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On the differential invariants of third-order ordinary differential equations $y''' = f(x, y, y', y'')$ via fiber preserving transformations

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Abstract

Bagderina [1] solved the equivalence problem of the third-order ordinary differential equations (ODEs), quadratic in the second-order derivative, via point transformations. However, the question is open for the general class $y''' = f(x, y, y', y'')$ which is not quadratic in the second-order derivative. In this paper, we use Lie's infinitesimal method to study the differential invariants of this general class under pseudo-group of fiber preserving equivalence transformations $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$. As a result, all the third order differential invariants of this group and the invariant differentiation operators are determined. This leads to simple necessary explicit conditions for third order differential equation to be equivalent to the canonical forms under the considered group of transformations.

Key words: Lie's infinitesimal method, differential invariants, third order ODEs, equivalence problem, fiber preserving transformations, normal forms, Lie symmetries.

1 Introduction

Differential invariants play a significant role in a broad range of problems arising in differential geometry, differential equations, mathematical physics, and applications. For instance, the differential invariants have been particularly useful in dealing with equivalence problem for geometric structures [2], classification of invariant differential equations and invariant variational problems [3, 4, 5, 6], integration of ordinary differential equations [4, 5] equivalence and symmetry properties of solutions [2], the construction of particular solutions to systems of partial differential equations [5, 7, 8, 9].

Lie [10] showed that every invariant system of differential equations [11] and every variational problem [12] can be directly expressed in terms of the differential invariants. Along with an illustration [11] of how differential invariants could be used to integrate ordinary differential equations, Lie succeeded in completely classifying all the differential invariants for all possible finite-dimensional Lie groups of point transformations in the case of one independent and one dependent variable.

Tresse [13] and Ovsiannikov [5] generalized Lie's preliminary results on invariant differentiations and existence of finite bases of differential invariants. For the general theory of differential invariants of Lie groups including algorithms of construction of differential invariants, the reader is referred to [2, 5]. Ibragimov [14, 15] developed a simple method for constructing invariants of families of linear and nonlinear differential equations admitting infinite equivalence transformation groups. Lie's infinitesimal method was applied to to solve the equivalence problem for several linear and nonlinear equations

[16, 17, 18, 19, 20, 21, 22, 23, 24]. Cartan's equivalence method [2, 25] is another systematic approach to solve the equivalence problem for differential equations. The linearization problem is a particular case of the equivalence problem.

Linearization criteria for the third-order ODEs which are at most cubic in the second-order derivative

$$y''' = a(x, y, y')y''^3 + b(x, y, y')y''^2 + c(x, y, y')y'' + d(x, y, y') \tag{1.1}$$

have been obtained in [26] by Cartan's method and then in [20] by direct approach.

Using Lie's infinitesimal method, Bagderina [1] solved the equivalence problem of the third-order ODEs which are at most quadratic in the second-order derivative

$$y''' = a(x, y, y')y''^2 + b(x, y, y')y'' + c(x, y, y') \tag{1.2}$$

with respect to the group of point equivalence transformations

$$\bar{x} = \phi(x, y), \bar{y} = \psi(x, y). \tag{1.3}$$

As an extension, in this paper, we use Lie's infinitesimal method to study the differential invariants of the third-order ODEs

$$y''' = f(x, y, y', y''), \tag{1.4}$$

which are not quadratic in the second-order derivative, under pseudo-group of fiber preserving equivalence transformations

$$\bar{x} = \phi(x), \bar{y} = \psi(x, y). \tag{1.5}$$

The structure of the paper is the following: In the next section, we give a short description of Lie's infinitesimal method to find the differential invariants and invariant differentiation operators of the class of ODEs (1.4) with respect to the general group of point equivalence transformations $\bar{x} = \phi(x, y), \bar{y} = \psi(x, y)$. In Sections 3, using the methods described in Section 2, first, we recover the infinitesimal point equivalence transformations. Then we find the third-order differential invariants and invariant differentiation operators of the class of ODEs (1.4), which are not quadratic in the second-order derivative, under two subgroups of the general group of point equivalence transformations.

Throughout this paper, we will use the notation $A = [a_1, a_2, ..., a_n]$ to express any differential operator $A = \sum_{j=1}^{n} a_j \frac{\partial}{\partial b_j}$. Also, we will denote y', y'' by p, q, respectively.

2 Description of Lie's infinitesimal method

In this section, we describe briefly the method used in this paper to derive differential invariants using point equivalence transformations. Consider the kth-order system of partial differential equations (PDEs) of n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$

$$E_\alpha(x, u, ..., u_{(k)}) = 0, \quad \alpha = 1, ..., m \ , \tag{2.6}$$

where $u_{(1)}, u_{(2)}, ..., u_{(k)}$ denote the collections of all first, second,..., kth-order partial derivatives, i.e., $u_i^\alpha = D_i(u^\alpha)$,
$u_{ij}^\alpha = D_j D_i(u^\alpha)$,...respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + ..., \quad i = 1, ..., n \ , \tag{2.7}$$

in which the summation convention is used.

Definition 2.1. *The Lie-Bäcklund operator* is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \quad \xi^i, \eta^\alpha \in A \ , \tag{2.8}$$

where A is the space of *differential functions*. The operator (2.8) is an abbreviated form of the infinite formal sum

$$\begin{aligned} X^{(s)} &= \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1 i_2 \dots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \\ &= \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \end{aligned} \tag{2.9}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} &\zeta^\alpha_i = D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j) = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \\ &\zeta^\alpha_{i_1 \dots i_s} = D_{i_s}(\zeta^\alpha_{i_1 \dots i_{s-1}}) - u^\alpha_{j i_1 \dots i_{s-1}} D_{i_s}(\xi^j) = D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \dots i_s}, \quad s > 1, \end{aligned} \tag{2.10}$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{2.11}$$

Definition 2.2. *The point equivalence transformation of a class of PDEs (2.6)* is an invertible transformation of the independent and dependent variables of the form:

$$\bar{x} = \phi(x, u), \bar{u} = \psi(x, u), \tag{2.12}$$

that maps every equation of the class into an equations of the same form,

$$E_\alpha(\bar{x}, \bar{u}, \dots, \bar{u}_{(k)}) = 0, \quad \alpha = 1, \dots, m. \tag{2.13}$$

In order to describe the Lie's infinitesimal method for deriving differential invariants using point equivalence transformations, we use as example the class (1.4). It is well known that the point equivalence transformation

$$\bar{x} = \phi(x, y), \bar{y} = \psi(x, y), \tag{2.14}$$

maps (1.4) into:

$$\bar{y}''' = \bar{f}(\bar{x}, \bar{y}, \bar{y}', \bar{y}''), \tag{2.15}$$

for arbitrary functions $\phi(x, y)$ and $\psi(x, y)$ where $\bar{f}$, in general, can be different from the original function f. The set of all equivalence transformations forms a group denoted by $\mathcal{E}$.

The standard procedure of Lie infinitesimal invariance criterion [5] is implemented in the next section to recover the continuous group of point equivalence transformations (2.14) of the class of third-order ODEs (1.4) with the corresponding infinitesimal point equivalence transformations

$$Y = \xi(x, y)D_x + W\partial_y + D_x(W)\partial_p + D_x^2(W)\partial_q + \mu(x, y, p, q, f)\partial_f, \tag{2.16}$$

where $\xi(x, y)$ and $\eta(x, y)$ are arbitrary functions defined through

$$\bar{x} = x + \epsilon\xi(x, y) + O(\epsilon^2) = \phi(x, y), \tag{2.17}$$

$$\bar{y} = y + \epsilon\eta(x, y) + O(\epsilon^2) = \psi(x, y), \tag{2.18}$$

and

$$\mu = \dot{D}_x^3(W) + \xi(x, y)\dot{D}_x f, \tag{2.19}$$

with $W = \eta - \xi p$ and $\dot{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + f\frac{\partial}{\partial q}$.

Definition 2.3. *An invariant of a class of third-order ODEs (1.4)* is a function of the form:

$$J = J(x, y, p, q, f), \tag{2.20}$$

which is invariant under the equivalence transformation (2.14).

Definition 2.4. *A differential invariant of order s of a class of third-order ODEs (1.4)* is a function of the form:

$$J = J(x, y, p, q, f, f_{(1)}, f_{(2)}, ..., f_{(s)}), \tag{2.21}$$

which is invariant under the equivalence transformation (2.14) where $f_{(1)}, f_{(2)}, ..., f_{(s)}$ denote the collections of all first, second,..., sth-order partial derivatives.

Definition 2.5. *An invariant system of order s of a class of third-order ODEs (1.4)* is the system of the form $E_\alpha(x, y, p, q, f, f_{(1)}, f_{(2)}, ..., f_{(s)}) = 0,\ \alpha = 1, ..., m$ that satisfies the condition:

$$Y^{(s)}E_\alpha(x, y, p, q, f, f_{(1)}, f_{(2)}, ..., f_{(s)}) = 0\ (mod\ E_\alpha = 0,\ \alpha = 1, ..., m),\ \alpha = 1, ..., m. \tag{2.22}$$

An invariant system with $\alpha = 1$ is called an invariant equation.

Now, according to the theory of invariants of infinite transformation groups [5], the invariant criterion

$$YJ(x, y, p, q, f) = 0, \tag{2.23}$$

should be split by the functions $\xi(x, y)$ and $\eta(x, y)$ and their derivatives. This gives rise to a homogeneous linear system of PDEs whose solution gives the required invariants.

It should be noted here that since the generator Y contains arbitrary functions $\xi(x, y)$ and $\eta(x, y)$, then the corresponding identity (2.23) leads to m linear PDEs for J, where m is the number of the arbitrary functions and their derivatives that appear in Y. We point out that these m PDEs are not necessarily linearly independent.

In order to determine the differential invariants of order s, we need to calculate the prolongations of the operator Y using (2.9) by considering f as a dependent variable and the variables x, y, p, q as independent variables:

$$\begin{aligned} Y^{(s)} &= Y(x)\tilde{D}_x + Y(y)\tilde{D}_y + Y(p)\tilde{D}_p + Y(q)\tilde{D}_q + \tilde{W}\tfrac{\partial}{\partial f} + \sum_{s\geq 1} \tilde{D}_{i_1}...\tilde{D}_{i_s}(\tilde{W})\tfrac{\partial}{\partial f_{i_1 i_2 ... i_s}}, \\ & i_1, i_2, ..., i_s \in \{x, y, p, q\}, \end{aligned} \tag{2.24}$$

where

$$\tilde{D}_k = \partial_k + f_k\partial_f + f_{ki}\partial_{f_i} + f_{kij}\partial_{f_{ij}} + ...,\quad i, j, k \in \{x, y, p, q\}. \tag{2.25}$$

in which $\tilde{W}$ is the *Lie characteristic function*

$$\tilde{W} = \mu - Y(x)f_x - Y(y)f_y - Y(p)f_p - Y(q)f_q. \tag{2.26}$$

The differential invariants are determined by the equations:

$$Y^{(s)}J(x, y, p, q, f, f_{(1)}, f_{(2)}, ..., f_{(s)}) = 0. \tag{2.27}$$

It should be noted here that since the generator $Y^{(s)}$ contains arbitrary functions $\xi(x, y)$ and $\eta(x, y)$, then the corresponding identity (2.27) leads to m linear PDEs for J, where m is the number of the arbitrary functions and their derivatives that appear in $Y^{(s)}$.

For simplicity, from here on, we denote the derivative of $f(x, y, p, q)$ with respect to the independent variables x, y, p, q as f_1, f_2, f_3, f_4. The same notation will be used for higher-order derivatives.

Now, in order to find all the third order differential invariants of third-order ODEs (1.4), one can solve the invariant criterion (2.27) with $s = 3$. However, for compactness of the derived differential invariants, one can replace any partial derivative with respect to x by total derivative with respect to x. So, we need to solve the following invariant criterion

$$\begin{gathered} Y^{(3)}J(x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, \\ f_{2,4,4}, f_{3,3,3}, f_{3,3,4}, f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}) = 0, \end{gathered} \tag{2.28}$$

by prolonging the infinitesimal operator $Y^{(3)}$ to work over the variables $d_{i,j}$ through the infinitesimals $Y^{(3)}d_{i,j}$ where

$$\begin{aligned} &d_{1,1} = \dot{D}_x f, d_{1,2} = \dot{D}_x f_2, d_{1,3} = \dot{D}_x f_3, d_{1,4} = \dot{D}_x f_4, d_{1,5} = \dot{D}_x f_{2,2}, \\ &d_{1,6} = \dot{D}_x f_{2,3}, d_{1,7} = \dot{D}_x f_{2,4}, d_{1,8} = \dot{D}_x f_{3,3}, d_{1,9} = \dot{D}_x f_{3,4}, d_{1,10} = \dot{D}_x f_{4,4}, \\ &d_{2,1} = \dot{D}_x^2 f, d_{2,2} = \dot{D}_x^2 f_2, d_{2,3} = \dot{D}_x^2 f_3, d_{2,4} = \dot{D}_x^2 f_4, d_{3,1} = \dot{D}_x^3 f. \end{aligned} \tag{2.29}$$

Definition 2.6. *An invariant differentiation operator of a class of third-order ODEs (1.4)* is a differential operator $\mathcal{D}$ that satisfies that if I is a differential invariant of ODE (1.4), then $\mathcal{D}I$ is its differential invariant too.

As it is shown in [5], the number of the independent invariant differentiation operators $\mathcal{D}$ equals to the number of independent variables x, y, p and q. The invariant differentiation operators $\mathcal{D}$ should take the form

$$\mathcal{D} = K\tilde{D}_x + L\tilde{D}_y + M\tilde{D}_p + N\tilde{D}_q, \tag{2.30}$$

with the coordinates K, L, M and N satisfying the non-homogeneous linear system

$$Y^{(3)}K = \mathcal{D}(Y(x)), \quad Y^{(3)}L = \mathcal{D}(Y(y)), \quad Y^{(3)}M = \mathcal{D}(Y(p)), \quad Y^{(3)}N = \mathcal{D}(Y(q)), \tag{2.31}$$

where K, L, M and N are functions of the following variables

$$x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, f_{2,4,4}, f_{3,3,3}, f_{3,3,4}, f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}. \tag{2.32}$$

In reality, the general solution of the system (2.31) gives both of the differential invariants and the differential operators. This general solution can be found by prolonging the infinitesimal operator $Y^{(3)}$ to work over the variables K, L, M and N through the infinitesimals $Y^{(3)}K, Y^{(3)}L, Y^{(3)}M$ and $Y^{(3)}N$ respectively. Then solving the invariant criterion

$$Y^{(3)}J(x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, f_{2,4,4}, f_{3,3,3}, f_{3,3,4}, f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}, K, L, M, N) = 0, \tag{2.33}$$

gives the implicit solution of the variables K, L, M and N with the differential invariants.

In this paper, we are interested in find the the third-order differential invariants and differential operators of the general class $y''' = f(x, y, y', y'')$ under a subgroup of point transformations (2.14), namely the fiber preserving transformations $x = \phi(x), y = \psi(x, y)$. So, according to the theory of invariants of infinite transformation groups [5], the invariant criterion (2.33) should be split by the functions $\xi(x)$ and $\eta(x, y)$ and their derivatives. This gives rise to a homogeneous linear system of partial differential equations (PDEs):

$$X_iJ = 0, i = 1 \dots 28, \quad T_iJ = 0, i = 1 \dots 7, \tag{2.34}$$

where $X_i, i = 1 \ldots 28$, are the differential operators corresponding to the coefficients of the following derivatives of $\eta(x, y)$ up to the sixth order in the invariant criterion

$$\eta, \eta_1, \eta_2, \eta_{1,1}, \eta_{1,2}, \eta_{2,2}, \eta_{1,1,1}, \eta_{1,1,2}, \eta_{1,2,2}, \eta_{2,2,2}, \eta_{1,1,1,1}, \eta_{1,1,1,2}, \eta_{1,1,2,2}, \eta_{1,2,2,2}, \eta_{2,2,2,2}, \eta_{1,1,1,1,1}, \eta_{1,1,1,1,2},$$
$$\eta_{1,1,1,2,2}, \eta_{1,1,2,2,2}, \eta_{1,2,2,2,2}, \eta_{2,2,2,2,2}, \eta_{1,1,1,1,1,1}, \eta_{1,1,1,1,1,2}, \eta_{1,1,1,1,2,2}, \eta_{1,1,1,2,2,2}, \eta_{1,1,2,2,2,2}, \eta_{1,2,2,2,2,2}, \eta_{2,2,2,2,2,2} \tag{2.35}$$

and $T_i, i = 1 \ldots 7$, are the differential operators corresponding to the coefficients of the following derivatives of $\xi(x)$ up to the sixth order in the invariant criterion

$$\xi, \xi_1, \xi_{1,1}, \xi_{1,1,1}, \xi_{1,1,1,1}, \xi_{1,1,1,1,1}, \xi_{1,1,1,1,1,1}. \tag{2.36}$$

The expressions for the differential operators $X_i, i = 1 \ldots 28$ and $T_i, i = 1 \ldots 7$ are too large therefore, these are given, in the Appendix A, after relabeling the variables

$$x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, f_{2,4,4}, f_{3,3,3}, f_{3,3,4},$$
$$f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}, K, L, M, N, \tag{2.37}$$

by the variables $z_i, i = 1 \ldots 43$, respectively.

Functionally independent solutions of system (2.34) provide all independent differential invariants of $y''' = f(x, y, y', y'')$ up to the third order under the fiber preserving transformation, as well as an implicit solution of the variables K, L, M and N which provide the differential operators via (2.30) as explained in the next section.

The existence of the solutions for system (2.34) is proved by showing that the differential operators $X_i, i = 1 \ldots 28$ and $T_i, i = 1 \ldots 7$ form a Lie algebra $\mathcal{L}_{35}$ [2, p.422, Theorem 14.1]. The nonzero commutators for the Lie algebra $\mathcal{L}_{35}$ are given in the Appendix B, after relabeling the differential operators $X_i, i = 1 \ldots 28$ and $T_i, i = 1 \ldots 7$ by the operators $e_i, i = 1 \ldots 35$, respectively.

Using Appendix B, it can be seen that the Lie algebra $\mathcal{L}_{35}$ is solvable and has the chain of Lie subalgebras $0 = \mathcal{G}_0 \lhd \mathcal{G}_1 \lhd \mathcal{G}_2 \lhd \mathcal{G}_3 \lhd \mathcal{G}_4 \lhd \mathcal{G}_5 \lhd \mathcal{G}_6 \lhd \mathcal{G}_7 \lhd \mathcal{G}_8 = \mathcal{L}_{35}$ with each an ideal in the next where

$$\begin{aligned}
\mathcal{G}_1 &= \{e_{22}, e_{23}, e_{24}, e_{25}, e_{26}, e_{27}, e_{28}\}, \\
\mathcal{G}_2 &= \mathcal{G}_1 \cup \{e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}\}, \\
\mathcal{G}_3 &= \mathcal{G}_2 \cup \{e_{11}, e_{12}, e_{13}, e_{14}, e_{15}\}, \\
\mathcal{G}_4 &= \mathcal{G}_3 \cup \{e_7, e_8, e_9, e_{10}\}, \\
\mathcal{G}_5 &= \mathcal{G}_4 \cup \{e_4, e_5, e_6\}, \\
\mathcal{G}_6 &= \mathcal{G}_5 \cup \{e_{33}, e_{34}, e_{35}\}, \\
\mathcal{G}_7 &= \mathcal{G}_6 \cup \{e_{31}, e_{32}\}, \\
\mathcal{G}_8 &= \mathcal{G}_7 \cup \{e_1, e_2, e_3, e_{29}, e_{30}\}.
\end{aligned} \tag{2.38}$$

In the next section, we solve the system (2.34) by using the chain (2.38). In more detail, as $\mathcal{G}_1$ is abelian, one can find its joint invariants by finding the invariants of its generators in any order. Since $\mathcal{G}_1$ is an ideal in $\mathcal{G}_2$, the algebra $\mathcal{G}_2$ operates on the joint invariants of $\mathcal{G}_1$. Moreover, as $\mathcal{G}_1$ is abelian, then the induced action of $\mathcal{G}_2$ on the joint invariants of $\mathcal{G}_1$ is also abelian. Continuing in this manner, we obtain the joint invariants of the full algebra.

The reader is referred to [14] and [15, Section 10] for more detailed description and examples illustrating the method.

3 Third-order differential invariants and invariant equations for $y''' = f(x, y, y', y'')$

3.1 The infinitesimal point equivalence transformations

In order to find continuous group of equivalence transformations of the class (1.4) we consider the arbitrary functions f that appear in our equation as a dependent variable and the variables $x, y, y' = p, y'' = q$ as independent variables and apply the Lie infinitesimal invariance criterion [5], that is we look for the infinitesimal ξ, η and μ of the equivalence operator Y:

$$Y = \xi(x, y)\partial_x + \eta(x, y)\partial_y + \mu(x, y, p, q, f)\partial_f, \tag{3.39}$$

such that its prolongation leaves the equation (1.4) invariant.

The prolongation of operator Y can be given using (2.9) as:

$$Y = \xi(x, y)D_x + W\partial_y + D_x(W)\partial_p + D_x^2(W)\partial_q + D_x^3(W)\partial_{y'''} + \mu(x, y, p, q, f)\partial_f, \tag{3.40}$$

where

$$D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + y'''\frac{\partial}{\partial q} + y^{(4)}\frac{\partial}{\partial y'''} + ...$$

is the operator of total derivative and $W = \eta(x, y) - \xi(x, y)p$ is the characteristic of infinitesimal operator $X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$.

So, Lie infinitesimal invariance criterion gives $\mu = \dot{D}_x^3(W) + \xi(x, y)\dot{D}_x f$ for arbitrary functions $\xi(x, y)$ and $\eta(x, y)$ where $\dot{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + f\frac{\partial}{\partial q}$.

Thus, equation (1.4) admit an infinite continuous group of equivalence transformations generated by the Lie algebra $\mathcal{L}_{\mathcal{E}}$ spanned by the following infinitesimal operators:

$$U = \xi(x, y)\frac{\partial}{\partial x} - pD_x(\xi)\partial_p - (2qD_x(\xi) + pD_x^2(\xi))\partial_q - (3fD_x(\xi) + 3qD_x^2(\xi) + p\dot{D}_x^3(\xi))\partial_f, \tag{3.41}$$

$$V = \eta(x, y)\partial_y + D_x(\eta)\partial_p + D_x^2(\eta)\partial_q + \dot{D}_x^3(\eta)\partial_f, \tag{3.42}$$

The infinitesimal point equivalence transformations (3.41)-(3.42) can be written in the finite form as in (2.17)-(2.18), respectively, where ϕ and ψ are arbitrary functions of the indicated variables.

3.2 Third-order differential invariants and invariant equations under the transformation $\bar{x} = x,\ \bar{y} = \psi(x, y)$

In this section, we derive all the third order differential invariants of the general class $y''' = f(x, y, y', y'')$ under a subgroup of point transformations (2.14), namely the transformations $\bar{x} = x,\ \bar{y} = \psi(x, y)$. Moreover, the invariant differentiation operators [5] are constructed in order to get some higher-order differential invariants from the lower-order ones. Precisely, we obtain the following theorem.

Theorem 3.1. *Let $y''' = f(x, y, y', y'')$ be the class of third order ODE with $f_{4,4,4} \neq 0$. All the third order differential invariants, under the transformations $\bar{x} = x,\ \bar{y} = \psi(x, y)$, are function of the following seventeen differential invariants*

$$\begin{array}{llllll} \alpha_1 = x, & \alpha_2 = \frac{\lambda_5}{\lambda_4}, & \alpha_3 = \frac{\lambda_6}{{\lambda_4}^2}, & \alpha_4 = \frac{\lambda_7}{{\lambda_4}^2}, & \alpha_5 = \frac{\lambda_8}{{\lambda_4}^2}, & \alpha_6 = \frac{\lambda_9}{{\lambda_4}^2}, \\ \alpha_7 = \frac{\lambda_{10}}{{\lambda_4}^2}, & \alpha_8 = \frac{\lambda_{11}}{{\lambda_4}^2}, & \alpha_9 = \lambda_{12}, & \alpha_{10} = \lambda_{13}, & \alpha_{11} = \frac{\lambda_{14}}{\lambda_4}, & \alpha_{12} = \frac{\lambda_{15}}{\lambda_4}, \\ \alpha_{13} = \frac{\lambda_{16}}{\lambda_4}, & \alpha_{14} = \frac{\lambda_{17}}{\lambda_4}, & \alpha_{15} = \frac{\lambda_{18}}{\lambda_4}, & \alpha_{16} = \lambda_{19}, & \alpha_{17} = \lambda_{20}, & \end{array} \tag{3.43}$$

where $\{\lambda_i\}_{i=4}^{20}$ are the following relative invariants

$$\begin{aligned}
\lambda_4 = {} & f_{4,4}, \\
\lambda_5 = {} & \tfrac{1}{3}\left(2\, f_{3,4} f_4 - 2 f_3 f_{4,4} - 6 f_{2,4} + 3 f_{3,3}\right), \\
\lambda_6 = {} & f_{4,4,4}, \\
\lambda_7 = {} & \tfrac{1}{3}\left(2 f_4 f_{4,4,4} + 3 f_{3,4,4}\right), \\
\lambda_8 = {} & \tfrac{1}{9}\left(4 f_{4,4,4} f_4^2 + 12 f_4 f_{3,4,4} + 4 f_4 f_{4,4}^2 + 9 f_{3,3,4} + 6 f_{3,4} f_{4,4}\right), \\
\lambda_9 = {} & \tfrac{1}{9}\left(2 f_{4,4,4} f_4^2 + 2 f_4 f_{4,4}^2 + 3 f_4 f_{3,4,4} + 3 f_{4,4,4} f_3 + 9 f_{2,4,4} + 3 f_{3,4} f_{4,4}\right), \\
\lambda_{10} = {} & \tfrac{1}{18}(-4 f_4^2 f_{3,4,4} + 4 f_4 f_{4,4,4} f_3 + 12 f_4 f_{2,4,4} - 12 f_{3,3,4} f_4 - 4 f_4 f_{3,4} f_{4,4} - 6 f_{3,4}^2 - 9 f_{3,3,3} \\
& + 4 f_{4,4}^2 f_3 + 6 f_3 f_{3,4,4} + 12 f_{2,4} f_{4,4} + 18 f_{2,3,4}), \\
\lambda_{11} = {} & \tfrac{1}{27}(-4 f_4 f_3 f_{4,4}^2 - 6 f_{4,4} f_{3,4} f_3 - 18 f_{4,4} f_{2,3} - 12 f_{4,4} f_{2,4} f_4 + 4 f_{4,4} f_{3,4} f_4^2 - 6 f_{4,4,4} f_3^2 - 36 f_3 f_{2,4,4} \\
& - 4 f_3 f_{4,4,4} f_4^2 + 9 f_3 f_{3,3,4} + 12 f_{3,4}^2 f_4 + 9 f_{3,4} f_{3,3} + 4 f_4^3 f_{3,4,4} - 12 f_4^2 f_{2,4,4} + 12 f_{3,3,4} f_4^2 \\
& + 9 f_4 f_{3,3,3} - 54 f_{2,2,4} + 27 f_{2,3,3}), \\
\lambda_{12} = {} & \tfrac{1}{3}(-f_4^2 - 3 f_3 + 3 \dot{D}_x f_4),
\end{aligned}$$

$$
\begin{aligned}
\lambda_{13} = & \ \tfrac{1}{9}(-2f_4^3 - 9f_4 f_3 + 6f_4 \dot{D}_x f_4 - 27f_2 + 9\dot{D}_x f_3), \\
\lambda_{14} = & \ \tfrac{1}{3}(3\dot{D}_x f_{4,4} + f_{4,4} f_4), \\
\lambda_{15} = & \ \tfrac{1}{9}(2f_{4,4} f_4^2 + 3f_3 f_{4,4} + 6f_4 \dot{D}_x f_{4,4} - 9f_{2,4} + 9\dot{D}_x f_{3,4}), \\
\lambda_{16} = & \ \tfrac{1}{9}(4f_4^2 \dot{D}_x f_{4,4} - 2f_4^2 f_{3,4} - 3f_4 f_{3,3} + 12f_4 \dot{D}_x f_{3,4} + 4f_4 f_{4,4} \dot{D}_x f_4 - 12f_{2,4} f_4 + 6f_{3,4} \dot{D}_x f_4 - 18f_{2,3} + 9\dot{D}_x f_{3,3}), \\
\lambda_{17} = & \ \tfrac{1}{27}(2f_{4,4} f_4^3 - 6f_4^2 f_{3,4} + 6f_4^2 \dot{D}_x f_{4,4} - 9f_4 f_{3,3} + 12f_4 f_3 f_{4,4} + 9f_4 \dot{D}_x f_{3,4} - 9f_{2,4} f_4 + 9f_3 \dot{D}_x f_{4,4} - 27f_{2,3} \\
& +27f_2 f_{4,4} + 27\dot{D}_x f_{2,4}), \\
\lambda_{18} = & \ \tfrac{1}{27}(-9f_3^2 f_{4,4} + 9f_3 \dot{D}_x f_{3,4} + 6f_3 f_4 \dot{D}_x f_{4,4} + 6f_3 f_{4,4} \dot{D}_x f_4 + 6f_3 f_{3,4} f_4 - 54f_3 f_{2,4} + 9f_3 f_{3,3} + 4f_{4,4} f_4^2 \dot{D}_x f_4 \\
& +6f_{3,4} f_4 \dot{D}_x f_4 + 18f_{2,4} \dot{D}_x f_4 + 27f_2 f_{3,4} - 2f_{3,4} f_4^3 - 30f_4^2 f_{2,4} + 18f_4 f_{4,4} f_2 + 4f_4^3 \dot{D}_x f_{4,4} + 12f_4^2 \dot{D}_x f_{3,4} \\
& -3f_4^2 f_{3,3} + 9f_4 \dot{D}_x f_{3,3} + 18f_4 \dot{D}_x f_{2,4} - 45f_4 f_{2,3} - 81f_{2,2} + 27\dot{D}_x f_{2,3}), \\
\lambda_{19} = & \ \tfrac{1}{9}(-2f_4^3 - 9f_4 f_3 - 27f_2 + 9\dot{D}_x^2 f_4), \\
\lambda_{20} = & \ \tfrac{1}{27}(-2f_4^4 - 12f_4^2 f_3 - 6f_4^2 \dot{D}_x f_4 + 18f_4 \dot{D}_x^2 f_4 - 27f_4 \dot{D}_x f_3 - 18f_3^2 + 9f_3 \dot{D}_x f_4 - 81\dot{D}_x f_2 + 27\dot{D}_x^2 f_3),
\end{aligned}
\tag{3.44}
$$

Moreover, the invariant differential operators are

$$
\begin{aligned}
\mathcal{D}_1 &= \tfrac{1}{f_{4,4}} \tilde{D}_q, \\
\mathcal{D}_2 &= \tfrac{1}{f_{4,4}} (3\tilde{D}_p + 2f_4 \tilde{D}_q), \\
\mathcal{D}_3 &= \tfrac{1}{f_{4,4}} (9\tilde{D}_y + 3f_4 \tilde{D}_p + (2{f_4}^2 + 3f_3)\tilde{D}_q), \\
\mathcal{D}_4 &= \tilde{D}_x + p\tilde{D}_y + q\tilde{D}_p + f\tilde{D}_q.
\end{aligned}
\tag{3.45}
$$

Proof. Functionally independent solutions of the subsystem

$$X_i J = 0, i = 1 \dots 28, \tag{3.46}$$

of (2.34) provide all independent differential invariants of $y''' = f(x, y, y', y'')$ up to the third order under the transformations $\bar{x} = x,\ \bar{y} = \psi(x, y)$, as well as an implicit solution of the variables K, L, M and N which provide the differential operators via (2.30).

The solution of system (3.46) is found in two steps using Maple through the chain (2.38). First we consider the following subsystem of equations (3.46)

$$X_i J = 0, i = 4 \dots 28. \tag{3.47}$$

In 43-dimensional space of variables $z_i, i = 1 \dots 43$, the rank of the system (3.47) is 19, so it has 24 functionally independent solutions which are given as:

$$
\begin{aligned}
\lambda_1 =\ & z_1,\\
\lambda_2 =\ & z_2,\\
\lambda_3 =\ & z_3,\\
\lambda_4 =\ & z_{14},\\
\lambda_5 =\ & \tfrac{2}{3}\, z_{13}z_8 - \tfrac{2}{3}\, z_7z_{14} - 2\, z_{11} + z_{12},\\
\lambda_6 =\ & z_{24},\\
\lambda_7 =\ & \tfrac{2}{3}\, z_8z_{24} + z_{23},\\
\lambda_8 =\ & \tfrac{4}{9}\, z_{24}{z_8}^2 + \tfrac{4}{9}\, z_8{z_{14}}^2 + \tfrac{4}{3}\, z_{23}z_8 + z_{22} + \tfrac{2}{3}\, z_{13}z_{14},\\
\lambda_9 =\ & \tfrac{2}{9}\, z_{24}{z_8}^2 + \tfrac{1}{3}\, z_{23}z_8 + \tfrac{2}{9}\, z_8{z_{14}}^2 + \tfrac{1}{3}\, z_{24}z_7 + z_{20} + \tfrac{1}{3}\, z_{13}z_{14}\\
\lambda_{10} =\ & -\tfrac{2}{9}\, {z_8}^2z_{23} - \tfrac{2}{3}\, z_{22}z_8 + \tfrac{2}{9}\, z_8z_{24}z_7 + \tfrac{2}{3}\, z_8z_{20} - \tfrac{2}{9}\, z_8z_{13}z_{14} - \tfrac{1}{3}\, {z_{13}}^2 - \tfrac{1}{2}\, z_{21} + \tfrac{1}{3}\, z_7z_{23} + \tfrac{2}{9}\, {z_{14}}^2z_7\\
& + \tfrac{2}{3}\, z_{14}z_{11} + z_{19},\\
\lambda_{11} =\ & \tfrac{4}{27}\, {z_8}^3z_{23} - \tfrac{4}{27}\, {z_8}^2z_{24}z_7 + \tfrac{4}{27}\, {z_8}^2z_{13}z_{14} + \tfrac{4}{9}\, z_{22}{z_8}^2 - \tfrac{4}{9}\, {z_8}^2z_{20} - \tfrac{4}{27}\, {z_{14}}^2z_8z_7 - \tfrac{4}{9}\, z_8z_{14}z_{11}\\
& + \tfrac{4}{9}\, {z_{13}}^2z_8 + \tfrac{1}{3}\, z_{21}z_8 - \tfrac{2}{9}\, z_{13}z_7z_{14} + \tfrac{1}{3}\, z_{12}z_{13} - \tfrac{2}{9}\, z_{24}{z_7}^2 + \tfrac{1}{3}\, z_7z_{22} - \tfrac{4}{3}\, z_7z_{20} - 2\, z_{17} - \tfrac{2}{3}\, z_{14}z_{10} + z_{18},\\
\lambda_{12} =\ & -\tfrac{1}{3}\, {z_8}^2 - z_7 + z_{28},\\
\lambda_{13} =\ & -\tfrac{2}{9}\, {z_8}^3 - z_7z_8 + \tfrac{2}{3}\, z_8z_{28} - 3\, z_6 + z_{27},\\
\lambda_{14} =\ & \tfrac{1}{3}\, z_{14}z_8 + z_{34},\\
\lambda_{15} =\ & \tfrac{2}{9}\, z_{14}{z_8}^2 + \tfrac{2}{3}\, z_8z_{34} + \tfrac{1}{3}\, z_7z_{14} - z_{11} + z_{33},\\
\lambda_{16} =\ & \tfrac{4}{9}\, {z_8}^2z_{34} - \tfrac{2}{9}\, {z_8}^2z_{13} + \tfrac{4}{9}\, z_8z_{14}z_{28} + \tfrac{4}{3}\, z_8z_{33} - \tfrac{4}{3}\, z_{11}z_8 - \tfrac{1}{3}\, z_8z_{12} + \tfrac{2}{3}\, z_{28}z_{13} - 2\, z_{10} + z_{32},\\
\lambda_{17} =\ & \tfrac{2}{27}\, z_{14}{z_8}^3 + \tfrac{4}{9}\, z_{14}z_7z_8 + z_6z_{14} - \tfrac{2}{9}\, {z_8}^2z_{13} + \tfrac{2}{9}\, {z_8}^2z_{34} + \tfrac{1}{3}\, z_8z_{33} - \tfrac{1}{3}\, z_{11}z_8 - \tfrac{1}{3}\, z_8z_{12} + \tfrac{1}{3}\, z_7z_{34} - z_{10} + z_{31},\\
\lambda_{18} =\ & \tfrac{4}{27}\, {z_8}^3z_{34} - \tfrac{2}{27}\, {z_8}^3z_{13} - \tfrac{10}{9}\, {z_8}^2z_{11} + \tfrac{4}{27}\, {z_8}^2z_{14}z_{28} - \tfrac{1}{9}\, {z_8}^2z_{12} + \tfrac{4}{9}\, {z_8}^2z_{33}\\
& + \tfrac{2}{9}\, z_8z_{13}z_7 + \tfrac{2}{3}\, z_8z_6z_{14} + \tfrac{1}{3}\, z_8z_{32} + \tfrac{2}{3}\, z_8z_{31} - \tfrac{5}{3}\, z_8z_{10} + \tfrac{2}{9}\, z_8z_7z_{34} + \tfrac{2}{9}\, z_8z_{28}z_{13}\\
& + \tfrac{2}{9}\, z_{14}z_{28}z_7 - \tfrac{1}{3}\, z_{14}{z_7}^2 + \tfrac{2}{3}\, z_{28}z_{11} + z_{30} - 3\, z_9 - 2\, z_7z_{11} + z_6z_{13} + \tfrac{1}{3}\, z_7z_{33} + \tfrac{1}{3}\, z_7z_{12},\\
\lambda_{19} =\ & -\tfrac{2}{9}\, {z_8}^3 - z_7z_8 - 3\, z_6 + z_{38},\\
\lambda_{20} =\ & -\tfrac{2}{27}\, {z_8}^4 - \tfrac{4}{9}\, {z_8}^2z_7 - \tfrac{2}{9}\, {z_8}^2z_{28} - z_8z_{27} + \tfrac{2}{3}\, z_8z_{38} - \tfrac{2}{3}\, {z_7}^2 + \tfrac{1}{3}\, z_{28}z_7 - 3\, z_{26} + z_{37},
\end{aligned}
\tag{3.48}
$$

and

$$
\begin{aligned}
\lambda_{21} =\ & z_{40},\\
\lambda_{22} =\ & z_{41},\\
\lambda_{23} =\ & \tfrac{1}{3}\, z_8z_{40}z_3 - \tfrac{1}{3}\, z_{41}z_8 - z_{40}z_4 + z_{42},\\
\lambda_{24} =\ & \tfrac{2}{3}\, z_{40}z_8z_4 + \tfrac{1}{3}\, z_{40}z_7z_3 - z_{40}z_5 - \tfrac{1}{3}\, z_7z_{41} - \tfrac{2}{3}\, z_8z_{42} + z_{43},
\end{aligned}
\tag{3.49}
$$

In variables $\lambda_i, i = i = 1 \dots 24$, the remaining non-zero operators in system (3.46) take the form

$$\begin{aligned} X_1 = \quad & [0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0], \\ X_2 = \quad & [0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\lambda_{21},0,0], \\ X_3 = \quad & [0,0,\lambda_3,-\lambda_4,-\lambda_5,-2\,\lambda_6,-2\,\lambda_7,-2\,\lambda_8,-2\,\lambda_9,-2\,\lambda_{10}, \\ & -2\,\lambda_{11},0,0,-\lambda_{14},-\lambda_{15},-\lambda_{16},-\lambda_{17},-\lambda_{18},0,0,0,\lambda_{22},\lambda_{23},\lambda_{24}], \end{aligned} \tag{3.50}$$

Finally, we consider the following subsystem of equations (2.34)

$$X_i J = 0, i = 1 \ldots 3. \tag{3.51}$$

In 24-dimensional space of variables $\lambda_i, i = 1 \ldots 24$, the rank of the system (3.51) is 3, so it has 21 functionally independent solutions which are given as:

$$\begin{array}{llllll} \alpha_1 = x, & \alpha_2 = \frac{\lambda_5}{\lambda_4}, & \alpha_3 = \frac{\lambda_6}{\lambda_4{}^2}, & \alpha_4 = \frac{\lambda_7}{\lambda_4{}^2}, & \alpha_5 = \frac{\lambda_8}{\lambda_4{}^2}, & \alpha_6 = \frac{\lambda_9}{\lambda_4{}^2}, \\ \alpha_7 = \frac{\lambda_{10}}{\lambda_4{}^2}, & \alpha_8 = \frac{\lambda_{11}}{\lambda_4{}^2}, & \alpha_9 = \lambda_{12}, & \alpha_{10} = \lambda_{13}, & \alpha_{11} = \frac{\lambda_{14}}{\lambda_4}, & \alpha_{12} = \frac{\lambda_{15}}{\lambda_4}, \\ \alpha_{13} = \frac{\lambda_{16}}{\lambda_4}, & \alpha_{14} = \frac{\lambda_{17}}{\lambda_4}, & \alpha_{15} = \frac{\lambda_{18}}{\lambda_4}, & \alpha_{16} = \lambda_{19}, & \alpha_{17} = \lambda_{20}, & \end{array} \tag{3.52}$$

and

$$\alpha_{18} = \lambda_{21}, \quad \alpha_{19} = -\lambda_4\left(\lambda_3\lambda_{21} - \lambda_{22}\right), \quad \alpha_{20} = \lambda_4\lambda_{23}, \quad \alpha_{21} = \lambda_4\lambda_{24}. \tag{3.53}$$

Here $\alpha_{18}, \alpha_{19}, \alpha_{20}$ and α_{21} are the only invariants depending on the variables K, L, M and N. Then the general solution of (2.31) , for $\xi = 0, \eta = \eta(x, y)$, can be given implicitly by back substitution as

$$\begin{aligned} & K = F_1, \\ & f_{4,4}\left(pK - L\right) = F_2, \\ & f_{4,4}\left(f_4(pK - L) - 3(qK - M)\right) = F_3, \\ & f_{4,4}\left(f_3(pK - L) + 2f_4(qK - M) - 3(fK - N)\right) = F_4. \end{aligned} \tag{3.54}$$

where F_1, F_2, F_3 and F_4 are the arbitrary functions of $\alpha_i, i = 1...17$.

Finally, solving system (3.54) gives the variables K, L, M and N in terms of four arbitrary functions F_1, F_2, F_3 and F_4 which provide four independent invariant differentiation operators $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and $\mathcal{D}_4$ via (2.30). □

3.3 Third-order differential invariants and invariant equations under the fiber preserving transformation $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$

This section is devoted to derive all the third order differential invariants of the general class $y''' = f(x, y, y', y'')$ under a subgroup of point transformations (2.14), namely the fiber preserving transformations $\bar{x} = \phi(x),\ \bar{y} = \psi(x, y)$. The invariant differentiation operators are also constructed. Precisely, we obtain the following theorem.

Theorem 3.2. *Let $y''' = f(x, y, y', y'')$ be the class of third order ODE with $f_{4,4,4} \neq 0$. All the third order differential invariants, under pseudo-group of fiber preserving transformations $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$, are function of the following eleven differential invariants*

$$\begin{array}{llllll} \beta_1 = \frac{\gamma_6\gamma_5}{\gamma_4{}^4}, & \beta_2 = \frac{\gamma_7\gamma_5}{\gamma_4{}^4}, & \beta_3 = \frac{\gamma_8\gamma_5{}^2}{\gamma_4{}^6}, & \beta_4 = \frac{\gamma_9\gamma_5{}^3}{\gamma_4{}^8}, & \beta_5 = \frac{\gamma_5{}^2\gamma_{10}}{\gamma_4{}^4}, & \beta_6 = \frac{\gamma_5\gamma_{11}}{\gamma_4{}^3}, \\ \beta_7 = \frac{\gamma_{12}\gamma_5{}^2}{\gamma_4{}^5}, & \beta_8 = \frac{\gamma_{13}\gamma_5{}^3}{\gamma_4{}^7}, & \beta_9 = \frac{\gamma_{14}\gamma_5{}^3}{\gamma_4{}^7}, & \beta_{10} = \frac{\gamma_{15}\gamma_5{}^4}{\gamma_4{}^9}, & \beta_{11} = \frac{\gamma_{16}\gamma_5{}^3}{\gamma_4{}^6}, & \end{array} \quad (3.55)$$

where $\{\gamma_i\}_{i=4}^{16}$ are relative invariants given by (3.59).

Moreover, the invariant differential operators are

$$\begin{array}{l} \mathcal{D}_1 = \frac{f_{4,4}}{f_{4,4,4}}\tilde{D}_q, \\ \mathcal{D}_2 = \frac{1}{f_{4,4}f_{4,4,4}}\left(f_{4,4,4}\tilde{D}_p - f_{3,4,4}\tilde{D}_q\right), \\ \mathcal{D}_3 = \frac{1}{f_{4,4}^4 f_{4,4,4}}\left(6f_{4,4}f_{4,4,4}^2\tilde{D}_y - 2f_{4,4}f_{4,4,4}(f_4 f_{4,4,4} + 3f_{3,4,4})\tilde{D}_p + (3f_{4,4}f_{3,4,4}^2 + f_{4,4,4}^2(3f_{3,3} + 2f_{3,4}f_4 - 6f_{2,4}))\tilde{D}_q\right), \\ \mathcal{D}_4 = \frac{f_{4,4,4}}{f_{4,4}^2}\left(\tilde{D}_x + p\tilde{D}_y + q\tilde{D}_p + f\tilde{D}_q\right). \end{array} \quad (3.56)$$

Proof. Functionally independent solutions of the system (2.34) provide all independent differential invariants of $y''' = f(x, y, y', y'')$ up to the third order under the fiber preserving transformations $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$, as well as an implicit solution of the variables K, L, M and N which provide the differential operators via (2.30).

In variables $\lambda_i, i = i = 1 \dots 24$, given in (3.44), the remaining non-zero operators in system

(2.34) take the form

$$
\begin{aligned}
X_1 = \;& [0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],\\
X_2 = \;& [0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\lambda_{21},0,0],\\
X_3 = \;& [0,0,\lambda_3,-\lambda_4,-\lambda_5,-2\,\lambda_6,-2\,\lambda_7,-2\,\lambda_8,-2\,\lambda_9,-2\,\lambda_{10},\\
& -2\,\lambda_{11},0,0,-\lambda_{14},-\lambda_{15},-\lambda_{16},-\lambda_{17},-\lambda_{18},0,0,0,\lambda_{22},\lambda_{23},\lambda_{24}],\\
T_1 = \;& [1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],\\
T_2 = \;& [0,0,-\lambda_3,\lambda_4,-\lambda_5,3\,\lambda_6,2\,\lambda_7,\lambda_8,\lambda_9,0,-\lambda_{11},-2\,\lambda_{12},-3\,\lambda_{13},0,\\
& -\lambda_{15},-2\,\lambda_{16},-2\,\lambda_{17},-3\,\lambda_{18},-3\,\lambda_{19},-4\,\lambda_{20},\lambda_{21},0,-\lambda_{23},-2\,\lambda_{24}],\\
T_3 = \;& [0,0,0,0,0,0,-\lambda_6,-\tfrac{2}{3}\,{\lambda_4}^2-2\,\lambda_7,-\lambda_7-\tfrac{1}{3}\,{\lambda_4}^2,\tfrac{1}{2}\,\lambda_8-\lambda_9,2\,\lambda_{10}+\tfrac{1}{3}\,\lambda_4\lambda_5,0,-\lambda_{12},0,\\
& -\lambda_{14},-\tfrac{2}{3}\,\lambda_4\lambda_{12}-2\,\lambda_{15},\lambda_5-\lambda_{15},-\lambda_{16}-\lambda_{17},-3\,\lambda_{12},-2\,\lambda_{13}-\lambda_{19},0,0,-\lambda_{21}\lambda_3+\lambda_{22},\lambda_{23}],\\
T_4 = \;& [0,0,0,0,\tfrac{2}{3}\,\lambda_4,0,0,0,-\tfrac{1}{3}\,\lambda_6,-\tfrac{1}{3}\,\lambda_7-\tfrac{2}{9}\,{\lambda_4}^2,-\tfrac{1}{3}\,\lambda_8+\tfrac{4}{3}\,\lambda_9,-2,0,0,\tfrac{2}{3}\,\lambda_4,0,\\
& -\tfrac{1}{3}\,\lambda_{14},-\tfrac{2}{9}\,\lambda_4\lambda_{12}-\tfrac{1}{3}\,\lambda_5-\tfrac{1}{3}\,\lambda_{15},0,\tfrac{5}{3}\,\lambda_{12},0,0,0,-\tfrac{1}{3}\,\lambda_{21}\lambda_3+\tfrac{1}{3}\,\lambda_{22}],\\
T_5 = \;& [0,0,0,0,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,0,-3,0,0,0,0,0],\\
T_6 = \;& [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1,0,0,0,0].
\end{aligned}
\tag{3.57}
$$

The solution of system (3.57) is found in two steps using Maple through the chain (2.38). First we consider the following subsystem of equations (3.57)

$$T_iJ = 0, i = 3\ldots 6. \tag{3.58}$$

In 24-dimensional space of variables $\lambda_i, i = 1\ldots 24$, the rank of the system (3.58) is 4, so it has 20 functionally independent solutions which are given as:

$$
\begin{aligned}
\gamma_1 &= \lambda_1, \\
\gamma_2 &= \lambda_2, \\
\gamma_3 &= \lambda_3, \\
\gamma_4 &= \lambda_4, \\
\gamma_5 &= \lambda_6, \\
\gamma_6 &= -\frac{2\,{\lambda_4}^2\lambda_7+3\,{\lambda_7}^2-3\,\lambda_6\lambda_8}{3\lambda_6}, \\
\gamma_7 &= -\frac{2\,{\lambda_4}^3\lambda_7+3\,{\lambda_7}^2\lambda_4-3\,{\lambda_6}^2\lambda_5-6\,\lambda_6\lambda_4\lambda_9}{6\lambda_6\lambda_4}, \\
\gamma_8 &= \frac{2\,\lambda_6\lambda_4\lambda_5+6\,\lambda_6\lambda_{10}-6\,\lambda_7\lambda_9+3\,\lambda_7\lambda_8}{6\lambda_6}, \\
\gamma_9 &= \frac{4\,\lambda_6{\lambda_4}^3\lambda_7\lambda_5+6\,{\lambda_4}^2{\lambda_6}^2\lambda_{11}+12\,{\lambda_4}^2\lambda_6\lambda_7\lambda_{10}-6\,{\lambda_4}^2{\lambda_7}^2\lambda_9+3\,{\lambda_4}^2{\lambda_7}^2\lambda_8+3\,{\lambda_6}^2\lambda_4\lambda_5\lambda_8-12\,{\lambda_6}^2\lambda_4\lambda_5\lambda_9+3\,\lambda_6\lambda_4\lambda_5{\lambda_7}^2-3\,{\lambda_6}^3{\lambda_5}^2}{6{\lambda_6}^2{\lambda_4}^2}, \\
\gamma_{10} &= \frac{3\,\lambda_5+\lambda_4\lambda_{12}}{\lambda_4}, \\
\gamma_{11} &= \lambda_{14}, \\
\gamma_{12} &= -\frac{\lambda_{14}\lambda_7+\lambda_6\lambda_5-\lambda_6\lambda_{15}}{\lambda_6}, \\
\gamma_{13} &= -\frac{-3\,\lambda_{16}{\lambda_6}^2+6\,\lambda_6\lambda_7\lambda_{15}+2\,\lambda_6\lambda_7\lambda_4\lambda_{12}-3\,\lambda_{14}{\lambda_7}^2}{3{\lambda_6}^2}, \\
\gamma_{14} &= \frac{2\,\lambda_4{\lambda_6}^2\lambda_{17}+2\,\lambda_4\lambda_6\lambda_7\lambda_5-2\,\lambda_4\lambda_6\lambda_7\lambda_{15}+\lambda_4\lambda_{14}{\lambda_7}^2+{\lambda_6}^2\lambda_5\lambda_{14}}{2{\lambda_6}^2\lambda_4} \\
\gamma_{15} &= \frac{2\,{\lambda_6}^3\lambda_4\lambda_5\lambda_{12}+6\,{\lambda_6}^3\lambda_4\lambda_{18}+3\,{\lambda_6}^3{\lambda_5}^2+3\,{\lambda_6}^3\lambda_5\lambda_{15}}{6\lambda_4{\lambda_6}^3}, \\
&\quad +\frac{-6\,\lambda_7{\lambda_6}^2\lambda_4\lambda_{17}-6\,\lambda_7{\lambda_6}^2\lambda_4\lambda_{16}-3\,\lambda_7{\lambda_6}^2\lambda_5\lambda_{14}-3\,\lambda_6\lambda_4\lambda_5{\lambda_7}^2+9\,{\lambda_7}^2\lambda_4\lambda_6\lambda_{15}+2\,{\lambda_7}^2{\lambda_4}^2\lambda_6\lambda_{12}-3\,\lambda_{14}{\lambda_7}^3\lambda_4}{6\lambda_4{\lambda_6}^3}, \\
\gamma_{16} &= -3\,\lambda_{13}+\lambda_{19}, \\
\gamma_{17} &= \lambda_{21}, \\
\gamma_{18} &= \lambda_{22}, \\
\gamma_{19} &= -\frac{\lambda_7\lambda_{21}\lambda_3-\lambda_7\lambda_{22}-\lambda_{23}\lambda_6}{\lambda_6}, \\
\gamma_{20} &= -\frac{-2\,\lambda_4{\lambda_6}^2\lambda_{24}-2\,\lambda_4\lambda_7\lambda_{23}\lambda_6+\lambda_4{\lambda_7}^2\lambda_{21}\lambda_3-\lambda_4{\lambda_7}^2\lambda_{22}-{\lambda_6}^2\lambda_5\lambda_{21}\lambda_3+{\lambda_6}^2\lambda_5\lambda_{22}}{2{\lambda_6}^2\lambda_4},
\end{aligned}
\tag{3.59}
$$

In variables $\gamma_i, i = 1 \ldots 20$, the remaining non-zero operators in system (3.57) take the form

$$
\begin{aligned}
X_1 &= [0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0], \\
X_2 &= [0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\gamma_{17},0,0], \\
X_3 &= [0,0,\gamma_3,-\gamma_4,-2\,\gamma_5,-2\,\gamma_6,-2\,\gamma_7,-2\,\gamma_8,-2\,\gamma_9,0,-\gamma_{11},-\gamma_{12},-\gamma_{13},-\gamma_{14},-\gamma_{15},0,0,\gamma_{18},\gamma_{19},\gamma_{20}], \\
T_1 &= [1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0], \\
T_2 &= ,[0,0,-\gamma_3,\gamma_4,3\,\gamma_5,\gamma_6,\gamma_7,0,-\gamma_9,-2\,\gamma_{10},0,-\gamma_{12},-2\,\gamma_{13},-2\,\gamma_{14},-3\,\gamma_{15},-3\,\gamma_{16},\gamma_{17},0,-\gamma_{19},-2\,\gamma_{20}]
\end{aligned}
\tag{3.60}
$$

Finally, we consider the following subsystem of equations (2.34)

$$X_iJ = 0, i = 1\dots 3, T_kJ = 0, k = 1\dots 2. \tag{3.61}$$

In 20-dimensional space of variables $\gamma_i, i = 1\dots 20$, the rank of the system (3.61) is 5, so it has 15 functionally independent solutions which are given as:

$$\begin{array}{llllll} \beta_1 = \frac{\gamma_6\gamma_5}{{\gamma_4}^4}, & \beta_2 = \frac{\gamma_7\gamma_5}{{\gamma_4}^4}, & \beta_3 = \frac{\gamma_8{\gamma_5}^2}{{\gamma_4}^6}, & \beta_4 = \frac{\gamma_9{\gamma_5}^3}{{\gamma_4}^8}, & \beta_5 = \frac{{\gamma_5}^2\gamma_{10}}{{\gamma_4}^4}, & \beta_6 = \frac{\gamma_5\gamma_{11}}{{\gamma_4}^3}, \\ \beta_7 = \frac{\gamma_{12}{\gamma_5}^2}{{\gamma_4}^5}, & \beta_8 = \frac{\gamma_{13}{\gamma_5}^3}{{\gamma_4}^7}, & \beta_9 = \frac{\gamma_{14}{\gamma_5}^3}{{\gamma_4}^7}, & \beta_{10} = \frac{\gamma_{15}{\gamma_5}^4}{{\gamma_4}^9}, & \beta_{11} = \frac{\gamma_{16}{\gamma_5}^3}{{\gamma_4}^6} & \end{array} \tag{3.62}$$

and

$$\beta_{12} = \frac{\gamma_{17}{\gamma_4}^2}{\gamma_5}, \quad \beta_{13} = -\frac{(\gamma_{17}\gamma_3-\gamma_{18}){\gamma_4}^3}{\gamma_5}, \quad \beta_{14} = \gamma_{19}\gamma_4, \quad \beta_{15} = \frac{\gamma_5\gamma_{20}}{\gamma_4}. \tag{3.63}$$

Here $\beta_{12}, \beta_{13}, \beta_{14}$ and β_{15} are the only invariants depending on the variables K, L, M and N. Then the general solution of (2.31), for $\xi = \xi(x), \eta = \eta(x, y)$, can be given implicitly by back substitution as

$$\begin{array}{l} \frac{f_{4,4}^2}{f_{4,4,4}}K = F_1, \\ \frac{f_{4,4}^3}{f_{4,4,4}}(pK - L) = F_2, \\ \frac{f_{4,4}}{f_{4,4,4}}((f_4f_{4,4,4} + 3f_{3,4,4})(pK - L) + 3f_{4,4,4}(qK - M)) = F_3, \\ \frac{1}{f_{4,4}^2f_{4,4,4}}((f_{4,4}f_{3,4,4}(2\,f_4f_{4,4,4} + 3f_{3,4,4}) - {f_{4,4,4}}^2(2f_{3,4}f_4 - 6\,f_{2,4} + 3\,f_{3,3}))(pK - L) \\ +6\,f_{4,4}f_{3,4,4}f_{4,4,4}(qK - M) + 6f_{4,4}{f_{4,4,4}}^2(\,fK - \,N)) = F_4. \end{array} \tag{3.64}$$

where F_1, F_2, F_3 and F_4 are the arbitrary functions of $\beta_i, i = 1...11$.

Finally, solving system (3.64) gives the variables K, L, M and N in terms of four arbitrary functions F_1, F_2, F_3 and F_4 which provide four independent invariant differentiation operators $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and $\mathcal{D}_4$ via (2.30). □

4 Conclusion

The third order ODEs $y''' = f(x, y, y', y'')$ admit the point equivalence transformations $\bar{x} = \phi(x, y), \bar{y} = \psi(x, y)$. It is found that $f_{4,4,4} = 0$ is an invariant equation with respect to these equivalence transformations. So, the third order ODEs $y''' = f(x, y, y', y'')$ can be classified into two main classes, $f_{4,4,4} = 0$ and $f_{4,4,4} \neq 0$. Bagderina [1] solved the equivalence problem of the class of third order ODEs $y''' = f(x, y, y', y'')$ with $f_{4,4,4} = 0$.

In the present study, we use Lie's infinitesimal method to study the differential invariants of the class of third order ODEs $y''' = f(x, y, y', y'')$ with $f_{4,4,4} \neq 0$ under pseudo-group of fiber preserving equivalence transformations $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$. The first main result, Theorem 3.1, deals with the special case $\bar{x} = x, \bar{y} = \psi(x, y)$ while the second main result, Theorem 3,2, presents the general case $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$.

As a consequence, we determine all the third order differential invariants of this group and the invariant differentiation operators. This provides simple necessary explicit conditions for third order differential equation to be equivalent to the canonical forms under the considered group of transformations.

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Appendix.A: The differential operators of the homogeneous linear system of PDEs (2.34)

$$
\begin{aligned}
X_1 &= [0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\
&\quad 0,0,0,0,0,0]\\
X_2 &= [0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\
&\quad 0,0,0,z_{40},0,0]\\
X_3 &= [0,0,z_3,z_4,z_5,0,0,0,-z_9,-z_{10},-z_{11},-z_{12},-z_{13},-z_{14},-2\,z_{15},-2\,z_{16},-2\,z_{17},-2\,z_{18},\\
&\quad -2\,z_{19},-2\,z_{20},-2\,z_{21},-2\,z_{22},-2\,z_{23},-2\,z_{24},z_{25},0,0,0,-z_{29},-z_{30},-z_{31},-z_{32},-z_{33},\\
&\quad -z_{34},z_{35},0,0,0,z_{39},0,z_{41},z_{42},z_{43}]\\
X_4 &= [0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\
&\quad 0,0,0,0,z_{40},0]\\
X_5 &= [0,0,0,2\,z_3,3\,z_4,-z_7,-2\,z_8,3,-2\,z_{10},-2\,z_{11}-z_{12},-z_{13},-4\,z_{13},-2\,z_{14},0,-3\,z_{16},-2\,z_{18}\\
&\quad -2\,z_{17},-2\,z_{19},-z_{21}-4\,z_{19},-2\,z_{20}-z_{22},-z_{23},-6\,z_{22},-4\,z_{23},-2\,z_{24},0,4\,z_5,-z_{27},-2\,z_{28},\\
&\quad 0,-z_9-2\,z_{30},-z_{32}-z_{10}-2\,z_{31},-z_{11}-z_{33},-z_{12}-4\,z_{33},-z_{13}-2\,z_{34},-z_{14},5\,z_{25},-z_{37},\\
&\quad -2\,z_{38},0,6\,z_{35},0,0,z_{41}+z_{40}z_3,2\,z_{42}+z_{40}z_4]\\
X_6 &= [0,0,0,{z_3}^2,3\,z_3z_4,z_5-z_8z_4-z_7z_3,3\,z_4-2\,z_8z_3,3\,z_3,z_6-2\,z_3z_{10}-2\,z_4z_{11},-z_4z_{13}-2\,z_3z_{11}\\
&\quad -z_3z_{12},-z_3z_{13}-z_4z_{14},-2\,z_8-4\,z_3z_{13},3-2\,z_3z_{14},0,-3\,z_3z_{16}-3\,z_4z_{17},-2\,z_4z_{19}-2\,z_3z_{17}\\
&\quad -2\,z_3z_{18}-z_{10},-z_{11}-2\,z_3z_{19}-2\,z_4z_{20},-z_{12}-z_3z_{21}-2\,z_{11}-4\,z_3z_{19}-z_4z_{22},-z_{13}-z_3z_{22}\\
&\quad -2\,z_3z_{20}-z_4z_{23},-z_4z_{24}-z_{14}-z_3z_{23},-6\,z_{13}-6\,z_3z_{22},-2\,z_{14}-4\,z_3z_{23},-2\,z_3z_{24},0,3\,{z_4}^2\\
&\quad +4\,z_5z_3,z_{25}-z_5z_8-z_4z_{28}-z_7z_4-z_3z_{27},3\,z_5-2\,z_3z_{28}-2\,z_8z_4,3\,z_4,-z_3z_9+z_{26}-2\,z_4z_{31}\\
&\quad -2\,z_3z_{30}-2\,z_4z_{10}-2\,z_5z_{11},-z_5z_{13}-z_4z_{12}-z_3z_{10}-2\,z_3z_{31}-2\,z_4z_{11}-z_3z_{32}-z_4z_{33},-z_4z_{13}\\
&\quad -z_3z_{11}-z_3z_{33}-z_5z_{14}-z_4z_{34},-2\,z_{28}-4\,z_4z_{13}-z_3z_{12}-4\,z_3z_{33},-2\,z_3z_{34}-z_3z_{13}-2\,z_4z_{14},\\
&\quad -z_3z_{14},10\,z_5z_4+5\,z_3z_{25},z_{35}-2\,z_5z_{28}-2\,z_4z_{27}-z_8z_{25}-z_3z_{37}-z_4z_{38}-z_5z_7,3\,z_{25}-2\,z_5z_8\\
&\quad -4\,z_4z_{28}-2\,z_3z_{38},3\,z_5,10\,{z_5}^2+6\,z_3z_{35}+15\,z_4z_{25},0,0,z_{41}z_3,z_{41}z_4+2\,z_{42}z_3]
\end{aligned}
$$

$$
\begin{aligned}
X_7 &= [0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\\
&\quad 0,0,0,0,0,z_{40}]\\
X_8 &= [0,0,0,0,3\,z_3,-z_8,3,0,-2\,z_{11},-z_{13},-z_{14},0,0,0,-3\,z_{17},-2\,z_{19},-2\,z_{20},-z_{22},-z_{23},\\
&\quad -z_{24},0,0,0,0,6\,z_4,-z_{28}-z_7,-2\,z_8,3,-2\,z_{10}-2\,z_{31},-2\,z_{11}-z_{12}-z_{33},-z_{13}-z_{34},\\
&\quad -4\,z_{13},-2\,z_{14},0,10\,z_5,-2\,z_{27}-z_{38},-4\,z_{28},0,15\,z_{25},0,0,0,z_{41}+2\,z_{40}z_3]\\
X_9 &= [0,0,0,0,3\,{z_3}^2,3\,z_4-2\,z_8z_3,6\,z_3,0,-4\,z_3z_{11}-z_7,-2\,z_3z_{13}-2\,z_8,3-2\,z_3z_{14},6,0,\\
&\quad 0,-6\,z_3z_{17}-3\,z_{10},-4\,z_3z_{19}-z_{12}-4\,z_{11},-4\,z_3z_{20}-z_{13},-2\,z_3z_{22}-4\,z_{13},-2\,z_3z_{23}\\
&\quad -2\,z_{14},-2\,z_3z_{24},0,0,0,0,12\,z_3z_4,-3\,z_8z_4-2\,z_7z_3-2\,z_3z_{28}+4\,z_5,-4\,z_8z_3+9\,z_4,\\
&\quad 6\,z_3,-6\,z_4z_{11}-4\,z_3z_{10}-4\,z_3z_{31}-z_{27}+z_6,-3\,z_4z_{13}-4\,z_3z_{11}-2\,z_3z_{12}-2\,z_3z_{33}\\
&\quad -2\,z_{28},-3\,z_4z_{14}-2\,z_3z_{13}-2\,z_3z_{34},-8\,z_3z_{13}-2\,z_8,-4\,z_3z_{14}+3,0,20\,z_5z_3+15\,{z_4}^2,\\
&\quad -2\,z_3z_{38}-3\,z_7z_4-4\,z_5z_8-4\,z_3z_{27}-6\,z_4z_{28}+5\,z_{25},-6\,z_8z_4-8\,z_3z_{28}+12\,z_5,9\,z_4,\\
&\quad 30\,z_3z_{25}+60\,z_5z_4,0,0,0,z_{40}{z_3}^2+2\,z_{41}z_3]\\
X_{10} &= [0,0,0,0,{z_3}^3,-z_8{z_3}^2+3\,z_3z_4,3\,{z_3}^2,0,-z_8z_4-2\,{z_3}^2z_{11}-z_7z_3+z_5,-{z_3}^2z_{13}-2\,z_8z_3\\
&\quad +3\,z_4,3\,z_3-{z_3}^2z_{14},6\,z_3,0,0,-3\,z_3z_{10}-3\,z_4z_{11}-3\,{z_3}^2z_{17}+2\,z_6,-2\,{z_3}^2z_{19}-4\,z_3z_{11}\\
&\quad -z_4z_{13}-z_3z_{12},-2\,{z_3}^2z_{20}-z_3z_{13}-z_4z_{14},-4\,z_3z_{13}-{z_3}^2z_{22}-2\,z_8,3-2\,z_3z_{14}-{z_3}^2z_{23},\\
&\quad -{z_3}^2z_{24},6,0,0,0,6\,{z_3}^2z_4,-{z_3}^2z_{28}+4\,z_5z_3-3\,z_8z_3z_4-z_7{z_3}^2+3\,{z_4}^2,-2\,z_8{z_3}^2+9\,z_3z_4,\\
&\quad 3\,{z_3}^2,-2\,{z_3}^2z_{31}-2\,{z_3}^2z_{10}+z_6z_3-6\,z_3z_4z_{11}-z_3z_{27}-z_4z_{28}-z_5z_8-z_7z_4+z_{25},-2\,z_8z_4\\
&\quad -2\,{z_3}^2z_{11}-2\,z_3z_{28}-3\,z_3z_4z_{13}-{z_3}^2z_{33}-{z_3}^2z_{12}+3\,z_5,-{z_3}^2z_{13}-3\,z_3z_4z_{14}-{z_3}^2z_{34}+3\,z_4,\\
&\quad -4\,{z_3}^2z_{13}-2\,z_8z_3+6\,z_4,3\,z_3-2\,{z_3}^2z_{14},0,15\,z_3{z_4}^2+10\,z_5{z_3}^2,-3\,z_8{z_4}^2-2\,{z_3}^2z_{27}+10\,z_5z_4\\
&\quad +5\,z_3z_{25}-4\,z_5z_8z_3-3\,z_7z_3z_4-6\,z_3z_4z_{28}-{z_3}^2z_{38},-4\,{z_3}^2z_{28}+12\,z_5z_3-6\,z_8z_3z_4+9\,{z_4}^2,\\
&\quad 9\,z_3z_4,15\,{z_3}^2z_{25}+60\,z_5z_3z_4+15\,{z_4}^3,0,0,0,z_{41}{z_3}^2]
\end{aligned}
$$

$$X_{11} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$$

$$X_{12} = [0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,4\,z_3,-z_8,3,0,-2\,z_{11},-z_{13}, -z_{14},0,0,0,10\,z_4,-z_7-2\,z_{28},-2\,z_8,3,20\,z_5,0,0,0,0]$$

$$X_{13} = [0,0,0,0,0,3\,z_3,0,0,-z_8,3,0,0,0,0,-3\,z_{11},-z_{13},-z_{14},0,0,0,0,0,0,0,6\,{z_3}^2,6\,z_4 -3\,z_8z_3,9\,z_3,0,-z_7-z_{28}-6\,z_3z_{11},-2\,z_8-3\,z_3z_{13},3-3\,z_3z_{14},6,0,0,30\,z_3z_4,10\,z_5 -6\,z_3z_{28}-3\,z_7z_3-6\,z_8z_4,18\,z_4-6\,z_8z_3,9\,z_3,45\,{z_4}^2+60\,z_5z_3,0,0,0,0]$$

$$X_{14} = [0,0,0,0,0,3\,{z_3}^2,0,0,3\,z_4-2\,z_8z_3,6\,z_3,0,0,0,0,-z_7-6\,z_3z_{11},-2\,z_3z_{13}-2\,z_8, 3-2\,z_3z_{14},6,0,0,0,0,0,0,4\,{z_3}^3,12\,z_3z_4-3\,z_8{z_3}^2,9\,{z_3}^2,0,4\,z_5-2\,z_7z_3-3\,z_8z_4 -6\,{z_3}^2z_{11}-2\,z_3z_{28},9\,z_4-4\,z_8z_3-3\,{z_3}^2z_{13},6\,z_3-3\,{z_3}^2z_{14},12\,z_3,0,0,30\,{z_3}^2z_4, -12\,z_8z_3z_4+15\,{z_4}^2+20\,z_5z_3-6\,{z_3}^2z_{28}-3\,z_7{z_3}^2,36\,z_3z_4-6\,z_8{z_3}^2,9\,{z_3}^2,90\,z_3{z_4}^2 +60\,z_5{z_3}^2,0,0,0,0]$$

$$X_{15} = [0,0,0,0,0,{z_3}^3,0,0,-z_8{z_3}^2+3\,z_3z_4,3\,{z_3}^2,0,0,0,0,-z_8z_4-z_7z_3-3\,{z_3}^2z_{11}+z_5, -{z_3}^2z_{13}-2\,z_8z_3+3\,z_4,3\,z_3-{z_3}^2z_{14},6\,z_3,0,0,0,0,0,0,{z_3}^4,6\,{z_3}^2z_4-z_8{z_3}^3,3\,{z_3}^3, 0,-{z_3}^2z_{28}-z_7{z_3}^2-3\,z_8z_3z_4+3\,{z_4}^2+4\,z_5z_3-2\,{z_3}^3z_{11},-{z_3}^3z_{13}+9\,z_3z_4-2\,z_8{z_3}^2, -{z_3}^3z_{14}+3\,{z_3}^2,6\,{z_3}^2,0,0,10\,{z_3}^3z_4,-z_7{z_3}^3-6\,z_8{z_3}^2z_4-2\,{z_3}^3z_{28}+10\,z_5{z_3}^2+15\,z_3{z_4}^2, 18\,{z_3}^2z_4-2\,z_8{z_3}^3,3\,{z_3}^3,45\,{z_3}^2{z_4}^2+20\,z_5{z_3}^3,0,0,0,0]$$

$$X_{16} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0]$$

$$X_{17} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,5\,z_3, -z_8,3,0,15\,z_4,0,0,0,0]$$

$$X_{18} = [0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,4\,z_3,0,0,-z_8,3,0,0,0,0, 10\,{z_3}^2,10\,z_4-4\,z_8z_3,12\,z_3,0,60\,z_3z_4,0,0,0,0]$$

$$X_{19} = [0,0,0,0,0,0,0,0,3\,z_3,0,0,0,0,0,-z_8,3,0,0,0,0,0,0,0,0,0,6\,{z_3}^2,0,0,6\,z_4-3\,z_8z_3,$$
$$9\,z_3,0,0,0,0,10\,{z_3}^3,30\,z_3z_4-6\,z_8{z_3}^2,18\,{z_3}^2,0,90\,{z_3}^2z_4,0,0,0,0]$$

$$X_{20} = [0,0,0,0,0,0,0,0,3\,{z_3}^2,0,0,0,0,0,3\,z_4-2\,z_8z_3,6\,z_3,0,0,0,0,0,0,0,0,0,4\,{z_3}^3,0,0,$$
$$12\,z_3z_4-3\,z_8{z_3}^2,9\,{z_3}^2,0,0,0,0,5\,{z_3}^4,-4\,z_8{z_3}^3+30\,{z_3}^2z_4,12\,{z_3}^3,0,60\,{z_3}^3z_4,0,0,0,0]$$

$$X_{21} = [0,0,0,0,0,0,0,0,{z_3}^3,0,0,0,0,0,-z_8{z_3}^2+3\,z_3z_4,3\,{z_3}^2,0,0,0,0,0,0,0,0,0,{z_3}^4,0,0,$$
$$6\,{z_3}^2z_4-z_8{z_3}^3,3\,{z_3}^3,0,0,0,0,{z_3}^5,10\,{z_3}^3z_4-z_8{z_3}^4,3\,{z_3}^4,0,15\,{z_3}^4z_4,0,0,0,0]$$

$$X_{22} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,$$
$$0,1,0,0,0,0]$$

$$X_{23} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,$$
$$0,6\,z_3,0,0,0,0]$$

$$X_{24} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,5\,z_3,$$
$$0,0,15\,{z_3}^2,0,0,0,0]$$

$$X_{25} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,4\,z_3,0,0,0,0,0,0,$$
$$10\,{z_3}^2,0,0,20\,{z_3}^3,0,0,0,0]$$

$$X_{26} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,3\,z_3,0,0,0,0,0,0,0,0,0,0,0,0,0,6\,{z_3}^2,0,0,0,0,0,$$
$$0,10\,{z_3}^3,0,0,15\,{z_3}^4,0,0,0,0]$$

$$X_{27} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,3\,{z_3}^2,0,0,0,0,0,0,0,0,0,0,0,0,0,4\,{z_3}^3,0,0,0,0,0,$$
$$0,5\,{z_3}^4,0,0,6\,{z_3}^5,0,0,0,0]$$

$$X_{28} = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,{z_3}^3,0,0,0,0,0,0,0,0,0,0,0,0,0,{z_3}^4,0,0,0,0,0,0,$$
$${z_3}^5,0,0,{z_3}^6,0,0,0,0]$$

$$
\begin{aligned}
T_1 &= [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 0, 0, 0, 0, 0] \\
T_2 &= [0, 0, -z_3, -2\,z_4, -3\,z_5, -3\,z_6, -2\,z_7, -z_8, -3\,z_9, -2\,z_{10}, -z_{11}, -z_{12}, 0, z_{14}, -3\,z_{15}, \\
&\quad -2\,z_{16}, -z_{17}, -z_{18}, 0, z_{20}, 0, z_{22}, 2\,z_{23}, 3\,z_{24}, -4\,z_{25}, -4\,z_{26}, -3\,z_{27}, -2\,z_{28}, -4\,z_{29}, \\
&\quad -3\,z_{30}, -2\,z_{31}, -2\,z_{32}, -z_{33}, 0, -5\,z_{35}, -5\,z_{36}, -4\,z_{37}, -3\,z_{38}, -6\,z_{39}, z_{40}, 0, -z_{42}, -2\,z_{43}] \\
T_3 &= [0, 0, 0, -z_3, -3\,z_4, 0, z_8, -3, 0, z_{11}, 0, 2\,z_{13}, z_{14}, 0, 0, z_{17}, 0, 2\,z_{19}, z_{20}, 0, 3\,z_{22}, 2\,z_{23}, \\
&\quad z_{24}, 0, -6\,z_5, -3\,z_6, -2\,z_7 + z_{28}, -z_8, -3\,z_9, -2\,z_{10} + z_{31}, -z_{11}, 2\,z_{33} - z_{12}, z_{34}, z_{14}, \\
&\quad -10\,z_{25}, -7\,z_{26}, -5\,z_{27} + z_{38}, -3\,z_{28}, -15\,z_{35}, 0, 0, -z_{40}z_3, -z_{42} - 2\,z_{40}z_4] \\
T_4 &= [0, 0, 0, 0, -z_3, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4\,z_4, 0, z_8, -3, 0, z_{11}, \\
&\quad 0, 2\,z_{13}, z_{14}, 0, -10\,z_5, -3\,z_6, -2\,z_7 + 2\,z_{28}, -z_8, -20\,z_{25}, 0, 0, 0, -z_{40}z_3] \\
T_5 &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, -1, 0, 0, 0, 0, 0, 0, 0, \\
&\quad -5\,z_4, 0, z_8, -3, -15\,z_5, 0, 0, 0, 0] \\
T_6 &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, \\
&\quad 0, -1, 0, -6\,z_4, 0, 0, 0, 0] \\
T_7 &= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&\quad 0, 0, -z_3, 0, 0, 0, 0]
\end{aligned}
$$

Appendix.B: The nonzero commutators for the Lie algebra $\mathcal{L}_{35}$ of the differential operators of the homogeneous linear system of PDEs (2.34)

$$
\begin{array}{lllll}
[e_2,e_3]=e_2, & [e_2,e_5]=2e_4, & [e_2,e_6]=e_5, & [e_2,e_8]=3e_7, & [e_2,e_9]=2e_8,\\
[e_2,e_{10}]=e_9, & [e_2,e_{12}]=4e_{11}, & [e_2,e_{13}]=3e_{12}, & [e_2,e_{14}]=2e_{13}, & [e_2,e_{15}]=e_{14},\\
[e_2,e_{17}]=5e_{16}, & [e_2,e_{18}]=4e_{17}, & [e_2,e_{19}]=3e_{18}, & [e_2,e_{20}]=2e_{19}, & [e_2,e_{21}]=e_{20},\\
[e_2,e_{23}]=6e_{22}, & [e_2,e_{24}]=5e_{23}, & [e_2,e_{25}]=4e_{24}, & [e_2,e_{26}]=3e_{25}, & [e_2,e_{27}]=2e_{26},\\
[e_2,e_{28}]=e_{27}, & [e_2,e_{30}]=-e_2, & [e_2,e_{31}]=-e_4, & [e_2,e_{32}]=-e_7, & [e_2,e_{33}]=-e_{11},\\
[e_2,e_{34}]=-e_{16}, & [e_2,e_{35}]=-e_{22}, & [e_3,e_4]=-e_4, & [e_3,e_6]=e_6, & [e_3,e_7]=-e_7,\\
[e_3,e_9]=e_9, & [e_3,e_{10}]=2e_{10}, & [e_3,e_{11}]=-e_{11}, & [e_3,e_{13}]=e_{13}, & [e_3,e_{14}]=2e_{14},\\
[e_3,e_{15}]=3e_{15}, & [e_3,e_{16}]=-e_{16}, & [e_3,e_{18}]=e_{18}, & [e_3,e_{19}]=2e_{19}, & [e_3,e_{20}]=3e_{20},\\
[e_3,e_{21}]=4e_{21}, & [e_3,e_{22}]=-e_{22}, & [e_3,e_{24}]=e_{24}, & [e_3,e_{25}]=2e_{25}, & [e_3,e_{26}]=3e_{26},\\
[e_3,e_{27}]=4e_{27}, & [e_3,e_{28}]=5e_{28}, & [e_4,e_5]=3e_7, & [e_4,e_6]=e_8, & [e_4,e_8]=6e_{11},\\
[e_4,e_9]=3e_{12}, & [e_4,e_{10}]=e_{13}, & [e_4,e_{12}]=10e_{16}, & [e_4,e_{13}]=6e_{17}, & [e_4,e_{14}]=3e_{18},\\
[e_4,e_{15}]=e_{19}, & [e_4,e_{17}]=15e_{22}, & [e_4,e_{18}]=10e_{23}, & [e_4,e_{19}]=6e_{24}, & [e_4,e_{20}]=3e_{25},\\
[e_4,e_{21}]=e_{26}, & [e_4,e_{30}]=-2e_4, & [e_4,e_{31}]=-3e_7, & [e_4,e_{32}]=-4e_{11}, & [e_4,e_{33}]=-5e_{16},\\
[e_4,e_{34}]=-6e_{22}, & [e_5,e_6]=e_9, & [e_5,e_7]=-4e_{11}, & [e_5,e_9]=2e_{13}, & [e_5,e_{10}]=2e_{14},\\
[e_5,e_{11}]=-5e_{16}, & [e_5,e_{13}]=3e_{18}, & [e_5,e_{14}]=4e_{19}, & [e_5,e_{15}]=3e_{20}, & [e_5,e_{16}]=-6e_{22},\\
[e_5,e_{18}]=4e_{24}, & [e_5,e_{19}]=6e_{25}, & [e_5,e_{20}]=6e_{26}, & [e_5,e_{21}]=4e_{27}, & [e_5,e_{30}]=-e_5,\\
[e_5,e_{31}]=-e_8, & [e_5,e_{32}]=-e_{12}, & [e_5,e_{33}]=-e_{17}, & [e_5,e_{34}]=-e_{23}, & [e_6,e_7]=-e_{12},\\
[e_6,e_8]=-e_{13}, & [e_6,e_{10}]=2e_{15}, & [e_6,e_{11}]=-e_{17}, & [e_6,e_{12}]=-e_{18}, & [e_6,e_{14}]=2e_{20},\\
[e_6,e_{15}]=5e_{21}, & [e_6,e_{16}]=-e_{23}, & [e_6,e_{17}]=-e_{24}, & [e_6,e_{19}]=2e_{26}, & [e_6,e_{20}]=5e_{27},\\
[e_6,e_{21}]=9e_{28}, & [e_7,e_8]=10e_{16}, & [e_7,e_9]=4e_{17}, & [e_7,e_{10}]=e_{18}, & [e_7,e_{12}]=20e_{22},\\
[e_7,e_{13}]=10e_{23}, & [e_7,e_{14}]=4e_{24}, & [e_7,e_{15}]=e_{25}, & [e_7,e_{30}]=-3e_7, & [e_7,e_{31}]=-6e_{11},\\
[e_7,e_{32}]=-10e_{16}, & [e_7,e_{33}]=-15e_{22}, & [e_8,e_9]=3e_{18}, & [e_8,e_{10}]=2e_{19}, & [e_8,e_{11}]=-15e_{22},\\
[e_8,e_{13}]=6e_{24}, & [e_8,e_{14}]=6e_{25}, & [e_8,e_{15}]=3e_{26}, & [e_8,e_{30}]=-2e_8, & [e_8,e_{31}]=-3e_{12},\\
[e_8,e_{32}]=-4e_{17}, & [e_8,e_{33}]=-5e_{23}, & [e_9,e_{10}]=2e_{20}, & [e_9,e_{11}]=-5e_{23}, & [e_9,e_{12}]=-4e_{24},\\
[e_9,e_{14}]=4e_{26}, & [e_9,e_{15}]=5e_{27}, & [e_9,e_{30}]=-e_9, & [e_9,e_{31}]=-e_{13}, & [e_9,e_{32}]=-e_{18},\\
[e_9,e_{33}]=-e_{24}, & [e_{10},e_{11}]=-e_{24}, & [e_{10},e_{12}]=-2e_{25}, & [e_{10},e_{13}]=-2e_{26}, & [e_{10},e_{15}]=5e_{28},\\
[e_{11},e_{30}]=-4e_{11}, & [e_{11},e_{31}]=-10e_{16}, & [e_{11},e_{32}]=-20e_{22}, & [e_{12},e_{30}]=-3e_{12}, & [e_{12},e_{31}]=-6e_{17},\\
[e_{12},e_{32}]=-10e_{23}, & [e_{13},e_{30}]=-2e_{13}, & [e_{13},e_{31}]=-3e_{18}, & [e_{13},e_{32}]=-4e_{24}, & [e_{14},e_{30}]=-e_{14},\\
[e_{14},e_{31}]=-e_{19}, & [e_{14},e_{32}]=-e_{25}, & [e_{16},e_{30}]=-5e_{16}, & [e_{16},e_{31}]=-15e_{22}, & [e_{17},e_{30}]=-4e_{17},\\
[e_{17},e_{31}]=-10e_{23}, & [e_{18},e_{30}]=-3e_{18}, & [e_{18},e_{31}]=-6e_{24}, & [e_{19},e_{30}]=-2e_{19}, & [e_{19},e_{31}]=-3e_{25},\\
[e_{20},e_{30}]=-e_{20}, & [e_{20},e_{31}]=-e_{26}, & [e_{22},e_{30}]=-6e_{22}, & [e_{23},e_{30}]=-5e_{23}, & [e_{24},e_{30}]=-4e_{24},\\
[e_{25},e_{30}]=-3e_{25}, & [e_{26},e_{30}]=-2e_{26}, & [e_{27},e_{30}]=-e_{27}, & [e_{30},e_{31}]=e_{31}, & [e_{30},e_{32}]=2e_{32},\\
[e_{30},e_{33}]=3e_{33}, & [e_{30},e_{34}]=4e_{34}, & [e_{30},e_{35}]=5e_{35}, & [e_{31},e_{32}]=2e_{33}, & [e_{31},e_{33}]=5e_{34},\\
[e_{31},e_{34}]=9e_{35}, & [e_{32},e_{33}]=5e_{35}. & & &
\end{array}
\tag{4.65}
$$

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