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A Direct Proof of the Joint Moment Generating Function of Sample Mean and Variance

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Abstract. Without assuming independence of sample mean and variance, or without using any conditional distribution, we present a direct derivation of the joint moment generating function for sample mean and variance for independently, identically and normally distributed random variables.

Key Words: Sample mean, sample variance, independence, moment generating function

MSC (2010). 62E10, 62H20, 62G35.

1. Introduction

The independence of sample mean and variance of independently, identically and normally distributed variables is essential in the basic definition of Student t-statistic, and also in the development of many statistical methods. It is usually proved using the independence of $\overline{X}$ and $(X_1-\overline{X},\ X_2-\overline{X},\ \ldots,X_n-\overline{X})$, (see e.g. Theorem 1, p.340, Rohatgi and Saleh, 2001), but this requires background on independence of functions of random variables (Theorem 2, p.121, Rohatgi and Saleh, 2001) that may not be easily accessible to beginning undergraduate students. There are several proofs of the independence of sample mean and variance, the simplest of which seems to be the one due to Shuster (1973) which uses moment generating function. It should however be noted that proofs accessible to undergraduate students have been an issue of discussion. See for example, Zehna (1991) and also American Statistician, 1992, Volume 46, No. 1, pp. 72-75. In this note we give a new proof of this independence that also uses moment generating function, but, unlike Shuster's proof, avoids the use of conditional distributions and seems to be more suitable for undergraduate students.

2. Some Preliminaries

Let $X_1, X_2,\ldots,X_n$ $(n=2,3,\cdots)$ have an arbitrary n-dimensional joint distribution. We define the sample mean $\overline{x}$ and variance s^2 by $n\overline{x}=\sum_{i=1}^{i=n}x_i$ and $(n-1)s^2=\sum_{i=1}^{i=n}(x_i-\overline{x})^2$, respectively. The sample variance can also be represented by

$$n(n-1)s^2=(n-1)\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}x_ix_j. \qquad (2.1)$$

Also for identically distributed observations $X_1, X_2, \ldots, X_n$ with common mean $\mu,$ we denote $\mu'_a \equiv E(X^a),$ the a-th moment of X and $\mu_a \equiv E(X-\mu)^a$, the centered moment of X order $a.$ The mean μ'_1 and variance $\mu_2 \equiv V(X)$ will be simply denoted by μ and $\sigma^2 = \mu'_2 - \mu^2$ respectively.

The moment generating function of $X \sim N(\mu, \sigma^2)$ is given by

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right), \quad -\infty < t < \infty.$$

Let $A = [a_{ij}]$ be a $n \times n$ positive definite symmetric matrix. Then the following integral is known:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} a_{ij}x_i x_j + \sum_{i=1}^{i=n} b_i x_i\right) dx_1 dx_2 \cdots dx_n = \frac{(2\pi)^{n/2}}{|A|^{1/2}} \exp\left(\frac{1}{2} \underset{\sim}{b}' A^{-1} \underset{\sim}{b}\right), \tag{2.2}$$

where $\underset{\sim}{b} = (b_1, b_2, \cdots b_n)'.$ The above expression usually appears as part of the moment generating function

$$M_{\underset{\sim}{X}}(\underset{\sim}{b}) = \exp\left(\frac{1}{2}\underset{\sim}{b}'\Sigma \underset{\sim}{b}\right), \tag{2.3}$$

of a random variable $\underset{\sim}{X}$ having multivariate normal distribution $N(\underset{\sim}{0}, \Sigma).$ See for example, Anderson (1984, p.21 and p.47).

For the proof of the following theorem dealing with the properties of a pattern matrix, see Rao (1973, p.67).

Theorem 2.1 Let the $n \times n$ matrix $\Gamma = [\gamma_{ij}]$ where $\gamma_{ii} = \alpha,$ for $i = 1, 2, \cdots, n,$ and $\gamma_{ij} = \beta,$ for $i = 1, 2, \cdots, n,$ and $j (\neq i) = 1, 2, \cdots, n.$ Then the following hold:

a. $|\Gamma| = (\alpha - \beta)^{n-1}[\alpha + (n-1)\beta],$

b. Γ^{-1} exists if and only if $\alpha \neq \beta$ and $\alpha \neq (1-n)\beta.$ Moreover, if γ^{ii} and γ^{ij} are the entries of $\Gamma^{-1},$ then

$$\gamma^{ii} = \frac{\alpha + (n-2)\beta}{[\alpha + (n-1)\beta](\alpha - \beta)}, \text{ and } \gamma^{ij} = \frac{-\beta}{[\alpha + (n-1)\beta](\alpha - \beta)}, \ i \neq j.$$

Corollary 2.1 Let the $n \times n$ matrix $\Gamma = [\gamma_{ij}]$ where $\gamma_{ii} = \alpha,$ for $i = 1, 2, \cdots, n,$ and $\gamma_{ij} = \beta,$ for $i = 1, 2, \cdots, n,$ and $j (\neq i) = 1, 2, \cdots, n.$ Further if $\alpha + (n-1)\beta = 1,$ then we have the following:

a. $|\Gamma| = (\alpha - \beta)^{n-1},$

b. $\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}\gamma_{ij}=n,$

c. $\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}\gamma^{ij}=n.$

Proof.

a. It is obvious from part (a) of Theorem 2.1.

b. Since $\alpha+(n-1)\beta=1,$ it follows that

$\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}=\sum_{i=1}^{n}\gamma_{ii}+\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\gamma_{ij}$ which equals n.

c. Since $\alpha+(n-1)\beta=1,$ it follows from Theorem 2.1 that $\gamma^{ii}=\dfrac{1-\beta}{\alpha-\beta},\quad (i=1,2,\cdots,n)$ and $\gamma^{ij}=\dfrac{-\beta}{\alpha-\beta},\ (i=1,2,\cdots,n,\ j(\neq i)=1,2,\cdots,n).$ Then

$\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma^{ij}=\sum_{i=1}^{n}\gamma^{ii}+\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\gamma^{ij}$ which equals n.

3. The Joint M.G.F. of Sample Mean and Variance

Without using any conditional distribution or assuming independence of $\overline{X}$ and $S^2,$ we present a direct proof of the joint moment generating function of $\overline{X}$ and $S^2,$ based on independently, identically and normally distributed random variables.

Theorem 3.1 Let the random variables $X_1, X_2, \cdots, X_n,\ (n\geq 2)$ be independently, identically and normally distributed with $E(X_1)=\mu$ and $Var(X_1)=\sigma^2.$ Then the joint moment generating function of the sample mean $\overline{X}$ and variance S^2 is given by

$$M_{\overline{X},S^2}(t_1,t_2)=\left[\exp\left(\mu t_1+\frac{\sigma^2 t_1^2}{2n}\right)\right]\left(1-\frac{2\sigma^2 t_2}{n-1}\right)^{-(n-1)/2},\quad n\geq 2. \tag{3.1}$$

In particular, $\overline{X}$ and S^2 are independent, and $(n-1)S^2/\sigma^2\sim\chi^2_{n-1}.$

Proof. The joint moment generating function of $\overline{X}$ and S^2 is given by

$$M_{\overline{X},S^2}(t_1,t_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(t_1\overline{x}+t_2 s^2\right)\prod_{i=1}^{n}f(x_i)dx_i,$$

which equals

$$M_{\bar{X},S^2}(t_1,t_2)=\frac{1}{(2\pi)^{n/2}\sigma^n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(t_1\bar{x}+t_2 s^2\right)\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]\prod_{i=1}^{n}dx_i.$$

Using the representation of s^2 given in (2.1) in the above integral, and the transformation $x_i=\mu+\sigma z_i$, $(i=1,2,\cdots,n)$ yields

$$M_{\bar{X},S^2}(t_1,t_2)=\frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(t_1\sigma\bar{z})\exp\left(\frac{t_2}{n}\sum_{i=1}^{n}z_i^2-\frac{t_2}{n(n-1)}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}z_i z_j\right)\exp\left(-\frac{1}{2}\sum_{i=1}^{n}z_i^2\right)\prod_{i=1}^{n}dz_i.$$

Let $\underset{\sim}{b}$ be the n-vector $\frac{\sigma t_1}{n}\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}'$, and $\Gamma=(\gamma_{ij})$ be the $n\times n$ matrix given by

$$\gamma_{ij}=\begin{cases}1-\dfrac{2\sigma^2 t_2}{n}, & \text{if } i=j\\[2ex] \dfrac{2\sigma^2 t_2}{n(n-1)}, & \text{if } i\neq j.\end{cases}$$

$$M_{\bar{X},S^2}(t_1,t_2)=\frac{\exp(t_1\mu)}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(\sum_{i=1}^{n}b_i z_i\right)\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}z_i z_j\right)\prod_{i=1}^{n}dz_i. \tag{3.2}$$

The equation (3.2) can then be written as

$$M_{\bar{X},S^2}(t_1,t_2)=\frac{\exp(t_1\mu)}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\underset{\sim}{z}'\Gamma\underset{\sim}{z}\right)\exp(\underset{\sim}{b}'\underline{z})d\underset{\sim}{z},$$

which, by (2.3), can be evaluated to be

$$M_{\bar{X},S^2}(t_1,t_2)=\exp(t_1\mu)\,|\,\Gamma\,|^{-1/2}\exp\left(-\frac{1}{2}\underset{\sim}{b}'\Gamma^{-1}\underset{\sim}{b}\right). \tag{3.3}$$

It is easy to check that

$$\underset{\sim}{b}'\Gamma^{-1}\underset{\sim}{b}=tr\Gamma^{-1}\underset{\sim}{b}\underset{\sim}{b}'=\frac{\sigma^2 t_1^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma^{ij}=\frac{\sigma^2 t_1^2}{n}, \tag{3.4}$$

where the last step follows by Corollary 2.1(c). Also by Corollary 2.1(a), we have

$|\,\Gamma\,|^{-1/2}=(\alpha-\beta)^{n-1}$ where $\alpha=1-\dfrac{2\sigma^2 t_2}{n}$ and $\beta=\dfrac{2\sigma^2 t_2}{n(n-1)}$. Since $\alpha-\beta=1-\dfrac{2\sigma^2 t_2}{n-1}$, we obtain

$$|\,\Gamma\,|^{-1/2}=\left(1-\frac{2\sigma^2 t_2}{n-1}\right)^{-(n-1)/2}. \tag{3.5}$$

Plugging (3.4) and (3.5) in (3.3), we obtain (3.1).

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