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Abstract

In [1], the Noether symmetries of the wave equation on a general spherically symmetric spacetime metric were provided in terms of five functions of the variables t and r subject to certain differential equations and constraints. In this work, these differential equations and constraints are solved for all the static spherically symmetric spacetimes admitting G_{10} or G_7 or G_6 as maximal isometry group. As a result, the Noether symmetries and the corresponding conservation laws for all static spherically symmetric spacetimes admitting higher symmetries are obtained explicitly.

Key words: Wave equation, spherically symmetric spacetimes, Noether symmetries, conservation laws.

1 Introduction

A complete classification of static spherically symmetric Lorentzian manifolds according to their isometries and metrics was obtained in [2, 3] where it was shown that these admit G_{10} or G_7 or G_6 or G_4 as maximal isometry group. In a recent work [4], a study of the Lie symmetries of the wave equation on static spherically symmetric spacetimes, admitting G_{10} or G_7 or G_6, is carried out. The aim of this paper is to investigate the Noether symmetries and conservation laws of the wave equation on these spaces; hence completing the symmetry analysis of the wave equation on static spherically symmetric spacetimes admitting higher symmetries. The standard procedure for determining the conservation laws for the variational systems is given by the well known Noether theorem [5]. This theorem requires a Lagrangian. There are approaches that do not require a Lagrangian or even assume the existence of a Lagrangian for differential equations. These approaches include direct construction methods for multipliers [6], partial Lagrangian [7] and the new conservation theorem [8].

For the case of wave equation on spherically symmetric spacetimes, Bokhari et al [1] provided the Noether symmetries in terms of five functions of the variables t and r subject to certain differential equations and constraints. Here we consider the wave equation on static spherically symmetric spacetimes admitting G_{10} or G_7 or G_6 and completely solve the differential equations and constraints of [1] to obtain the corresponding Noether symmetries and conservation laws. For references related to the applications and various approaches of obtaining conservation laws, the reader is referred to [9, 10, 11, 12, 13].

Using the d'Alembert's operator $\Box_g u = \frac{\partial}{\partial x_i}(\sqrt{\mid g \mid} g^{ik} \frac{\partial u}{\partial x_k})$, the wave equation $\Box_g u = 0$ on spherically symmetric spacetime with the metric

$$ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - e^{\mu(r,t)} d\theta^2 - e^{\mu(r,t)} \sin^2 \theta d\varphi^2 \quad (1.1)$$

can be written as

$$\frac{\partial}{\partial t}\left(m_1 \frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial r}\left(m_2 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(m_3 \frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(m_4 \frac{\partial u}{\partial \phi}\right) = 0 \quad (1.2)$$

where $m_1 = e^{\left(\mu - \frac{\nu}{2} + \frac{\lambda}{2}\right)} \sin\theta, m_2 = -e^{\left(\mu + \frac{\nu}{2} - \frac{\lambda}{2}\right)} \sin\theta, m_3 = -e^{\left(\frac{\nu}{2} + \frac{\lambda}{2}\right)} \sin\theta$ and $m_4 = -e^{\left(\frac{\nu}{2} + \frac{\lambda}{2}\right)} \csc\theta$.

Throughout the paper we use a convention in which derivatives of u with respect to t, r, θ and ϕ are respectively represented by u_1,u_2,u_3 and u_4. In addition, we will use the notation

$$X = [\xi_1, \xi_2, \xi_3, \xi_4, \eta]$$

to represent the Noether symmetries

$$X = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial r} + \xi_3 \frac{\partial}{\partial \theta} + \xi_4 \frac{\partial}{\partial \phi} + \eta \frac{\partial}{\partial u}$$

of the wave equation (1.2).

The Lagrangian of the wave equation (1.2) is given by the expression

$$2L = m_1 {u_1}^2 + m_2 {u_2}^2 + m_3 {u_3}^2 + m_4 {u_4}^2. \quad (1.3)$$

As mentioned above, the system and the constraints that can be used to determine the Noether symmetries of the wave equation on general spherically symmetric spacetimes were provided in [1]. Introducing the auxiliary function

$$Z = (k_2 \sin\phi - k_1 \cos\phi) \sin\theta - k_7 \cos\theta \quad (1.4)$$

in the results of [1], the Noether symmetries of of the wave equation on static spherically symmetric spacetimes in terms of the functions $k_1(t,r), k_2(t,r), k_7(t,r), k_8(t,r), k_9(t,r)$ and the three constants d_1, d_2, d_3 can be given as

$$\begin{aligned} \xi_1 =& \ \mathrm{e}^{\mu-\nu} Z_t + k_8 \\ \xi_2 =& \ -\mathrm{e}^{\mu-\lambda} Z_r + k_9 \\ \xi_3 =& \ Z_\theta + d_2 \sin\phi - d_1 \cos\phi \\ \xi_4 =& \ Z_\phi \csc^2\theta + (d_1 \sin\phi + d_2\cos\phi)\cot\theta + d_3 \\ \eta =& \ \alpha(t,r,\theta,\phi)u + \beta(t,r,\theta,\phi) \end{aligned} \tag{1.5}$$

where

$$\alpha = \ -e^{\mu-\nu} Z_{tt} - \tfrac{1}{2} e^{\mu-\nu}(2\mu_t - \nu_t) Z_t + \tfrac{1}{2} e^{\mu-\lambda} \nu_r Z_r - \tfrac{1}{2}(\nu_t k_8 + \nu_r k_9 + 2\, k_{8,t}) \tag{1.6}$$

and β is an arbitrary solution for the wave equation. The corresponding gauge terms are given as [1]

$$B_1 = \tfrac{1}{2}\ \sin\theta\ \mathrm{e}^{\mu-\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_t u^2 + 2\,\beta_t u) \tag{1.7}$$

$$B_2 = -\tfrac{1}{2}\ \sin\theta\ \mathrm{e}^{\mu+\frac{\nu}{2}-\frac{\lambda}{2}}(\alpha_r u^2 + 2\,\beta_r u) \tag{1.8}$$

$$B_3 = -\tfrac{1}{2}\ \sin\theta \mathrm{e}^{\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_\theta u^2 + 2\,\beta_\theta u) \tag{1.9}$$

$$B_4 = -\tfrac{1}{2\sin\theta} \mathrm{e}^{\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_\phi u^2 + 2\,\beta_\phi u) \tag{1.10}$$

So the problem of determining the Noether symmetries is reduced to finding five functions of two variables (t and r), namely, k_1, k_2, k_7, k_8, k_9 by first solving the following system [1]

$$\begin{aligned} &2\,k_{1,rt} + (\mu_t - \lambda_t)k_{1,r} + (\mu_r - \nu_r)k_{1,t} = 0 \\ &2\,k_{2,rt} + (\mu_t - \lambda_t)k_{2,r} + (\mu_r - \nu_r)k_{2,t} = 0 \\ &2\,k_{7,rt} + (\mu_t - \lambda_t)k_{7,r} + (\mu_r - \nu_r)k_{7,t} = 0 \\ &2\,\mathrm{e}^\nu k_{1,rr} + 2\,\mathrm{e}^\lambda k_{1,tt} + \mathrm{e}^\nu(2\mu_r - \nu_r - \lambda_r)k_{1,r} + \mathrm{e}^\lambda(2\mu_t - \nu_t - \lambda_t)\,k_{1,t} = 0 \\ &2\,\mathrm{e}^\nu k_{2,rr} + 2\,\mathrm{e}^\lambda k_{2,tt} + \mathrm{e}^\nu(2\mu_r - \nu_r - \lambda_r)k_{2,r} + \mathrm{e}^\lambda(2\mu_t - \nu_t - \lambda_t)\,k_{2,t} = 0 \\ &2\,\mathrm{e}^\nu k_{7,rr} + 2\,\mathrm{e}^\lambda k_{7,tt} + \mathrm{e}^\nu(2\mu_r - \nu_r - \lambda_r)k_{7,r} + \mathrm{e}^\lambda(2\mu_t - \nu_t - \lambda_t)\,k_{7,t} = 0 \\ &2\,\mathrm{e}^\mu\mathrm{e}^\lambda k_{1,tt} + \mathrm{e}^\mu\mathrm{e}^\nu(\mu_r - \nu_r)k_{1,r} + \mathrm{e}^\mu\mathrm{e}^\lambda(\mu_t - \nu_t)k_{1,t} + 2\mathrm{e}^\lambda\mathrm{e}^\nu k_1 = 0 \\ &2\,\mathrm{e}^\mu\mathrm{e}^\lambda k_{2,tt} + \mathrm{e}^\mu\mathrm{e}^\nu(\mu_r - \nu_r)k_{2,r} + \mathrm{e}^\mu\mathrm{e}^\lambda(\mu_t - \nu_t)k_{2,t} + 2\mathrm{e}^\lambda\mathrm{e}^\nu k_2 = 0 \\ &2\,\mathrm{e}^\mu\mathrm{e}^\lambda k_{7,tt} + \mathrm{e}^\mu\mathrm{e}^\nu(\mu_r - \nu_r)k_{7,r} + \mathrm{e}^\mu\mathrm{e}^\lambda(\mu_t - \nu_t)k_{7,t} + 2\mathrm{e}^\lambda\mathrm{e}^\nu k_7 = 0 \\ &\nu_r k_9 - \lambda_t k_8 - \lambda_r k_9 - 2\,k_{9,r} + \nu_t k_8 + 2\,k_{8,t} = 0 \\ &\mu_t k_8 + \mu_r k_9 - \nu_t k_8 - \nu_r k_9 - 2\,k_{8,t} = 0 \\ &\mathrm{e}^\lambda k_{9,t} - \mathrm{e}^\nu k_{8,r} = 0 \end{aligned} \tag{1.11}$$

and then by using the constraints [1]

$$\begin{aligned} &2\,\mathrm{e}^{\mu+\nu} R_{1,rr} - 2\,\mathrm{e}^{\mu+\lambda} R_{1,tt} + \mathrm{e}^{\mu+\nu}(2\mu_r + \nu_r - \lambda_r)R_{1,r} - \mathrm{e}^{\mu+\lambda}(2\mu_t - \nu_t + \lambda_t)R_{1,t} - 4\ \mathrm{e}^{\nu+\lambda} R_1 = 0 \\ &2\,\mathrm{e}^{\mu+\nu} R_{2,rr} - 2\,\mathrm{e}^{\mu+\lambda} R_{2,tt} + \mathrm{e}^{\mu+\nu}(2\mu_r + \nu_r - \lambda_r)R_{2,r} - \mathrm{e}^{\mu+\lambda}(2\mu_t - \nu_t + \lambda_t)R_{2,t} - 4\ \mathrm{e}^{\nu+\lambda} R_2 = 0 \\ &2\,\mathrm{e}^{\mu+\nu} R_{3,rr} - 2\,\mathrm{e}^{\mu+\lambda} R_{3,tt} + \mathrm{e}^{\mu+\nu}(2\mu_r + \nu_r - \lambda_r)R_{3,r} - \mathrm{e}^{\mu+\lambda}(2\mu_t - \nu_t + \lambda_t)R_{3,t} - 4\ \mathrm{e}^{\nu+\lambda} R_3 = 0 \\ &2\,\mathrm{e}^{\nu} R_{4,rr} - 2\,\mathrm{e}^{\lambda} R_{4,tt} + \mathrm{e}^{\nu}(2\mu_r + \nu_r - \lambda_r)R_{4,r} - \mathrm{e}^{\lambda}(2\mu_t - \nu_t + \lambda_t)R_{4,t} = 0, \end{aligned} \tag{1.12}$$

where

$$\begin{aligned}
R_1(t,r) &= \mathrm{e}^{\mu-\nu}(2\,k_{1,t,t} - k_{1,t}\nu_t + 2\,\mu_t k_{1,t}) - \mathrm{e}^{\mu-\lambda}\nu_r k_{1,r}\\
R_2(t,r) &= \mathrm{e}^{\mu-\nu}(2\,k_{2,t,t} - \nu_t k_{2,t} + 2\,\mu_t k_{2,t}) - \mathrm{e}^{\mu-\lambda}\nu_r k_{2,r}\\
R_3(t,r) &= \mathrm{e}^{\mu-\nu}(2\,k_{7,t,t} - k_{7,t}\nu_t + 2\,\mu_t k_{7,t}) - \mathrm{e}^{\mu-\lambda}\nu_r k_{7,r}\\
R_4(t,r) &= \nu_t k_8 + \nu_r k_9 + 2\,k_{8,t}.
\end{aligned} \tag{1.13}$$

The three constants d_1, d_2, d_3 in (1.5) lead to the common Noether symmetries, for all spherically symmetric spacetimes, which are

$$\begin{aligned}
N_1 &= [0,0,-\cos\phi, \cot\theta\sin\phi, 0]\\
N_2 &= [0,0,\sin\phi, \cot\theta\cos\phi, 0]\\
N_3 &= [0,0,0,1,0]
\end{aligned}$$

and generate the Lie algebra $so(3)$. These are isometries of spacetime consisting of spatial rotations. So, using the Noether theorem [5], the corresponding conservation laws of angular momentum can be given as

$$T_1 = \left[\begin{array}{l} -m_1\,u_1\,u_3\,\cos\varphi + m_1\,u_1\,u_4\,\cot\theta\sin\varphi,\ -m_2\,u_2\,u_3\cos\varphi + m_2\,u_2\,u_4\,\cot\theta\sin\varphi,\\ \frac{1}{2}S_3\cos\varphi + m_3\,u_3\,u_4\,\cot\theta\sin\varphi, -m_4\,u_3\,u_4\cos\varphi - \frac{1}{2}S_4\,\cot\theta\sin\varphi \end{array}\right]$$

$$T_2 = \left[\begin{array}{l} m_1\,u_1\,u_3\,\sin\varphi + m_1\,u_1\,u_4\,\cot\theta\cos\varphi,\ m_2\,u_2\,u_3\sin\varphi + m_2\,u_2\,u_4\,\cot\theta\cos\varphi,\\ -\frac{1}{2}S_3\sin\varphi + m_3\,u_3\,u_4\,\cot\theta\cos\varphi, m_4\,u_3\,u_4\sin\varphi - \frac{1}{2}S_4\,\cot\theta\cos\varphi \end{array}\right]$$

$$T_3 = \left[m_1\,u_1\,u_4,\ m_2\,u_2\,u_4,\ m_3\,u_3\,u_4,\ -\frac{1}{2}S_4\right]$$

where $S_i = \sum_{j=1}^{4} \varepsilon_{i\,j} m_j u_j^2$ with $\varepsilon_{i\,j} = \left\{\begin{array}{rl} 1 & i \neq j\\ -1 & i = j \end{array}\right.$.

It follows directly from the system (1.11) and the constraints (1.12) that $k_1 = k_2 = k_7 = k_9 = 0, k_8 = 1$ is a solution for all *static* spherically symmetric spacetimes. Hence, the fourth common Noether symmetry, for all *static* spherically symmetric spacetimes, is $N_4 = [1,0,0,0,0]$. This is an isometry of the spacetime that corresponds to the time translation. The conservation law of energy corresponding to this Noether symmetry is

$$T_4 = [-\frac{1}{2}S_1, m_2 u_1 u_2, m_3 u_1 u_3, m_4 u_1 u_4]$$

In the subsequent sections, we provide the values of k_1, k_2, k_7, k_8, k_9, and the Noether symmetries for all static spherically symmetric spacetimes admitting G_{10} or G_7 or G_6 as maximal isometry group. For each case, we provide explicit expressions of all the corresponding conservations laws for the wave equation.

2 G_{10} spaces

In this section we apply our approach to the three spherically spacetimes admitting G_{10}, namely the Minkowski spacetime, de Sitter spacetime and anti de Sitter spacetime.

2.1 Minkowski spacetime

The spherically symmetric form of the metric on the Minkowski spacetime is given by $\nu = 0$, $\lambda = 0$ and $\mu = \ln r^2$ in the metric (1.1). The isometry group is the Poincare group, $SO(1,3) \bigotimes_s \mathbb{R}^4$, and the conformal symmetries are given by the 15 parameter conformal group, containing the above group, a homothety and four special conformal isometries.

Using (1.2) and (1.3) the corresponding wave equation is

$$\frac{\partial}{\partial t}(r^2 u_t \sin\theta) - \frac{\partial}{\partial r}(r^2 u_r \sin\theta) - \frac{\partial}{\partial \theta}(u_\theta \sin\theta) - \frac{\partial}{\partial \phi}\left(\frac{u_\phi}{\sin\theta}\right) = 0 \tag{2.1}$$

which is the Euler Lagrange equation of the Lagrangian

$$L = \frac{1}{2}\left(r^2 \sin\theta \ {u_t}^2 - r^2 \sin\theta \ {u_r}^2 - \sin\theta \ {u_\theta}^2 - \frac{{u_\phi}^2}{\sin\theta}\right) \tag{2.2}$$

Solving the system (1.11) for $\nu = 0$, $\lambda = 0$ and $\mu = \ln r^2$ and then imposing the constraints (1.12) gives the solution k_1, k_2, k_7, k_8, k_9

$$k_1 = C_1\frac{t}{r} + C_2\frac{1}{r} + C_3\frac{r^2 - t^2}{r}, k_2 = C_4\frac{t}{r} + C_5\frac{1}{r} + C_6\frac{r^2 - t^2}{r},$$
$$k_7 = C_7\frac{t}{r} + C_8\frac{1}{r} + C_9\frac{r^2 - t^2}{r}, k_8 = C_{10}\frac{r^2 + t^2}{2} + C_{11}t + C_{12}, k_9 = C_{10}tr + C_{11}r.$$

Hence the Noether symmetry algebra of the wave equation (2.1) on Minkowski spacetime is the 15 dimensional algebra generated by $\{N_1, N_2, N_3, N_4\}$, described in Section 1, and

$$\begin{aligned}
X_1 &= [-r\sin\theta\cos\phi, -t\sin\theta\cos\phi, -\frac{t\cos\phi\cos\theta}{r}, \frac{t\sin\phi}{r\sin\theta}, 0] \\
X_2 &= [0, -\cos\phi\sin\theta, -\frac{\cos\phi\cos\theta}{r}, \frac{\sin\phi}{r\sin\theta}, 0] \\
X_3 &= [2\ r\ t\cos\phi\sin\theta, \left(r^2 + t^2\right)\sin\theta\cos\phi, \frac{(t^2 - r^2)\cos\theta\cos\phi}{r}, -\frac{(t^2 - r^2)\sin\phi}{r\sin\theta}, -2\ ru\sin\theta\cos\phi] \\
X_4 &= [r\sin\theta\sin\phi, t\sin\theta\sin\phi, \frac{t\cos\theta\sin\phi}{r}, \frac{t\cos\phi}{r\sin\theta}, 0] \\
X_5 &= [0, \sin\phi\sin\theta, \frac{\sin\phi\cos\theta}{r}, \frac{\cos\phi}{r\sin\theta}, 0] \\
X_6 &= [-2\ r\ t\sin\phi\sin\theta, -\left(r^2 + t^2\right)\sin\theta\sin\phi, \frac{(r^2 - t^2)\cos\theta\sin\phi}{r}, \frac{(r^2 - t^2)\cos\phi}{r\sin\theta}, 2ru\sin\theta\sin\phi] \\
X_7 &= [-r\cos\theta, -t\cos\theta, \frac{t\sin\theta}{r}, 0, 0] \\
X_8 &= [0, -\cos\theta, \frac{\sin\theta}{r}, 0, 0] \\
X_9 &= [2\ r\ t\cos\theta, (t^2 + r^2)\cos\theta, \frac{(r^2 - t^2)\sin\theta}{r}, 0, -2\ r\ u\cos\theta] \\
X_{10} &= [\frac{(t^2 + r^2)}{2}, r\ t, 0, 0, -t\ u] \\
X_{11} &= [t, r, 0, 0, -u]
\end{aligned}$$

Applying the Noether theorem [5], we determine all the conservations laws for the wave equation on Minkowski spacetime. The conservation laws are written in terms of m_1, m_2, m_3, m_4 where for this case $m_1 = r^2 \sin\theta, m_2 = -r^2 \sin\theta, m_3 = -\sin\theta, m_4 = -csc\theta$.

The Noether symmetries X_2, X_5, X_8 are isometries which correspond to the translations

$$X_2 = [0, 1, 0, 0, 0], X_5 = [0, 0, 1, 0, 0], X_8 = [0, 0, 0, 1, 0]$$

in x, y, z respectively in the Cartesian coordinates. So, these yield the conservation laws of linear momentum which are

$$TC^2 = \begin{bmatrix} \frac{m_1 u_1}{r\sin\theta}(u_4 \sin\phi - u_3 \cos\phi\cos\theta\sin\theta - ru_2\cos\phi\ \sin^2\theta), \frac{m_2 u_2}{r\sin\theta}(u_4\sin\phi - u_3\cos\phi\cos\theta\sin\theta) + \frac{\sin\theta\cos\phi}{2}S_2, \\ \frac{m_3 u_3}{r\sin\theta}(u_4\sin\phi - ru_2\cos\phi \sin^2\theta) + \frac{\cos\theta\cos\phi}{2r}S_3, -\frac{m_4 u_4\cos\phi}{r}(u_3\cos\theta + ru_2\sin\theta) - \frac{\sin\phi}{2r\sin\theta}S_4 \end{bmatrix}$$

$$TC^5 = \begin{bmatrix} \frac{m_1 u_1}{r\sin\theta}(u_4 \cos\phi + u_3 \sin\phi\cos\theta\sin\theta + ru_2\sin\phi\ \sin^2\theta), \frac{m_2 u_2}{r\sin\theta}(u_4\cos\phi + u_3\sin\phi\cos\theta\sin\theta) - \frac{\sin\theta\sin\phi}{2}S_2, \\ \frac{m_3 u_3}{r\sin\theta}(u_4\cos\phi + ru_2\sin\phi \sin^2\theta) - \frac{\cos\theta\sin\phi}{2r}S_3, \frac{m_4 u_4\sin\phi}{r}(u_3\cos\theta + ru_2\sin\theta) - \frac{\cos\phi}{2r\sin\theta}S_4 \end{bmatrix}$$

$$TC^8 = [\frac{m_1 u_1}{r}(u_3\sin\theta - ru_2\cos\theta), \frac{m_2}{r}u_2 u_3\sin\theta + \frac{S_2}{2}\cos\theta, -m_3 u_2 u_3\cos\theta - \frac{S_3}{2r}\sin\theta, \frac{m_4 u_4}{r}(u_3\sin\theta - ru_2\cos\theta)]$$

The Noether symmetries X_1, X_4, X_7 are isometries which correspond to the Lorentz transformations

$$X_1 = [x, t, 0, 0, 0], X_4 = [y, 0, t, 0, 0], X_7 = [z, 0, 0, t, 0]$$

in the Cartesian coordinates. The corresponding conservation laws are

$$TC^1 = \begin{bmatrix} \frac{m_1 u_1 t}{r\sin\theta}(u_4\sin\phi - u_3\cos\phi\cos\theta\sin\theta - ru_2\cos\phi\ \sin^2\theta) + \frac{r\sin\theta\cos\phi}{2}S_1, \\ \frac{m_2 u_2}{r\sin\theta}(tu_4\sin\phi - tu_3\cos\phi\cos\theta\sin\theta - r^2 u_1\cos\phi\ \sin^2\theta) + \frac{t\sin\theta\cos\phi}{2}S_2, \\ \frac{m_3 u_3}{r\sin\theta}(tu_4\sin\phi - tru_2\cos\phi \sin^2\theta - r^2 u_1\cos\phi\ \sin^2\theta) + \frac{t\cos\theta\cos\phi}{2r}S_3, \\ -\frac{m_4 u_4}{r}(tu_3\cos\phi\cos\theta + tru_2\cos\phi\sin\theta + r^2 u_1\cos\phi\ \ \sin\theta) - \frac{t\sin\phi}{2r\sin\theta}S_4 \end{bmatrix}$$

$$TC^4 = \begin{bmatrix} \frac{m_1 u_1 t}{r\sin\theta}(u_4\cos\phi + u_3\sin\phi\cos\theta\sin\theta + ru_2\sin\phi\ \sin^2\theta) - \frac{r\sin\theta\sin\phi}{2}S_1, \\ \frac{m_2 u_2}{r\sin\theta}(tu_4\cos\phi + tu_3\sin\phi\cos\theta\sin\theta + r^2 u_1\sin\phi\ \sin^2\theta) - \frac{t\sin\theta\sin\phi}{2}S_2, \\ \frac{m_3 u_3}{r\sin\theta}(tu_4\cos\phi + tru_2\sin\phi \sin^2\theta + r^2 u_1\sin\phi\ \sin^2\theta) - \frac{t\cos\theta\sin\phi}{2r}S_3, \\ \frac{m_4 u_4}{r}(tu_3\sin\phi\cos\theta + tru_2\sin\phi\sin\theta + r^2 u_1\sin\phi\ \ \sin\theta) - \frac{t\cos\phi}{2r\sin\theta}S_4 \end{bmatrix}$$

$$TC^7 = \begin{bmatrix} -\frac{m_1 u_1}{r}(rtu_2\cos\theta - \ t\sin\theta u_3) + \frac{r\,S_1}{2}\cos\theta, -\frac{m_2 u_2}{2r}(2u_1 r^2\cos\theta - t\sin\theta u_3) + \frac{t\,S_2}{2}\cos\theta, \\ -m_3 u_3\cos\theta(r\,u_1 + t\,u_2) - \frac{t\,S_3}{2r}\sin\theta, -\frac{m_4 u_4}{r}(r^2 u_1\cos\theta + r\,t\,u_2\cos\theta - t\,u_3\sin\theta) \end{bmatrix}$$

The Noether symmetries $X_3, X_6, X_9, X_{10}, X_{11}$ are conformal symmetries of the spacetime. The conservation laws corresponding to these Noether symmetries are

$$TC^3 = \begin{bmatrix} 2m_1 u_1\,(\cos\varphi\sin\theta\,(r\,u + u_2 H - u_3\cos\theta\csc\theta N) + u_4\sin\varphi\ \csc\theta N) - r\,t\cos\varphi\sin\theta\ S_1, \\ 2m_2 u_2\,(\cos\varphi\sin\theta\,(r\,u + \ r\,t\,u_1 - \ u_3\cos\theta\csc\theta N) + u_4\sin\varphi\ \csc\theta\,N) + r^2\,u^2\cos\varphi\sin^2\theta - \cos\varphi\sin\theta H\ S_2, \\ 2m_3 u_3\,(\cos\varphi\sin\theta\,(r\,u + \ r\,t\,u_1 + u_2 H) + u_4\sin\varphi\csc\theta\,N) + r\cos\varphi\cos\theta\ \sin\theta\,u^2 + \cos\varphi\cos\theta N S_3, \\ 2m_4 u_4\cos\varphi\sin\theta\,(r\,u + \ r\,t\,u_1 + u_2 H - u_3\cos\theta\csc\theta\,N) - r\sin\varphi\ u^2 - \sin\varphi\csc\theta N S_4 \end{bmatrix}$$

where $H = \frac{t^2+r^2}{2}$ and $N = \frac{r^2-t^2}{2r}$.

$$TC^6 = \left[\begin{array}{l} 2m_1u_1\left(-\sin\varphi\sin\theta\left(r\,u + u_2H - u_3\cos\theta\csc\theta N\right) + u_4\cos\varphi\,\csc\theta N\right) + r\,t\sin\varphi\sin\theta\;S_1, \\ 2m_2u_2\left(-\sin\varphi\sin\theta\left(r\,u + \,r\,t\,u_1 - \,u_3\cos\theta\csc\theta N\right) + u_4\cos\varphi\,\csc\theta\,N\right) - r^2\,u^2\sin\varphi\sin^2\theta + \sin\varphi\sin\theta H\,S_2, \\ 2m_3u_3\left(-\sin\varphi\sin\theta\left(r\,u + \,r\,t\,u_1 + u_2H\right) + u_4\cos\varphi\csc\theta\,N\right) - r\sin\varphi\cos\theta\,\sin\theta\,u^2 - \sin\varphi\cos\theta NS_3, \\ -2m_4u_4\sin\varphi\sin\theta\left(r\,u + \,r\,t\,u_1 + u_2H - u_3\cos\theta\csc\theta\,N\right) - r\cos\varphi\;u^2 - \cos\varphi\csc\theta NS_4 \end{array}\right]$$

$$TC^9 = \left[\begin{array}{l} m_1u_1\left(2Nu_3\sin\theta + 2\cos\theta\left(Hu_2 + ru\right)\right) - rt\cos\theta S_1, \\ m_2u_2\left(2Nu_3\sin\theta + 2\cos\theta\left(rtu_1 + ru\right)\right) + r^2u^2\sin\theta\cos\theta - H\cos\theta S_2, \\ 2m_3u_3\cos\theta\left(Hu_2 + rtu_1 + ru\right) - ru^2\sin^2\theta - N\sin\theta S_3, \\ m_4u_4\left(2Nu_3\sin\theta + 2\cos\theta\left(Hu_2 + rtu_1 + ru\right)\right) \end{array}\right]$$

$$TC^{10} = \left[\begin{array}{l} m_1u_1t(ru_2 + u) - \frac{1}{2}r^2u^2\sin\theta - \frac{1}{2}HS_1, m_2u_2\left(Hu_1 + ut\right) - \frac{tr}{2}S_2, \\ m_3u_3\left(Hu_1 + rtu_2 + tu\right), m_4u_4\left(Hu_1 + rtu_2 + tu\right) \end{array}\right]$$

$$TC^{11} = [m_1u_1(ru_2 + u) - \frac{t}{2}S_1, m_2u_2(tu_1 + u) - \frac{r}{2}S_2, m_3u_3(tu_1 + ru_2 + u), m_4u_4(tu_1 + ru_2 + u)]$$

2.2 De Sitter spacetime

The spherically symmetric metric on de Sitter spacetime is of the form (1.1) with $\nu = \ln(\alpha r^2 + 1)$, $\lambda = -\ln(\alpha r^2 + 1)$ and $\mu = \ln r^2$ where $\alpha = -c^2 < 0$. The isometry group of de Sitter spacetime is $SO(1,4)$, see [3]. Using (1.2) and (1.3) the corresponding wave equation is

$$\frac{\partial}{\partial t}(\frac{r^2u_t\sin\theta}{\alpha r^2 + 1}) - \frac{\partial}{\partial r}(r^2u_r\sin\theta(\alpha r^2 + 1)) - \frac{\partial}{\partial\theta}(u_\theta\sin\theta) - \frac{\partial}{\partial\phi}\left(\frac{u_\phi}{\sin\theta}\right) = 0 \tag{2.3}$$

which is the Euler Lagrange equation of the Lagrangian

$$L = \frac{1}{2}\left(\frac{r^2\sin\theta\;{u_t}^2}{\alpha r^2 + 1} - r^2\sin\theta(\alpha r^2 + 1)\;{u_r}^2 - \sin\theta\;{u_\theta}^2 - \frac{{u_\phi}^2}{\sin\theta}\right) \tag{2.4}$$

Solving the system (1.11) for $\nu = \ln(1 - c^2r^2)$, $\lambda = -\ln(1 - c^2r^2)$ and $\mu = \ln r^2$ and then imposing the constraints (1.12) gives the solution k_1, k_2, k_7, k_8, k_9

$$k_1 = C_1\frac{Be^{-ct}}{cr} + C_2\frac{Be^{ct}}{cr}, k_2 = C_3\frac{Be^{-ct}}{cr} + C_4\frac{Be^{ct}}{cr},$$
$$k_7 = C_5\frac{Be^{-ct}}{cr} + C_6\frac{Be^{ct}}{cr}, k_8 = C_7, k_9 = 0.$$

Hence the Noether symmetry algebra of the wave equation (2.3) on de Sitter spacetime is the 10 dimensional algebra, isomorphic to its isometry algebra, generated by

$\{N_1, N_2, N_3, N_4\}$ and

$$
\begin{aligned}
X_1 &= [\frac{r\cos\phi\sin\theta \mathrm{e}^{-ct}}{B}, -\frac{B\cos\phi\sin\theta\mathrm{e}^{-ct}}{c}, -\frac{B\cos\theta\cos\phi\mathrm{e}^{-ct}}{cr}, \frac{B\sin\phi\mathrm{e}^{-ct}}{cr\sin\theta}, 0] \\
X_2 &= [-\frac{r\cos\phi\sin\theta\mathrm{e}^{ct}}{B}, -\frac{B\cos\phi\sin\theta\mathrm{e}^{ct}}{c}, -\frac{B\mathrm{e}^{ct}\cos\theta\cos\phi}{cr}, \frac{B\mathrm{e}^{ct}\sin\phi}{cr\sin\theta}, 0] \\
X_3 &= [-\frac{r\sin\phi\sin\theta\mathrm{e}^{-ct}}{B}, \frac{B\sin\phi\sin\theta\mathrm{e}^{-ct}}{c}, \frac{B\cos\theta\sin\phi\mathrm{e}^{-ct}}{cr}, \frac{B\cos\phi\mathrm{e}^{-ct}}{cr\sin\theta}, 0] \\
X_4 &= [\frac{r\sin\phi\sin\theta\mathrm{e}^{ct}}{B}, \frac{B\sin\phi\sin\theta\mathrm{e}^{ct}}{c}, \frac{B\mathrm{e}^{ct}\cos\theta\sin\phi}{cr}, \frac{B\mathrm{e}^{ct}\cos\phi}{cr\sin\theta}, 0] \\
X_5 &= [\frac{r\cos\theta\mathrm{e}^{-ct}}{B}, -\frac{B\cos\theta\mathrm{e}^{-ct}}{c}, \frac{B\sin\theta\mathrm{e}^{-ct}}{cr}, 0, 0] \\
X_6 &= [-\frac{r\cos\theta\mathrm{e}^{ct}}{B}, -\frac{B\cos\theta\mathrm{e}^{ct}}{c}, \frac{B\sin\theta\mathrm{e}^{ct}}{cr}, 0, 0]
\end{aligned}
$$

where $B = \sqrt{1 - c^2r^2}$.

The corresponding conservation laws for the wave equation for this metric, determined using Noether theorem [5], are given below in terms of m_1, m_2, m_3, m_4 where $m_1 = \frac{r^2}{1-c^2r^2}\sin\theta, m_2 = -r^2(1-c^2r^2)\sin\theta, m_3 = -\sin\theta, m_4 = -\csc\theta$.

$$
TC^1 = \begin{bmatrix}
-\frac{B}{cr}m_1u_1\,e^{-c\,t}\,(u_3\cos\theta\cos\varphi - u_4\csc\theta\sin\varphi) - \frac{B}{c}m_1u_1u_2\,e^{-c\,t}\,\sin\theta\cos\varphi - \frac{r}{2\,B}\,e^{-c\,t}\sin\theta\cos\varphi\,S_1, \\
-\frac{B}{cr}m_2u_2\,e^{-c\,t}\,(u_3\cos\theta\cos\varphi - u_4\csc\theta\sin\varphi) + \frac{r}{B}m_2u_1u_2\,e^{-c\,t}\sin\theta\cos\varphi + \frac{B}{2c}e^{-c\,t}\sin\theta\cos\varphi\,S_2, \\
-\frac{B}{cr}m_3u_3\,e^{-c\,t}\,(r\,u_2\sin\theta\cos\varphi - u_4\csc\theta\sin\varphi) + \frac{r}{B}m_3u_1u_3\,e^{-c\,t}\sin\theta\cos\varphi + \frac{B}{2cr}e^{-c\,t}\cos\theta\cos\varphi\,S_3, \\
-\frac{B}{cr}m_4u_4\,e^{-c\,t}\,(r\,u_2\sin\theta + u_3\cos\theta)\cos\varphi + \frac{r}{B}m_4u_1u_4\,e^{-c\,t}\sin\theta\cos\varphi - \frac{B}{2cr}e^{-c\,t}\csc\theta\sin\varphi\;S_4
\end{bmatrix}
$$

$$
TC^2 = \begin{bmatrix}
-\frac{B}{cr}m_1u_1\,e^{c\,t}\,(u_3\cos\theta\cos\varphi - u_4\csc\theta\sin\varphi) - \frac{B}{c}m_1u_1u_2\,e^{c\,t}\,\sin\theta\cos\varphi + \frac{r}{2\,B}\,e^{c\,t}\sin\theta\cos\varphi\,S_1, \\
-\frac{B}{cr}m_2u_2\,e^{c\,t}\,(u_3\cos\theta\cos\varphi - u_4\csc\theta\sin\varphi) - \frac{r}{B}m_2u_1u_2\,e^{c\,t}\sin\theta\cos\varphi + \frac{B}{2c}\,e^{c\,t}\sin\theta\cos\varphi\,S_2, \\
-\frac{B}{cr}m_3u_3\,e^{c\,t}\,(r\,u_2\sin\theta\cos\varphi - u_4\csc\theta\sin\varphi) - \frac{r}{B}m_3u_1u_3\,e^{c\,t}\sin\theta\cos\varphi + \frac{B}{2cr}\,e^{c\,t}\cos\theta\cos\varphi\,S_3, \\
-\frac{B}{cr}m_4u_4\,e^{c\,t}\,(r\,u_2\sin\theta + u_3\cos\theta)\cos\varphi - \frac{r}{B}m_4u_1u_4\,e^{c\,t}\sin\theta\cos\varphi - \frac{B}{2cr}\,e^{c\,t}\csc\theta\sin\varphi\;S_4
\end{bmatrix}
$$

$$
TC^3 = \begin{bmatrix}
\frac{B}{cr}m_1u_1\;e^{-c\,t}\,(u_3\cos\theta\sin\varphi + u_4\csc\theta\cos\varphi) + \frac{B}{c}m_1u_1u_2\;e^{-c\,t}\;\sin\theta\sin\varphi + \frac{r}{2\,B}\;e^{-c\,t}\sin\theta\sin\varphi\,S_1, \\
\frac{B}{cr}m_2u_2\;e^{-c\,t}\,(u_3\cos\theta\sin\varphi + u_4\csc\theta\cos\varphi) - \frac{r}{B}m_2u_1u_2\;e^{-c\,t}\sin\theta\sin\varphi - \frac{B}{2c}e^{-c\,t}\sin\theta\sin\varphi\,S_2, \\
\frac{B}{cr}m_3u_3\;e^{-c\,t}\,(r\,u_2\sin\theta\sin\varphi + u_4\csc\theta\cos\varphi) - \frac{r}{B}m_3u_1u_3\;e^{-c\,t}\sin\theta\sin\varphi - \frac{B}{2cr}\;e^{-c\,t}\cos\theta\sin\varphi\,S_3, \\
\frac{B}{cr}m_4u_4\;e^{-c\,t}\,(r\,u_2\sin\theta + u_3\cos\theta)\sin\varphi - \frac{r}{B}m_4u_1u_4\;e^{-c\,t}\sin\theta\sin\varphi - \frac{B}{2cr}\;e^{-c\,t}\csc\theta\cos\varphi S_4
\end{bmatrix}
$$

$$
TC^4 = \begin{bmatrix}
\frac{B}{cr}m_1u_1\;e^{c\,t}\,(u_3\cos\theta\sin\varphi + u_4\csc\theta\cos\varphi) + \frac{B}{c}m_1u_1u_2\;e^{c\,t}\;\sin\theta\sin\varphi - \frac{r}{2\,B}\;e^{c\,t}\sin\theta\sin\varphi\,S_1, \\
\frac{B}{cr}m_2u_2\;e^{c\,t}\,(u_3\cos\theta\sin\varphi + u_4\csc\theta\cos\varphi) + \frac{r}{B}m_2u_1u_2\;e^{c\,t}\sin\theta\sin\varphi - \frac{B}{2c}e^{c\,t}\sin\theta\sin\varphi\,S_2, \\
\frac{B}{cr}m_3u_3\;e^{c\,t}\,(r\,u_2\sin\theta\sin\varphi + u_4\csc\theta\cos\varphi) + \frac{r}{B}m_3u_1u_3\;e^{c\,t}\sin\theta\sin\varphi - \frac{B}{2cr}\;e^{c\,t}\cos\theta\sin\varphi\,S_3, \\
\frac{B}{cr}m_4u_4\;e^{c\,t}\,(r\,u_2\sin\theta + u_3\cos\theta)\sin\varphi + \frac{r}{B}m_4u_1u_4\;e^{c\,t}\sin\theta\sin\varphi - \frac{B}{2cr}\;e^{c\,t}\csc\theta\cos\varphi S_4
\end{bmatrix}
$$

$$
TC^5 = \begin{bmatrix}
\frac{B}{cr}m_1u_1u_3\;e^{-c\,t}\sin\theta - \frac{B}{c}m_1u_1u_2\;e^{-c\,t}\cos\theta - \frac{r}{2B}\;e^{-c\,t}\cos\theta\,S_1, \\
\frac{B}{cr}m_2u_2u_3\;e^{-c\,t}\sin\theta + \frac{r}{B}m_2u_1u_2\;e^{-c\,t}\cos\theta + \frac{B}{2c}\;e^{-c\,t}\cos\theta\,S_2, \\
-\frac{B}{c}m_3u_2u_3\;e^{-c\,t}\cos\theta + \frac{r}{B}m_3u_1u_3\;e^{-c\,t}\cos\theta - \frac{B}{2cr}\;e^{-c\,t}\sin\theta\,S_3, \\
\frac{r}{B}m_4u_1u_4\;e^{-c\,t}\cos\theta - \frac{B}{c}m_4u_2u_4\;e^{-c\,t}\cos\theta + \frac{B}{cr}m_4u_3u_4\;e^{-c\,t}\sin\theta
\end{bmatrix}
$$

$$
TC^6 = \begin{bmatrix}
\frac{B}{cr}m_1u_1u_3\;e^{c\,t}\sin\theta - \frac{B}{c}m_1u_1u_2\;e^{c\,t}\cos\theta + \frac{r}{2B}\;e^{c\,t}\cos\theta\,S_1, \\
\frac{B}{cr}m_2u_2u_3\;e^{c\,t}\sin\theta - \frac{r}{B}m_2u_1u_2\;e^{c\,t}\cos\theta + \frac{B}{2c}\;e^{c\,t}\cos\theta\,S_2, \\
-\frac{B}{c}m_3u_2u_3\;e^{c\,t}\cos\theta - \frac{r}{B}m_3u_1u_3\;e^{c\,t}\cos\theta - \frac{B}{2cr}\;e^{c\,t}\sin\theta\,S_3, \\
-\frac{r}{B}m_4u_1u_4\;e^{c\,t}\cos\theta - \frac{B}{c}m_4u_2u_4\;e^{c\,t}\cos\theta + \frac{B}{cr}m_4u_3u_4\;e^{c\,t}\sin\theta
\end{bmatrix}
$$

2.3 Anti de Sitter spacetime

The spherically symmetric metric on anti de Sitter spacetime is of the form (1.1) with $\nu = \ln(\alpha r^2 + 1)$, $\lambda = -\ln(\alpha r^2 + 1)$ and $\mu = \ln r^2$ where $\alpha = c^2 > 0$. The isometry group of anti de Sitter spacetime is $SO(2,3)$, see [3]. The corresponding wave equation and Lagrangian are obtained by using $\alpha = c^2 > 0$ in (2.3) and (2.4). Solving the system (1.11) for $\nu = \ln(1 + c^2r^2)$, $\lambda = -\ln(1 + c^2r^2)$ and $\mu = \ln r^2$ and then imposing the constraints (1.12) gives the solution k_1, k_2, k_7, k_8, k_9

$$k_1 = C_1\frac{A\sin ct}{cr} + C_2\frac{A\cos ct}{cr}, k_2 = C_3\frac{A\sin ct}{cr} + C_4\frac{A\cos ct}{cr},$$
$$k_7 = C_5\frac{A\sin ct}{cr} + C_6\frac{A\cos ct}{cr}, k_8 = C_7, k_9 = 0.$$

Hence the Noether symmetry algebra of the wave equation (2.3) on anti de Sitter spacetime is the 10 dimensional algebra, isomorphic to its isometry algebra, generated by $\{N_1, N_2, N_3, N_4\}$ and

$$\begin{aligned}
X_1 &= [-\frac{r\cos ct\sin\theta\cos\phi}{A}, -\frac{A\sin ct\sin\theta\cos\phi}{c}, -\frac{A\sin ct\cos\theta\cos\phi}{rc}, \frac{\sin ct\sin\phi A}{rc\sin\theta}, 0] \\
X_2 &= [\frac{r\sin ct\sin\theta\cos\phi}{A}, -\frac{A\cos ct\sin\theta\cos\phi}{c}, -\frac{A\cos\theta\cos\phi\cos ct}{rc}, \frac{\cos ct\sin\phi A}{rc\sin\theta}, 0] \\
X_3 &= [\frac{r\cos ct\sin\theta\sin\phi}{A}, \frac{A\sin ct\sin\theta\sin\phi}{c}, \frac{A\sin ct\cos\theta\sin\phi}{rc}, \frac{\sin ct\cos\phi A}{rc\sin\theta}, 0] \\
X_4 &= [-\frac{r\sin ct\sin\theta\sin\phi}{A}, \frac{A\cos ct\sin\theta\sin\phi}{c}, \frac{A\cos\theta\sin\phi\cos ct}{rc}, \frac{\cos ct\cos\phi A}{rc\sin\theta}, 0] \\
X_5 &= [-\frac{r\cos ct\cos\theta}{A}, -\frac{A\sin ct\cos\theta}{c}, \frac{A\sin ct\sin\theta}{rc}, 0, 0] \\
X_6 &= [\frac{r\sin ct\cos\theta}{A}, -\frac{A\cos ct\cos\theta}{c}, \frac{A\cos ct\sin\theta}{rc}, 0, 0]
\end{aligned}$$

where $A = \sqrt{1 + c^2r^2}$.

The corresponding conservation laws for the wave equation for this metric, determined using Noether theorem [5], are given below in terms of m_1, m_2, m_3, m_4 where $m_1 = \frac{r^2}{1+c^2r^2}\sin\theta, m_2 = -r^2(1 + c^2r^2)\sin\theta, m_3 = -\sin\theta, m_4 = -\csc\theta$.

$$TC^1 = \begin{bmatrix} -\frac{A}{cr}m_1u_1\sin(ct)\left(u_3\cos\theta\cos\varphi - u_4\csc\theta\sin\varphi\right) - \frac{A}{c}m_1u_1u_2\sin(ct)\sin\theta\cos\varphi + \frac{r}{2A}\cos(ct)\sin\theta\cos\varphi\, S_1, \\ -\frac{A}{cr}m_2u_2\sin(ct)\left(u_3\cos\theta\cos\varphi - u_4\csc\theta\sin\varphi\right) - \frac{r}{A}m_2u_1u_2\cos(ct)\sin\theta\cos\varphi + \frac{A}{2c}\sin(ct)\sin\theta\cos\varphi\, S_2, \\ -\frac{A}{cr}m_3u_3\sin(ct)\left(r\,u_2\sin\theta\cos\varphi - u_4\csc\theta\sin\varphi\right) - \frac{r}{A}m_3u_1u_3\cos(ct)\sin\theta\cos\varphi + \frac{A}{2cr}\sin(ct)\cos\theta\cos\varphi\, S_3, \\ -\frac{A}{cr}m_4u_4\sin(ct)\left(r\,u_2\sin\theta + u_3\cos\theta\right)\cos\varphi - \frac{r}{A}m_4u_1u_4\cos(ct)\sin\theta\cos\varphi - \frac{A}{2cr}\sin(ct)\csc\theta\sin\varphi S_4 \end{bmatrix}$$

$$TC^2 = \begin{bmatrix} -\frac{A}{cr}m_1u_1\cos(ct)\left(u_3\cos\theta\cos\varphi - u_4\csc\theta\sin\varphi\right) - \frac{A}{c}m_1u_1u_2\cos(ct)\sin\theta\cos\varphi - \frac{r}{2A}\sin(ct)\sin\theta\cos\varphi\, S_1, \\ -\frac{A}{cr}m_2u_2\cos(ct)\left(u_3\cos\theta\cos\varphi - u_4\csc\theta\sin\varphi\right) + \frac{r}{A}m_2u_1u_2\sin(ct)\sin\theta\cos\varphi + \frac{A}{2c}\cos(ct)\sin\theta\cos\varphi\, S_2, \\ -\frac{A}{cr}m_3u_3\cos(ct)\left(r\,u_2\sin\theta\cos\varphi - u_4\csc\theta\sin\varphi\right) + \frac{r}{A}m_3u_1u_3\sin(ct)\sin\theta\cos\varphi + \frac{A}{2cr}\cos(ct)\cos\theta\cos\varphi\, S_3, \\ -\frac{A}{cr}m_4u_4\cos(ct)\left(r\,u_2\sin\theta + u_3\cos\theta\right)\cos\varphi + \frac{r}{A}m_4u_1u_4\sin(ct)\sin\theta\cos\varphi - \frac{A}{2cr}\cos(ct)\csc\theta\sin\varphi S_4 \end{bmatrix}$$

$$TC^3 = \begin{bmatrix} \frac{A}{cr}m_1u_1\sin(ct)\left(u_3\cos\theta\sin\varphi+u_4\csc\theta\cos\varphi\right)+\frac{A}{c}m_1u_1u_2\sin(ct)\,\sin\theta\sin\varphi-\frac{r}{2A}\cos(ct)\sin\theta\sin\varphi\,S_1, \\ \frac{A}{cr}m_2u_2\sin(ct)\left(u_3\cos\theta\sin\varphi+u_4\csc\theta\cos\varphi\right)+\frac{r}{A}m_2u_1u_2\cos(ct)\sin\theta\sin\varphi-\frac{A}{2c}\sin(ct)\sin\theta\sin\varphi\,S_2, \\ \frac{A}{cr}m_3u_3\sin(ct)\left(r\,u_2\sin\theta\sin\varphi+u_4\csc\theta\cos\varphi\right)+\frac{r}{A}m_3u_1u_3\cos(ct)\sin\theta\sin\varphi-\frac{A}{2cr}\sin(ct)\cos\theta\sin\varphi\,S_3, \\ \frac{A}{cr}m_4u_4\sin(ct)\left(r\,u_2\sin\theta+u_3\cos\theta\right)\sin\varphi+\frac{r}{A}m_4u_1u_4\cos(ct)\sin\theta\sin\varphi-\frac{A}{2cr}\sin(ct)\csc\theta\cos\varphi S_4 \end{bmatrix}$$

$$TC^4 = \begin{bmatrix} \frac{A}{cr}m_1u_1\cos(ct)\left(u_3\cos\theta\sin\varphi+u_4\csc\theta\cos\varphi\right)+\frac{A}{c}m_1u_1u_2\cos(ct)\,\sin\theta\sin\varphi+\frac{r}{2A}\sin(ct)\sin\theta\sin\varphi\,S_1, \\ \frac{A}{cr}m_2u_2\cos(ct)\left(u_3\cos\theta\sin\varphi+u_4\csc\theta\cos\varphi\right)-\frac{r}{A}m_2u_1u_2\sin(ct)\sin\theta\sin\varphi-\frac{A}{2c}\cos(ct)\sin\theta\sin\varphi\,S_2, \\ \frac{A}{cr}m_3u_3\cos(ct)\left(r\,u_2\sin\theta\sin\varphi+u_4\csc\theta\cos\varphi\right)-\frac{r}{A}m_3u_1u_3\sin(ct)\sin\theta\sin\varphi-\frac{A}{2cr}\cos(ct)\cos\theta\sin\varphi\,S_3, \\ \frac{A}{cr}m_4u_4\cos(ct)\left(r\,u_2\sin\theta+u_3\cos\theta\right)\sin\varphi-\frac{r}{A}m_4u_1u_4\sin(ct)\sin\theta\sin\varphi-\frac{A}{2cr}\cos(ct)\csc\theta\cos\varphi S_4 \end{bmatrix}$$

$$TC^5 = \begin{bmatrix} \frac{A}{cr}m_1u_1u_3\sin(ct)\sin\theta-\frac{A}{c}m_1u_1u_2\sin(ct)\cos\theta+\frac{r}{2A}\cos(ct)\cos\theta\,S_1, \\ \frac{A}{cr}m_2u_2u_3\sin(ct)\sin\theta-\frac{r}{A}m_2u_1u_2\cos(ct)\cos\theta+\frac{A}{2c}\sin(ct)\cos\theta\,S_2, \\ -\frac{A}{c}m_3u_2u_3\sin(ct)\cos\theta-\frac{r}{A}m_3u_1u_3\cos(ct)\cos\theta-\frac{A}{2cr}\sin(ct)\sin\theta\,S_3, \\ -\frac{r}{A}m_4u_1u_4\cos(ct)\cos\theta-\frac{A}{c}m_4u_2u_4\sin(ct)\cos\theta+\frac{A}{cr}m_4u_3u_4\sin(ct)\sin\theta \end{bmatrix}$$

$$TC^6 = \begin{bmatrix} \frac{A}{cr}m_1u_1u_3\cos(ct)\sin\theta-\frac{A}{c}m_1u_1u_2\cos(ct)\cos\theta-\frac{r}{2A}\sin(ct)\cos\theta\,S_1, \\ \frac{A}{cr}m_2u_2u_3\cos(ct)\sin\theta+\frac{r}{A}m_2u_1u_2\sin(ct)\cos\theta+\frac{A}{2c}\cos(ct)\cos\theta\,S_2, \\ -\frac{A}{c}m_3u_2u_3\cos(ct)\cos\theta+\frac{r}{A}m_3u_1u_3\sin(ct)\cos\theta-\frac{A}{2cr}\cos(ct)\sin\theta\,S_3, \\ \frac{r}{A}m_4u_1u_4\sin(ct)\cos\theta-\frac{A}{c}m_4u_2u_4\cos(ct)\cos\theta+\frac{A}{cr}m_4u_3u_4\cos(ct)\sin\theta \end{bmatrix}$$

3 G_7 spaces

This section provides Noether symmetries and conservation laws for the wave equation on static spherically symmetric spacetimes admitting G_7 which consists of two spacetimes according to the classification of spherically symmetric spacetimes [3]. These spacetimes are anti Einstein and Einstein spacetimes with the metric of the form (1.1) for $\nu = 0$, $\lambda = -\ln(\alpha r^2 + 1)$ and $\mu = \ln r^2$ where $\alpha = c^2 > 0$ for anti Einstein and $\alpha = -c^2 < 0$ for Einstein spacetime. Hence the corresponding wave equation and Lagrangian are given by

$$\frac{\partial}{\partial t}\left(\frac{r^2 u_t \sin\theta}{\sqrt{\alpha r^2+1}}\right)-\frac{\partial}{\partial r}\left(r^2 u_r \sin\theta\sqrt{\alpha r^2+1}\right)-\frac{\partial}{\partial\theta}\frac{u_\theta\sin\theta}{\sqrt{\alpha r^2+1}}-\frac{\partial}{\partial\phi}\left(\frac{u_\phi}{\sin\theta\sqrt{\alpha r^2+1}}\right)=0 \quad (3.1)$$

which is the Euler Lagrange equation of the Lagrangian

$$L=\frac{1}{2}\left(\frac{r^2\sin\theta\ {u_t}^2}{\sqrt{\alpha r^2+1}}-r^2\sin\theta\sqrt{\alpha r^2+1}\ {u_r}^2-\frac{\sin\theta}{\sqrt{\alpha r^2+1}}\ {u_\theta}^2-\frac{{u_\phi}^2}{\sin\theta\sqrt{\alpha r^2+1}}\right) \quad (3.2)$$

Solving the system (1.11) for $\nu = 0$, $\lambda = -\ln(\alpha r^2 + 1)$ and $\mu = \ln r^2$ and then imposing the constraints (1.12) gives the solution k_1, k_2, k_7, k_8, k_9 and hence the Noether symmetry algebras of the wave equation (3.1) on static spherically symmetric spacetimes admitting G_7 can be determined using (1.5), (1.6).

3.1 Einstein universe

The solutions k_1, k_2, k_7, k_8, k_9 are

$$k_1 = C_1 \frac{B}{r}, k_2 = C_2 \frac{B}{r}, k_7 = C_3 \frac{B}{r}, k_8 = C_4, k_9 = 0.$$

where $B = \sqrt{1 - c^2 r^2}$.

Hence the Noether symmetry algebra of the wave equation (3.1) with $\alpha = -c^2 < 0$ is the 7 dimensional algebra, isomorphic to its isometry algebra, generated by $\{N_1, N_2, N_3, N_4\}$ and

$$\begin{aligned} X_1 &= [0, -B\cos\phi\sin\theta, -\frac{B\cos\theta\cos\phi}{r}, \frac{B\sin\phi}{r\sin\theta}, 0] \\ X_2 &= [0, B\sin\phi\sin\theta, \frac{B\cos\theta\sin\phi}{r}, \frac{B\cos\phi}{r\sin\theta}, 0] \\ X_3 &= [0, -B\cos\theta, \frac{B\sin\theta}{r}, 0, 0] \end{aligned}$$

The corresponding conservation laws for the wave equation for the Einstein metric, determined using Noether theorem [5], are given below in terms of m_1, m_2, m_3, m_4 where $m_1 = \frac{r^2}{B}\sin\theta, m_2 = -r^2 B\sin\theta, m_3 = -\frac{\sin\theta}{B}, m_4 = -\frac{\csc\theta}{B}$.

$$TC^1 = \begin{bmatrix} -\frac{B}{r} m_1 u_1 \cos\varphi\, (r\, u_2 \sin\theta + u_3 \cos\theta\,) + \frac{B}{r} m_1 u_1 u_4 \csc\theta \sin\varphi, \\ -\frac{B}{r} m_2 u_2\, (\, u_3 \cos\theta \cos\varphi - u_4 \csc\theta \sin\varphi) + \frac{B}{2} \sin\theta \cos\varphi\, S_2, \\ -\frac{B}{r} m_3 u_3\, (r\, u_2 \sin\theta \cos\varphi - u_4 \csc\theta \sin\varphi) + \frac{B}{2r} \cos\theta\, \cos\varphi\, S_3, \\ -\frac{B}{r} m_4 u_4 \cos\varphi\, (r\, u_2 \sin\theta + u_3 \cos\theta) - \frac{B}{2r} \csc\theta\, \sin\varphi S_4 \end{bmatrix}$$

$$TC^2 = \begin{bmatrix} \frac{B}{r} m_1 u_1 \sin\varphi\, (r\, u_2 \sin\theta + u_3 \cos\theta\,) + \frac{B}{r} m_1 u_1 u_4 \csc\theta \cos\varphi, \\ \frac{B}{r} m_2 u_2\, (\, u_3 \cos\theta \sin\varphi + u_4 \csc\theta \cos\varphi) - \frac{B}{2} \sin\theta \sin\varphi\, S_2, \\ \frac{B}{r} m_3 u_3\, (r\, u_2 \sin\theta \sin\varphi + u_4 \csc\theta \cos\varphi) - \frac{B}{2r} \cos\theta \sin\varphi\, S_3, \\ \frac{B}{r} m_4 u_4 \sin\varphi\, (r\, u_2 \sin\theta + u_3 \cos\theta) - \frac{B}{2r} \csc\theta \cos\varphi\, S_4 \end{bmatrix}$$

$$TC^3 = \begin{bmatrix} -\frac{B}{r} m_1 u_1\, (r\, u_2 \cos\theta - u_3 \sin\theta)\,, \frac{B}{r} m_2 u_2 u_3 \sin\theta + \frac{B}{2} \cos\theta\, S_2, \\ -B\, m_3 u_2 u_3 \cos\theta - \frac{B}{2r} \sin\theta\, S_3, -\frac{B}{r} m_4 u_4\, (r\, u_2 \cos\theta - u_3 \sin\theta) \end{bmatrix}$$

3.2 Anti Einstein universe

The solutions k_1, k_2, k_7, k_8, k_9 are

$$k_1 = C_1 \frac{A}{r}, k_2 = C_2 \frac{A}{r}, k_7 = C_3 \frac{A}{r}, k_8 = C_4, k_9 = 0.$$

where $A = \sqrt{1 + c^2 r^2}$.

Hence the Noether symmetry algebra of the wave equation (3.1) with $\alpha = c^2 > 0$ is the

7 dimensional algebra, isomorphic to its isometry algebra, generated by $\{N_1, N_2, N_3, N_4\}$ and

$$\begin{aligned}
X_1 &= [0, -A\cos\phi\sin\theta, -\frac{A\cos\theta\cos\phi}{r}, \frac{A\sin\phi}{r\sin\theta}, 0] \\
X_2 &= [0, A\sin\phi\sin\theta, \frac{A\cos\theta\sin\phi}{r}, \frac{A\cos\phi}{r\sin\theta}, 0] \\
X_3 &= [0, -A\cos\theta, \frac{A\sin\theta}{r}, 0, 0]
\end{aligned}$$

The corresponding conservation laws for the wave equation for the anti Einstein metric, determined using Noether theorem [5], are given below in terms of m_1, m_2, m_3, m_4 where $m_1 = \frac{r^2}{A}\sin\theta, m_2 = -r^2 A\sin\theta, m_3 = -\frac{\sin\theta}{A}, m_4 = -\frac{\csc\theta}{A}$.

$$TC^1 = \begin{bmatrix} -\frac{A}{r}m_1u_1\cos\varphi\,(r\,u_2\sin\theta + u_3\cos\theta\,) + \frac{A}{r}m_1u_1u_4\csc\theta\sin\varphi, \\ -\frac{A}{r}m_2u_2\,(\,u_3\cos\theta\cos\varphi - u_4\csc\theta\sin\varphi) + \frac{A}{2}\sin\theta\cos\varphi\,S_2, \\ -\frac{A}{r}m_3u_3\,(r\,u_2\sin\theta\cos\varphi - u_4\csc\theta\sin\varphi) + \frac{A}{2r}\cos\theta\,\cos\varphi\,S_3, \\ -\frac{A}{r}m_4u_4\cos\varphi\,(r\,u_2\sin\theta + u_3\cos\theta) - \frac{A}{2r}\csc\theta\,\sin\varphi S_4 \end{bmatrix}$$

$$TC^2 = \begin{bmatrix} \frac{A}{r}m_1u_1\sin\varphi\,(r\,u_2\sin\theta + u_3\cos\theta\,) + \frac{A}{r}m_1u_1u_4\csc\theta\cos\varphi, \\ \frac{A}{r}m_2u_2\,(\,u_3\cos\theta\sin\varphi + u_4\csc\theta\cos\varphi) - \frac{A}{2}\sin\theta\sin\varphi\,S_2, \\ \frac{A}{r}m_3u_3\,(r\,u_2\sin\theta\sin\varphi + u_4\csc\theta\cos\varphi) - \frac{A}{2r}\cos\theta\sin\varphi\,S_3, \\ \frac{A}{r}m_4u_4\sin\varphi\,(r\,u_2\sin\theta + u_3\cos\theta) - \frac{A}{2r}\csc\theta\cos\varphi\,S_4 \end{bmatrix}$$

$$TC^3 = \begin{bmatrix} -\frac{A}{r}m_1u_1\,(r\,u_2\cos\theta - u_3\sin\theta)\,, \frac{A}{r}m_2u_2u_3\sin\theta + \frac{A}{2}\cos\theta\,S_2, \\ -A\,m_3u_2u_3\cos\theta - \frac{A}{2r}\sin\theta\,S_3, -\frac{A}{r}m_4u_4\,(r\,u_2\cos\theta - u_3\sin\theta) \end{bmatrix}$$

4 Six isometry spaces

In this section we obtain Noether symmetries of wave equation on static spherically symmetric spacetimes admitting G_6 which consists of three spacetimes according to the classification of spherically symmetric spacetimes [3] with the metric of the form (1.1) for $\nu = \ln G(r)^2$, $\lambda = 0$ and $\mu = \ln a^2$. Precisely we will consider the wave equation on spacetimes of

- Bertotti-Robinson type with $G(r) = \cosh(A + m^2 r)$
- Petrov type D with $G(r) = \cos(C + m^2 r)$
- Petrov type D with $G(r) = B + r$

where A, B and C are arbitrary constants.

Hence the corresponding wave equation and Lagrangian are given by

$$\frac{\partial}{\partial t}\left(\frac{a^2 u_t\sin\theta}{G(r)}\right) - \frac{\partial}{\partial r}\left(a^2 u_r\sin\theta G(r)\right) - \frac{\partial}{\partial\theta}\left(u_\theta\sin\theta G(r)\right) - \frac{\partial}{\partial\phi}\left(\frac{u_\phi G(r)}{\sin\theta}\right) = 0 \quad (4.1)$$

$$L=\frac{a^2{u_t}^2\sin\theta}{2\;G(r)}-\frac{G(r)}{2}\left(a^2{u_r}^2\sin\theta+\sin\theta\;{u_\theta}^2+\frac{{u_\phi}^2}{\sin\theta}\right) \tag{4.2}$$

Solving the system (1.11) for $\nu=\ln G(r)^2$, $\lambda=0$ and $\mu=\ln a^2$ and then imposing the constraints (1.12) gives the solution k_1,k_2,k_7,k_8,k_9 and hence the Noether symmetry algebras of the wave equation (4.1) on static spherically symmetric spacetimes admitting G_6 can be determined using (1.5), (1.6).

4.1 Bertotti Robinson spacetime

The solutions k_1,k_2,k_7,k_8,k_9 are

$$\begin{aligned}
&k_1=k_2=k_7=0,\\
&k_8=\tanh(A+m^2r)(C_1\cos(m^2t)-C_2\;\sin(m^2t))+C_3,\\
&k_9=C_1\sin(m^2t)+C_2\cos(m^2t).
\end{aligned}$$

Hence the Noether symmetry algebra of the wave equation (4.1) with $G(r)=\cosh(A+m^2r)$ is the 6 dimensional algebra, isomorphic to its isometry algebra, generated by $\{N_1,N_2,N_3,N_4\}$ and

$$\begin{aligned}
X_1&=[\tanh(A+m^2r)\cos(m^2t),\sin(m^2t),0,0,0]\\
X_2&=[-\tanh(A+m^2r)\sin(m^2t),\cos(m^2t),0,0,0]
\end{aligned}$$

The corresponding conservation laws for the wave equation for this metric, determined using Noether theorem [5], are given below in terms of m_1,m_2,m_3,m_4 where $m_1=\frac{a^2}{\cosh(A+m^2r)}\sin\theta$, $m_2=-a^2\cosh(A+m^2r)\sin\theta$, $m_3=-\cosh(A+m^2r)\sin\theta$, $m_4=-\cosh(A+m^2r)\csc\theta$.

$$TC^1=\left[\begin{array}{l}
m_1u_1u_2\sin\left(m^2t\right)-\frac{1}{2}\tanh\left(A+m^2r\right)\cos\left(m^2t\right)S_1,\\
m_2u_1u_2\tanh\left(A+m^2r\right)\cos\left(m^2t\right)-\frac{1}{2}\sin\left(m^2t\right)S_2,\\
m_3u_1u_3\tanh\left(A+m^2r\right)\cos\left(m^2t\right)+m_3u_2u_3\sin\left(m^2t\right),\\
m_4u_1u_4\tanh\left(A+m^2r\right)\cos\left(m^2t\right)+m_4u_2u_4\sin\left(m^2t\right)
\end{array}\right]$$

$$TC^2=\left[\begin{array}{l}
m_1u_1u_2\cos\left(m^2t\right)+\frac{1}{2}\tanh\left(A+m^2r\right)\sin\left(m^2t\right)S_1,\\
-m_2u_1u_2\tanh\left(A+m^2r\right)\sin\left(m^2t\right)-\frac{1}{2}\cos\left(m^2t\right)S_2,\\
-m_3u_1u_3\tanh\left(A+m^2r\right)\sin\left(m^2t\right)+m_3u_2u_3\cos\left(m^2t\right),\\
-m_4u_1u_4\tanh\left(A+m^2r\right)\sin\left(m^2t\right)+m_4u_2u_4\cos\left(m^2t\right)
\end{array}\right]$$

4.2 Petrov type D with $G(r)=\cos(C+m^2r)$

The solutions k_1,k_2,k_7,k_8,k_9 are

$$\begin{aligned}
&k_1=k_2=k_7=0,\\
&k_8=\tan(C+m^2r)(-C_1\mathrm{e}^{-m^2t}+C_2\;\mathrm{e}^{m^2t})+C_3,\\
&k_9=C_1\;\mathrm{e}^{-m^2t}+C_2\,\mathrm{e}^{m^2t}.
\end{aligned}$$

Hence the Noether symmetry algebra of the wave equation (4.1) with $G(r) = \cos(C + m^2 r)$ is the 6 dimensional algebra, isomorphic to its isometry algebra, generated by $\{N_1, N_2, N_3, N_4\}$ and

$$\begin{aligned} X_1 &= [-\tan(C+m^2r)\ \mathrm{e}^{-m^2t}, \mathrm{e}^{-m^2t}, 0, 0, 0] \\ X_2 &= [\ \tan(C+m^2r)\ \mathrm{e}^{m^2t}, \mathrm{e}^{m^2t}, 0, 0, 0] \end{aligned}$$

The corresponding conservation laws for the wave equation for this metric, determined using Noether theorem [5], are given below in terms of m_1, m_2, m_3, m_4 where $m_1 = \frac{a^2}{\cos(C+m^2r)} \sin\theta, m_2 = -a^2\cos(C+m^2r)\sin\theta, m_3 = -\cos(C+m^2r)\sin\theta, m_4 = -\cos(C + m^2r)\csc\theta$.

$$TC^1 = \left[\begin{array}{l} m_1u_1u_2\mathrm{e}^{-m^2t} + \frac{1}{2}\tan\left(C+m^2r\right)\mathrm{e}^{-m^2t}S_1, \\ -m_2u_1u_2\tan\left(C+m^2r\right)\mathrm{e}^{-m^2t} - \frac{1}{2}\mathrm{e}^{-m^2t}S_2, \\ -m_3u_1u_3\tan\left(C+m^2r\right)\mathrm{e}^{-m^2t} + m_3u_2u_3\mathrm{e}^{-m^2t}, \\ -m_4u_1u_4\tan\left(C+m^2r\right)\mathrm{e}^{-m^2t} + m_4u_2u_4\mathrm{e}^{-m^2t} \end{array} \right]$$

$$TC^2 = \left[\begin{array}{l} m_1u_1u_2\mathrm{e}^{m^2t} - \frac{1}{2}\tan\left(C+m^2r\right)\mathrm{e}^{m^2t}S_1, \\ m_2u_1u_2\tan\left(C+m^2r\right)\mathrm{e}^{m^2t} - \frac{1}{2}\mathrm{e}^{m^2t}S_2, \\ m_3u_1u_3\tan\left(C+m^2r\right)\mathrm{e}^{m^2t} + m_3u_2u_3\mathrm{e}^{m^2t}, \\ m_4u_1u_4\tan\left(C+m^2r\right)\mathrm{e}^{m^2t} + m_4u_2u_4\mathrm{e}^{m^2t} \end{array} \right]$$

4.3 Petrov type D with $G(r) = B + r$

The solutions k_1, k_2, k_7, k_8, k_9 are

$$\begin{aligned} &k_1 = k_2 = k_7 = 0, \\ &k_8 = C_1\,\frac{\mathrm{e}^{-t}}{B+r} - C_2\,\frac{\mathrm{e}^{t}}{B+r} + C_3, \\ &k_9 = C_1\,\mathrm{e}^{-t} + C_2\,\mathrm{e}^{t}. \end{aligned}$$

Hence the Noether symmetry algebra of the wave equation (4.1) with $G(r) = B + r$ is the 6 dimensional algebra, isomorphic to its isometry algebra, generated by $\{N_1, N_2, N_3, N_4\}$ and

$$\begin{aligned} X_1 &= [\frac{\mathrm{e}^{-t}}{B+r}, \mathrm{e}^{-t}, 0, 0, 0] \\ X_2 &= [\frac{\mathrm{e}^{t}}{B+r}, -\mathrm{e}^{t}, 0, 0, 0] \end{aligned}$$

The corresponding conservation laws for the wave equation for this metric, determined using Noether theorem [5], are given below in terms of m_1, m_2, m_3, m_4 where $m_1 = \frac{a^2}{B+r}\sin\theta, m_2 = -a^2(B+r)\sin\theta, m_3 = -(B+r)\sin\theta, m_4 = -(B+r)\csc\theta$.

$$TC^1 = \left[\begin{array}{l} m_1\,u_1u_2\mathrm{e}^{-t} - \frac{\mathrm{e}^{-t}}{2\,(B+r)}S_1, m_2u_1u_2\frac{\mathrm{e}^{-t}}{(B+r)} - \frac{\mathrm{e}^{-t}}{2}S_2, \\ m_3u_1u_3\frac{\mathrm{e}^{-t}}{(B+r)} + m_3u_2u_3\mathrm{e}^{-t}, m_4u_1u_4\frac{\mathrm{e}^{-t}}{(B+r)} + m_4u_2u_4\mathrm{e}^{-t} \end{array} \right]$$

$$TC^2 = \left[\begin{array}{l} m_1\, u_1 u_2 \mathrm{e}^t + \frac{\mathrm{e}^t}{2\,(B+r)} S_1, -m_2 u_1 u_2 \frac{\mathrm{e}^t}{(B+r)} - \frac{\mathrm{e}^t}{2} S_2, \\ -m_3 u_1 u_3 \frac{\mathrm{e}^t}{(B+r)} + m_3 u_2 u_3 \mathrm{e}^t, -m_4 u_1 u_4 \frac{\mathrm{e}^t}{(B+r)} + m_4 u_2 u_4 \mathrm{e}^t \end{array}\right]$$

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