\alpha^2}\times 2^{(a/\alpha)-1}\beta^{((a+b)/\alpha)-2}\,\Gamma\left(1-(a/\alpha)\right)\Gamma\left(1-(b/\alpha)\right),$$

$$I_a(1)\,I_b(2) = \frac{1}{\alpha^2}\times 2^{(b/\alpha)-1}\beta^{((a+b)/\alpha)-2}\,\Gamma\left(1-(a/\alpha)\right)\Gamma\left(1-(b/\alpha)\right),$$

where $\max\{a,b\} < \alpha.$

Using the above results in equation (3.2), and performing a bit of simplification, we have

$$E(X^a Y^b) = \beta^{((a+b)/\alpha)}\ \Gamma\left(1-(a/\alpha)\right)\Gamma\left(1-(b/\alpha)\right)\left[(1+\rho)+2^{(a+b)/\alpha}\rho-2^{a/\alpha}\rho-2^{b/\alpha}\rho\right],$$

which equals the equation (3.1).

By putting $\rho = 0$ in equation (3.1), we have

$$\mu'_{a,b;0}(X,Y) = \beta^{(a+b)/\alpha}\ \Gamma\left(1-(a/\alpha)\right)\Gamma\left(1-(b/\alpha)\right),$$

or, $\mu'_{a,b;0}(X,Y) = \mu'_{a,0;0}(X,Y)\mu'_{0,b;0}(X,Y),\ \ \max\{a,b\} \le \alpha,$ which is the product of a-th and b-th moments of two independent univariate Type II Gumbel distributions with shape parameter α and scale parameter β respectively with density function in (2.7). Also if $b = 0,$ in (3.1), then $\mu'_{a,0;\rho}(X,Y) = \beta^{a/\alpha}\Gamma\left(1-(a/\alpha)\right),\ a \le \alpha,$ which is the a-th moment $\mu'_a(X)$ of X as expected.

With the notation in (3.1), the correlation coefficient between X and Y is

$$\rho_{X,Y} = \frac{\mu_{1,1;\rho}(X,Y)}{\sqrt{\mu_{2,0;\rho}(X,Y)\mu_{0,2;\rho}(X,Y)}}.$$

From (3.1), we have

$$\begin{aligned}\mu_{1,1;\rho}(X,Y) &= \mu'_{1,1;\rho}(X,Y) - \mu'_{1,0;\rho}(X,Y)\mu'_{0,1;\rho}(X,Y)\\ &= \beta^{2/\alpha}\Gamma^2\left(1-(1/\alpha)\right)\left[1+\rho(2^{1/\alpha}-1)^2(2^{b/\alpha}-1)\right] - \beta^{1/\alpha}\Gamma\left(1-(1/\alpha)\right)\times\beta^{1/\alpha}\Gamma\left(1-(1/\alpha)\right),\end{aligned}$$

$$\mu_{1,1;\rho}(X,Y) = (2^{1/\alpha}-1)^2\beta^{2/\alpha}\Gamma^2\left(1-(1/\alpha)\right)\rho,$$

and

$$\mu'_{2,0;\rho}(X,Y) = \mu'_{0,2;\rho}(X,Y) = \beta^{2/\alpha}\Gamma\left(1-(2/\alpha)\right).$$

Hence the correlation coefficient between X and Y is explicitly given by

$$\rho_{X,Y} = \frac{(2^{1/\alpha}-1)^2\Gamma^2\left(1-(1/\alpha)\right)}{\Gamma\left(1-(2/\alpha)\right)}\rho, \tag{3.4}$$

where $\alpha > 2$ and $-1 < \rho < 1$. Notice that the correlation coefficient does not depend on the parameter β.

The following theorem is obvious.

Theorem 3.2 Let X and Y have a Bivariate Type II Gumbel Probability Model with density function (2.5). Then for any positive integer a, the a-th moment of $T = X + Y$ is given by

$$\mu_1'(T) = \beta^{a/\alpha} \sum_{i=1}^{a} \Gamma\left(1-(i/\beta)\right)\Gamma\left(1-((a-i)/\alpha)\right)\left[1+\rho\left(2^{i/\alpha}-1\right)\left(2^{(a-i)/\alpha}-1\right)\right], \tag{3.5}$$

where $\alpha \geq a-1$, $\beta \geq a$ and $|\rho| < 1$.

Proof. Since $T^a = \sum_{i=1}^{a} \binom{a}{i} X^i Y^{a-i}$, it follows that $\mu_1' = \sum_{i=1}^{a} \binom{a}{i} \mu'_{i,a-i;\rho}(X,Y)$

where

$$\mu'_{i,a-i;\rho}(X,Y) = \beta^{a/\alpha}\Gamma\left(1-(i/\alpha)\right)\Gamma\left(1-\{a-(i/\alpha)\}\right)\left[1+\rho(2^{i/\alpha}-1)(2^{(a-i)/\alpha}-1)\right].$$

4. Distribution of the Product

Theorem 4.1. If X and Y are two correlated Type-II Gumbel variates having joint density as defined in equation (2.5). Then the density function of $V = XY$ is given by

$$f_V(v;\alpha,\beta;\rho) = \frac{2\alpha\beta^2}{v^{\alpha+1}}\left[(1+\rho)K_0\left(2\beta v^{-\alpha/2}\right) + 4\rho\left\{K_0\left(4\beta v^{-\alpha/2}\right) - K_0\left(2\sqrt{2}\beta v^{-\alpha/2}\right)\right\}\right], \tag{4.1}$$

where, $\alpha > 0$, $\beta > 0$, $-1 < \rho < 1$ and $K(.)$ is the modified Bessel function of second kind as defined in equations (1.4) to (1.6).

Proof. It follows from the equation (2.5) that

$$(\alpha\beta)^{-2} v^{\alpha+1} f_V(v;\alpha,\beta;\rho)$$
$$= (1+\rho)\int_0^\infty \exp\left(-\beta(x^{-\alpha} + (v/x)^{-\alpha})\right)\frac{1}{x}dx + 4\rho\int_0^\infty \exp\left(-2\beta(x^{-\alpha} + (v/x)^{-\alpha})\right)\frac{1}{x}dx$$
$$-2\rho\int_0^\infty \exp\left(-\beta(2x^{-\alpha} + (v/x)^{-\alpha})\right)\frac{1}{x}dx - 2\rho\int_0^\infty \exp\left(-\beta(x^{-\alpha} + 2(v/x)^{-\alpha})\right)\frac{1}{x}dx.$$

Transforming $x^{-\alpha} = t$ we have

$$f_V(v;\alpha,\beta;\rho) = \alpha\beta^2 v^{-(\alpha+1)}\Big[\left(1+\rho\right)I_0(\beta v^{-\alpha},\beta) + 4\rho I_0(2\beta v^{-\alpha},2\beta) \\ -2\rho I_0(\beta v^{-\alpha},2\beta) - 2\rho I_0(2\beta v^{-\alpha},\beta)\Big]. \tag{4.3}$$

where $I_p(\beta,\gamma)$ is defined by the equation (1.5).

The integrals in equation (4.3) above are simplified below by equation (1.5):

$$I_0(\beta v^{-\alpha},\beta) = 2K_0\left(2\beta v^{-\alpha/2}\right),$$
$$I_0(2\beta,2\beta v^{-\alpha}) = 2K_0\left(4\beta v^{-\alpha/2}\right),$$
$$I_0(\beta v^{-\alpha},2\beta) = 2K_0\left(2\sqrt{2}\beta v^{-\alpha/2}\right),$$
$$I_0(2\beta v^{-\alpha},\beta) = 2K_0\left(2\sqrt{2}\beta v^{-\alpha/2}\right).$$

Then from (4.3), we have

$$f_V(v;\alpha,\beta;\rho) = \alpha\beta^2 v^{-(\alpha+1)}\Big[\left(1+\rho\right)I_0(\beta v^{-\alpha},\beta) + 4\rho I_0(2\beta v^{-\alpha},2\beta) \\ -2\rho I_0(\beta v^{-\alpha},2\beta) - 2\rho I_0(2\beta v^{-\alpha},\beta)\Big],$$

which, by virtue of the equation (1.5), simplifies to the equation (4.1).

The following moment is derived from the equation (3.1) by putting $b = a$.

Theorem 4.2 Let V have the density function given by the equation (4.1). Then for any real number a, the a-th moment $\mu'_{a;\rho}(V)$ is given by

$$\mu'_a(V) = \beta^{2a/\alpha}\left(\Gamma\left(1-(a/\alpha)\right)\right)^2\left(1+\rho(2^{a/\alpha}-1)^2\right), \tag{4.4}$$

where $a < \alpha$ and $1+\rho(2^{a/\alpha}-1)^2 > 0$.

Corollary 4.1. Let V have the density function given by (4.1). Then the first four raw moments of V for $-1 \le \rho < 1$ are as follows:

a. $\mu'_1(V) = \beta^{(2/\alpha)}\left(\Gamma\left(1-(1/\alpha)\right)\right)^2\left(1+\rho(2^{1/\alpha}-1)^2\right), \quad \alpha > 1,$

b. $\mu'_2(V) = \beta^{4/\alpha}\left(\Gamma\left(1-(2/\alpha)\right)\right)^2\left(1+\rho(4^{1/\alpha}-1)^2\right), \quad \alpha > 2,$

c. $\mu'_3(V) = \beta^{6/\alpha}\left(\Gamma\left(1-(3/\alpha)\right)\right)^2\left(1+\rho(8^{1/\alpha}-1)^2\right), \quad \alpha > 3,$

d. $\mu'_4(V) = \beta^{8/\alpha}\left(\Gamma\left(1-(4/\alpha)\right)\right)^2\left(1+\rho(16^{1/\alpha}-1)^2\right), \quad \alpha > 4,$

function of $\beta > 0$.

5. Distribution of the Ratio

Theorem 5.1. If X and Y are two correlated Type II Gumbel variates having joint density function given by equation (2.5), then the density function of $W = X/Y$ is given by

$$f_W(W;\alpha,\beta;\rho) = \alpha w^{\alpha-1}\left[(1+2\rho)(1+w^{\alpha})^{-2} - 2\rho\{(2+w^{\alpha})^{-2} + (1+2w^{\alpha})^{-2}\}\right], \qquad (5.1)$$

where $w > 0$, $\alpha > 0$ and $-1 < \rho < 1$.

Proof. It follows from (2.5) that

$$(\alpha\beta)^{-2} f_W(W;\alpha,\beta;\rho) = (1+\rho)M(1,1) + 4\rho M(2,2 \;-)2\rho M(2,1) - 2\rho M(1,2\;,$$

$$\text{where } M(c,d) = \int_0^{\infty} (x\times(x/w))^{-(\alpha+1)} \exp\left(-\beta(cx^{-\alpha} + d(x/w)^{-\alpha})\right)\frac{x}{w^2}d\;..$$

Transforming $x^{-\alpha} = t$, we have

$$\alpha^{-1}\beta^{-2}w^{1-\alpha} f_W(W;\alpha,\beta;\rho)$$
$$= (1+\rho)w^{\alpha-1}\int_0^{\infty} t\exp\left(-\beta(1+w^{\alpha})t\right)dt + 4\rho w^{\alpha-1}\int_0^{\infty} t\exp\left(-2\beta(1+w^{\alpha})t\right)dt$$
$$-2\rho w^{\alpha-1}\int_0^{\infty} t\exp\left(-\beta(2+w^{\alpha})t\right)dt - 2\rho\int_0^{\infty} t\exp\left(-\beta(1+2w^{-\alpha})t\right)dt.$$

Evaluating the above gamma integrals, we havethe equation (5.1).

By putting $b = -a$ in (3.1), we have the following theorem.

Theorem 5.2 Let W have the density function given by (5.1). Then for any real number a, the a-th moment of W is given by

$$\mu'_{a;\rho}(W) = \Gamma\left(1-(a/\alpha)\right)\Gamma\left(1+(a/\alpha)\right)\left[1+\rho(2^{a/\alpha}-1)(2^{-a/\alpha}-1)\right], \qquad (5.1)$$

where $a < \alpha$ and $-1 < \rho < 1$.

Corollary 5.1: The first four raw moments of W are given by

a. $\mu'_1(W) = \Gamma\left(1-(1/\alpha)\right)\Gamma\left(1+(1/\alpha)\right)\left[1+\rho(2^{1/\alpha}-1)(2^{-1/\alpha}-1)\right]$,

b. $\mu'_2(W) = \Gamma\left(1-(2/\alpha)\right)\Gamma\left(1+(2/\alpha)\right)\left[1+\rho(4^{1/\alpha}-1)(4^{-1/\alpha}-1)\right]$,

c. $\mu_3'(W) = \Gamma\left(1-(3/\alpha)\right)\Gamma\left(1+(3/\alpha)\right)\left[1+\rho(8^{1/\alpha}-1)(8^{-1/\alpha}-1)\right],$

d. $\mu_4'(W) = \Gamma\left(1-(4/\alpha)\right)\Gamma\left(1+(4/\alpha)\right)\left[1+\rho(16^{1/\alpha}-1)(16^{-1/\alpha}-1)\right],$

where $\alpha > 4$ and $-1 < \rho < 1.$

6. Reliability function

Suppose that X represents the strength of a device and Y represents its stress. The component fails at the instant if the stress applied to it exceeds the strength and the component will function satisfactorily if reverse situation arises, i.e., if $X > Y$. Thus $\mathrm{P}(X > Y)$ defines reliability function for a device. See for example Krishnamoorthi (1992) and Troyer (2006).

The reliability function for the bivariate Gumbel II distribution may be denoted by $R(\alpha,\beta,\rho) = P(X > Y).$ By defining $W = X/Y,$ the reliability function can be written as $P(W > 1) = 1 - P(W \le 1).$

Theorem 6.1 Let $R(\alpha,\beta,\rho)$ be the reliability for the density function given by the equation (2.5). Then we have

$$R(\alpha,\beta,\rho) = \frac{1}{2} - \frac{4\rho}{3}. \tag{6.1}$$

where $\alpha > 0, \beta > 0$ and $-1 < \rho < 1.$

Proof. Then it follows from (5.1) that

$$R(\alpha,\beta,\rho) = 1 - \alpha\int_{w=0}^{1} w^{\alpha-1}\left[\left(1+2\rho\right)(1+w^{\alpha})^{-2} - 2\rho\{(2+w^{\alpha})^{-2} + (1+2w^{\alpha})^{-2}\}\right]dw,$$

Transforming $w^{\alpha} = t,$ we have

$$1 - R(\alpha,\beta,\rho) = \left(1+2\rho\right)\int_1^2 y^{-2}dy - 2\rho\int_2^3 y^{-2}dy + \rho\int_1^3 y^{-2}dy.$$

Since the right hand side simplifies to $\frac{1}{2} + \frac{4\rho}{3}$, the equation (6.1) is proved.

The reliability function in equation (6.1) implies that $P(W \le 1) = \frac{1}{2} + \frac{4\rho}{3}.$ That is, $W = 1$ is the $100\left(\frac{1}{2} + \frac{4\rho}{3}\right)$-th percentile of the distribution of W. Obviously, if $\rho = 0, W = 1$ will be the median of the distribution of W.

7. Conclusion

The results in the paper lay the foundation of data analysis of parallel system, standby system and series system where the component variables are correlated.

Acknowledgement

The author Anwar H Joarder is grateful to King Fahd University of Petroleum and Minerals, Saudi Arabia for providing excellent research facilities.

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