King Fahd University of Petroleum & Minerals

DEPARTMENT OF MATHEMATICAL SCIENCES

Technical Report Series

TR 430

Nov 2013

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A NOTE ON REAL ALGEBRAIC GROUPS

HASSAN AZAD AND INDRANIL BISWAS

Abstract. The efficacy of using complexifications to understand the structure of real algebraic groups is demonstrated. In particular, the following two results are proved:

(1) Let G be a connected solvable linear group whose eigenvalues are all real. If the complexification $G^{\mathbb{C}}$ of G is algebraic and operates algebraically on a complex variety V, and some G orbit in V is compact, then this orbit is a point.

(2) If L is a connected subgroup of a connected real linear semisimple group G such that the complexification $L^{\mathbb{C}}$ of L is algebraic and $L^{\mathbb{C}}$ contains a maximal torus of $G^{\mathbb{C}}$, then L contains a maximal torus of G which complexifies to a maximal torus of $G^{\mathbb{C}}$.

1. Introduction

The usefulness of the point of view of algebraic groups for understanding complex Lie groups is demonstrated in the book of Onishchik–Vinberg [OV]. In [Ne1], [Ne2], algebraic groups are also used to study Lie groups. In this note we prove, in the same spirit, some results on real algebraic groups.

It one uses this point of view, many results in real Lie groups follow easily from known results in complex algebraic groups – for example the existence of Cartan subgroups of real semisimple groups and the structure theorem of real parabolic subgroups.

Let G be a solvable connected linear group whose eigenvalues are all real. If the complexification $G^{\mathbb{C}}$ of G is algebraic and operates algebraically on a complex variety V, and some G orbit in V is compact, then this orbit is a point. (See Proposition 5.)

If L is a connected subgroup of a connected real linear semisimple group G such that the complexification $L^{\mathbb{C}}$ of L is algebraic and $L^{\mathbb{C}}$ contains a maximal torus of $G^{\mathbb{C}}$, then L contains a maximal torus of G which complexifies to a maximal torus of $G^{\mathbb{C}}$. (See Proposition 6.)

A connected subgroup C of a real algebraic group G – as defined in the next section – is callled a torus of G if its complexification $C^{\mathbb{C}}$ is an algebraic torus of G in the sense of [OV, p. 133].

2. Real algebraic groups

Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a connected Lie group such that the connected subgroup of $\mathrm{GL}(n, \mathbb{C})$ with the Lie algebra $\mathrm{Lie}(G) + \sqrt{-1} \cdot \mathrm{Lie}(G)$ is an algebraic subgroup of $\mathrm{GL}(n, \mathbb{C})$. We will denote this subgroup by $G^{\mathbb{C}}$. The group $G^{\mathbb{C}}$ is generated by the complex one-parameter subgroups $\{\exp(zX) \mid z \in \mathbb{C}\}$, $X \in \mathfrak{g}$.

2000 *Mathematics Subject Classification.* 20G20, 22E15, 14L30.

Key words and phrases. Real algebraic group, maximal torus, solvable group, group action.

The set of real points of $G^{\mathbb{C}}$ is a real algebraic group in the sense of [OV], whose connected component containing the identity element is the group G. However, the proofs of the results given below do not need the machinery of real algebraic groups developed in [OV], [Vi].

Classical references for this subject are the papers of Mostow [Mo] and Matsumoto [Ma].

One of the main technical tools is the following version of Lie's theorem for real solvable groups.

Proposition 1. *Let $G \subset \mathrm{GL}(n, \mathbb{R})$ be a connected solvable group. Then G is real conjugate to a subgroup of a block triangular group, whose unipotent part is the unipotent radical of a standard real parabolic subgroup of $\mathrm{GL}(n, \mathbb{R})$ given by mutually orthogonal roots and whose semisimple part is a maximal torus of the same parabolic subgroup.*

Proof. By Lie's theorem, there is a common eigenvector, say v, with $g(v) = \chi(g) \cdot v$ for all $g \in G$, where χ is a character. If $\chi(g) = \overline{\chi(g)}$ for all $g \in G$, then we can find a real non-zero common eigenvector. We may assume that v is real in this case.

If $\chi(g) \neq \overline{\chi(g)}$ for some g, then v and $\overline{v}$ are linearly independent over $\mathbb{C}$ and the real and imaginary parts of v give a G–invariant real plane on which the matrix g is of the form

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

where $x\,, y \in \mathbb{R}$ with $x + \sqrt{-1} \cdot y = \chi(g)$.

Thus replacing the G–module $\mathbb{R}^n$ by $\mathbb{R}^n/W$, where W is a G–invariant line or plane and repeating the argument, we see that the matrices in G can be simultaneously conjugated by a real transformation to block triangular form whose diagonal entries are in the multiplicative group of nonzero reals or in the group of real matrices

$$\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x^2 + y^2 \neq 0\}.$$

Block triangular non-singular matrices with exactly k many 2×2 blocks along the diagonal give a real form of a standard parabolic subgroup of $\mathrm{GL}(n, \mathbb{C})$ corresponding to k orthogonal roots in the Dynkin diagram of type A_{n-1}.

The group G is in the connected component of the real points of this group, so the connected component is a block triangular group whose diagonal 1×1 blocks consists of positive reals and diagonal 2×2 blocks are of the form

$$\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x^2 + y^2 \neq 0\}.$$

The real points of the unipotent radical of such a parabolic group are normalized by the Levi–complement of the group. In particular, they are normalized by the maximal tori of the parabolic group. □

Lemma 2. *If a real semisimple element of $G^{\mathbb{C}}$ has all positive eigenvalues, then the element is in G.*

Proof. By conjugation with a real matrix, we may assume that this real semisimple element is a diagonal matrix g. Let H denote the Zariski closure in $G^{\mathbb{C}}$ of the subgroup generated by this element g. Then there are finitely many characters

$$\chi_j(z_1, \cdots, z_n) = \prod_{i=1}^{n} z_i^{m_{i,j}}, \quad 1 \leq j \leq N, \ m_{i,j} \in \mathbb{Z},$$

of the diagonal subgroup such that

$$H = \{(z_1, \cdots, z_n) \mid \chi_j(z_1, \cdots, z_n) = 1, \ 1 \leq j \leq N\}$$

(see [Bo, p. 17, Proposition]).

Thus if g has all positive eigenvalues, then for any $1 \leq j \leq N$,

$$\chi_j(g^{1/k}) = 1$$

for all $k \in \mathbb{Z}$. Therefore,

$$\{\exp(t \log(g)) \mid t \in \mathbb{R}\}$$

is a real one-parameter subgroup in $G^{\mathbb{C}}$ and hence it is in G. □

Corollary 3. *Any real element $g \in G^{\mathbb{C}}$ with positive eigenvalues is in a real 1–parameter subgroup of G.*

Proof. Let σ denote the complex conjugation. Observe that if $g = su$ is the Jordan decomposition of g, then s and u are in $(G^{\mathbb{C}})^{\sigma}$. By Lemma 2, the 1–parameter subgroup generated by $\log(s) + \log(u)$ is in G and g is in this subgroup. □

The following proposition will be used in the proof of Proposition 6.

Proposition 4. *If $s \in G$ is semisimple, and H is the set of real points of the Zariski closure of the group generated by s, then H has only finitely many connected components.*

Proof. This follows by arguing as in Proposition 1 and using [Ca]. □

Proposition 5. *Let G be a connected solvable linear group with real eigenvalues such that $G^{\mathbb{C}}$ is an algebraic group. If $G^{\mathbb{C}}$ operates algebraically on a complex variety V, and some G orbit in V is compact, then this orbit is a point.*

Proof. By abuse of terminology and for convenience, we will call, in this proof, a connected subgroup of $\mathrm{GL}(n, \mathbb{R})$ an algebraic group if its complexification in $\mathrm{GL}(n, \mathbb{R})$ is an algebraic group.

The group G can be embedded into the group of real upper triangular matrices and as G is connected, it is in the connected component containing the identity element of the group of real upper triangular matrices. Thus all eigenvalues of the elements of G are positive.

Take a point $p \in V$. The stabilizer in $G^{\mathbb{C}}$ for p is an algebraic group. Let

$$G_p \subset G$$

be the stabilizer of p for the action of G. Since elements of G_p have all positive eigenvalues, the subgroup G_p is connected by Corollary 3.

Let T denote the connected component containing the identity element of the group of real upper triangular matrices. It has a filtration of real algebraic subgroups

$$T = T_0 \supset T_1 \supset T_2 \supset \cdots \supset T_{\ell-1} \supset T_\ell = 1,$$

where $T_{i+1} \subset T_i$ is a normal subgroup of codimension one. Let $G_i := G \bigcap T_i$ be the intersection. So we have a filtration of algebraic subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_{\ell-1} \supset G_\ell = 1,$$

such that each successive quotient is a group of dimension at most one.

The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is algebraic in the sense that the semisimple and nilpotent parts of any element of $\mathfrak{g}$ also lie in $\mathfrak{g}$. Recall that $\dim G_i/G_{i+1} \leq 1$. Hence any nonzero element X in the Lie algebra of G_i/G_{i+1} is either semisimple or nilpotent. Thus as

$$G_i = \{\exp(tX) \mid t \in \mathbb{R}\} G_{i+1},$$

using arguments similar to those in Lemma 2 we see that

$$\{\exp(tX) \mid t \in \mathbb{R}\} \cap G_{i+1} = 1.$$

Therefore,

$$G_i \cong \{\exp(tX) \mid t \in \mathbb{R}\} \times G_{i+1}.$$

Thus, topologically G is a cell. Here we have used that if H and K are Lie subgroups of G with $H \bigcap K = 1$, and HK is a closed subgroup of G, then HK is homeomorphic to $H \times K$. This implies that H and K are closed in the Euclidean topology of G.

Let U be the group of unipotents in G_p (recall that $G_p \subset G$ is the stabilizer of the point $p \in V$). Then

$$G/U \longrightarrow G/G_p$$

is a fibration with total space a cell and the fiber G_p/U is also a cell. Now one can argue as on p 450 of Hilgert-Neeb [HN, p. 885, Proposition 2.2], cases 1 & 2, to conclude that G/G_p is a cell and-as the orbit is compact-it must be a point. Or one can use the following topological argument to show that if the base G/G_p is compact, then G/G_p is a point. To prove this consider the long exact sequence of homotopies for the fibration $G/U \longrightarrow G/G_p$. Since G/U and the fiber G_p/U are contractible, the exact sequence gives that $\pi_i(G/G_p) = 0$ for all $i \geq 1$. Therefore,

$$H_i(G/G_p, \mathbb{Z}) = 0$$

for all $i \geq 1$ by the Hurewicz theorem. Since G_p is connected, the adjoint action of G_p on $\mathrm{Lie}(G)/\mathrm{Lie}(G_p)$ preserves its orientation. Hence G/G_p is orientable. We have

$$H_{\dim G/G_p}(G/G_p, \mathbb{Z}) = \mathbb{Z}$$

because G/G_p is a compact orientable manifold. Hence $\dim G/G_p = 0$, implying that G/G_p is a point. □

Proposition 6. *If L is a connected subgroup of a connected real linear semisimple group G such that the complexification $L^{\mathbb{C}}$ is algebraic and $L^{\mathbb{C}}$ contains a maximal torus of $G^{\mathbb{C}}$, then L contains a maximal torus of G whose complexification is a maximal torus of $G^{\mathbb{C}}$.*

Proof. Let $\mathfrak{l}$ be the Lie algebra of L. Then $\mathfrak{l}$ contains both the semisimple and nilpotent part of any element in it. If $\mathfrak{l}$ contains only nilpotent elements, then $\mathfrak{l}$ can be conjugated over the reals to strictly upper triangular form and the same goes for the complexification $\mathfrak{l}^{\mathbb{C}}$ and $L^{\mathbb{C}}$. Thus L has a connected abelian subgroup consisting only of semisimple elements. Let S be such a maximal subgroup.

Using Proposition 1, the real points of the connected component, containing the identity, of the Zariski closure of S contains only semisimple elements. Thus S equals this group. The Lie algebras of both S and $Z(S)$ complexify to Lie algebras of algebraic groups.

The group $Z(S^{\mathbb{C}})/S^{\mathbb{C}}$ is reductive algebraic and the $Z(S)$ orbit of the point

$$\xi_0 = eS^{\mathbb{C}} \in Z(S^{\mathbb{C}})/S^{\mathbb{C}}$$

for the left–translation action is embedded as a totally real subgroup. Moreover, $Z(S)/S$ fibers over the $Z(S)$ orbit of ξ_0 with a finite fiber. Thus if the Lie algebra of $Z(S)/S$ is nontrivial, it must have a nonzero semisimple element. This would contradict the maximality of S. Hence we conclude that the connected component $Z(S)$ containing the identity is S.

As the Lie algebra of $Z(S^{\mathbb{C}})$ coincides with the complexification of the Lie algebra of $Z(S)$, and the Lie algebra of $Z(S)$ is also the Lie algebra of S, we see that $Z(S^{\mathbb{C}}) = S^{\mathbb{C}}$.

Finally, $Z(S^{\mathbb{C}})$ contains a maximal torus, say T, of $L^{\mathbb{C}}$ and therefore of $G^{\mathbb{C}}$ – by assumption. But as $Z(S^{\mathbb{C}}) = S^{\mathbb{C}}$, it is itself an algebraic torus. Therefore, $T = S^{\mathbb{C}}$, □

Corollary 7. *Every semisimple element of* Lie(G) *lies in some maximal toral subalgebra of* Lie(G)*.*

Proof. Let X be a real semisimple element of Lie(G). Then X is tangent to the connected component H of the real points of the Zariski closure of the one-parameter subgroup generated by X. Therefore, as the centralizer $Z(H^{\mathbb{C}})$ of $H^{\mathbb{C}}$ in $G^{\mathbb{C}}$ contains a maximal torus of $G^{\mathbb{C}}$, we conclude that the centralizer $Z(H)$ of H in G contains a maximal torus, say C, of G. But then CH is a connected group and the connected component, containing the identity element, of the real points of its Zariski closure consists of semisimple elements. Therefore, $CH = C$ and H is contained in C. Consequently, X is in the Lie algebra of C. □

For a different proof of Corollary 7, see e.g. [He, p. 420].

Another consequence of Proposition 6 is the following result of [Mo].

Proposition 8 ([Mo])**.** *Let P be a subgroup of a real semisimple group G with $P^{\mathbb{C}}$ a parabolic subgroup of $G^{\mathbb{C}}$. Then P contains a noncompact maximal torus of G.*

Proof. By Proposition 6, the above subgroup P contains a maximal torus, say C, of G. If C is compact, then, as $P^{\mathbb{C}}$ is conjugation invariant, and conjugation maps any root to its negative, the unipotent radical of $P^{\mathbb{C}}$ contains a subgroup whose Lie algebra is $\mathrm{sl}(2, \mathbb{C})$. This contradiction proves that C must contain a closed real diagonalizable subgroup. □

Following [Ma] and [Mo] the structure of real parabolic subgroups can be explained using known results about complex parabolic groups. As shown in Proposition 8, if P is a parabolic subgroup of G, then P contains a maximal torus T of G of the form $T = AC$, where A is split and C is compact. By Proposition 1, both A and C are real algebraic.

The torus $T^{\mathbb{C}}$ and the unipotent radical $R_u(P^{\mathbb{C}})$ of $P^{\mathbb{C}}$ are both conjugation invariant. Take a Borel subgroup of $P^{\mathbb{C}}$ containing $T^{\mathbb{C}}$. This defines a positive system of roots. The $T^{\mathbb{C}}$ invariant Levi component of $P^{\mathbb{C}}$ is given by a subset Π of simple roots. There is an element t in the Lie algebra of $T^{\mathbb{C}}$ such that

$$\alpha(t) = 0$$

if $\alpha \in \Pi$, and $\alpha(t) = 1$ is α is in the complement of Π in the set of simple roots.

Using the invariance of $P^{\mathbb{C}}$ under conjugation, we may assume that, in fact, t is in the Lie algebra of A. Thus the connected component of $Z(t)$, the centralizer of t, is the $T^{\mathbb{C}}$ invariant Levi complement of $P^{\mathbb{C}}$ and $\alpha(t)$ is positive if α is a root of $R_u(P^{\mathbb{C}})$.

Thus the Lie algebra of P is spanned by the nonnegative eigenspaces of the element $t \in \text{Lie}(A)$.

On the other hand, if $t \in \text{Lie}(A)$ and G is semisimple, then $\text{ad}(t)$ has real eigenvalues. Replacing t by $-t$, we may assume that it has at least one positive eigenvalue. If T is a maximal torus containing A and we take roots relative to $T^{\mathbb{C}}$, then the roots which take nonnegative values on t give a parabolic subalgebra whose Levi complement corresponds to roots that vanish on t and whose nil radical is given by roots that take positive values on t: here we are using the fact that if an integrally closed set S of roots is such that it contains at least one of the roots r and $-r$ for every root r, then S contains a simple set of roots. This parabolic subgroup is conjugation invariant and the Lie algebra of its real points is given by the nonnegative eigenspaces of t.

Other, more recent applications of the real algebraic groups are given in [Ch].

Acknowledgements

The first author thanks KFUPM for funding Research Project IN131058. The second author acknowledges the support of the J. C. Bose Fellowship.

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