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ABSTRACT. A constructive version of the Frobenius integrability theorem – that can be programmed effectively – is given. This is used in computing invariants of groups of low ranks and recover examples from a recent paper of Boyko, Patera and Popoyvich [BPP].

1. INTRODUCTION

The effective computation of local invariants of Lie algebras of vector fields is one of the main technical tools in applications of Lie’s symmetry method to several problems in differential equations – notably their classification and explicit solutions of natural equations of mathematical physics, as shown, e.g., in several papers of Ibragimov [Ib2], [Ib3], and Olver [Ol1].

The main aim of this paper is to give a constructive procedure that reduces the determination of joint local invariants of any finite dimensional Lie algebra of vector fields — indeed any finite number of vector fields — to that of a commuting family of vector fields. It is thus a constructive version of the Frobenius integrability theorem – [Ol1, p. 422], [Le, p. 472], [Ib1, p. 92–94] – which can also be programmed effectively. This is actually valid for any field of scalars. A paper close to this paper is [Pe1].

We illustrate the main result by computing joint invariants for groups of low rank as well as examples from Boyko et al [BPP], where the authors have used the method of moving frames, [Ol2], to obtain invariants.

It is stated in [BPP] that solving the first order system of differential equations is not practicable. However, it is practicable for at least two reasons. The local joint invariants in any representation of a Lie algebra as an algebra of vector fields are the same as those of a commuting family of operators. Moreover, one needs to take only operators that are generators for the full algebra. For example, if the the Lie algebra is semisimple with Dynkin diagram having n nodes, then one needs just $2n$ basic operators to determine invariants.

Another reason is that software nowadays can handle symbolic computations very well.

The main results of the paper are as follows:

Theorem 1. *Let \mathcal{L} be a finite dimensional Lie algebra of vector fields defined on some open subset U of \mathbb{R}^n . Let X_1, \dots, X_d be a basis of \mathcal{L} . Then the following hold:*

- (1) *The algebra of operators whose coefficient matrix is the matrix of functions obtained from the coefficients of X_1, \dots, X_d by reducing it to reduced row echelon form is abelian.*
- (2) *The local joint invariants of \mathcal{L} are the same as those of the above abelian algebra.*

Theorem 2. *Let X_1, X_2, \dots, X_d be vector fields defined on some open subset of \mathbb{R}^n . Then the joint invariants of X_1, X_2, \dots, X_d are given by the following algorithm:*

- (1) *[Step 1] Find the row reduced echelon form of X_1, X_2, \dots, X_d , and let Y_1, \dots, Y_r be the corresponding vector fields. If this is a commuting family then stop. Otherwise go to:*

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- (2) [Step 2] If some $[Y_i, Y_j] \neq 0$, then set $Y_{r+1} := [Y_i, Y_j]$. Go to Step 1 and substitute Y_1, \dots, Y_r, Y_{r+1} in place of X_1, X_2, \dots, X_d .

This process terminates in at most n iterations. If V_1, \dots, V_m are the commuting vector fields at the end of the above iterative process, the joint invariants of X_1, X_2, \dots, X_d coincide with the joint invariants of V_1, \dots, V_m .

2. SOME EXAMPLES AND PROOF OF THEOREM 1 AND THEOREM 2

Before proving Theorem 1, we give some examples in detail, because these examples contain all the key ideas of a formal proof and of computation of local joint invariants of vector fields.

2.1. Example: The rotations in \mathbb{R}^3 . The group $\text{SO}(3)$ has one basic invariant in its standard representation, namely $x^2 + y^2 + z^2$, which is clear from geometry. Let us recover this by Lie algebra calculations in a manner that is applicable to all Lie groups.

The fundamental vector fields given by rotations in the coordinate planes are

$$I = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad J = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \quad \text{and} \quad K = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}.$$

The coefficients matrix is

$$\begin{pmatrix} y & -x & 0 \\ 0 & z & -y \\ z & 0 & -x \end{pmatrix}. \quad (1)$$

This is a singular matrix, so its rank is at most two. On the open subset U where $yz \neq 0$, the rank is two. The rank is two everywhere except at the origin but we are only interested in the rank on some open set.

The differentiable functions on U simultaneously annihilated by I, J, K are clearly the same as those of the operators whose coefficient matrix is obtained from (1) reducing it to its row echelon form. Since I, J generate the infinitesimal rotations, we may delete the last row in (1). The reduced row echelon form of (1) is

$$\begin{pmatrix} 1 & 0 & \frac{-x}{z} \\ 0 & 1 & \frac{-y}{z} \end{pmatrix}.$$

The operators whose matrix of coefficients is this matrix are

$$X := \frac{\partial}{\partial x} - \frac{x}{z} \frac{\partial}{\partial z} \quad \text{and} \quad Y := \frac{\partial}{\partial y} - \frac{y}{z} \frac{\partial}{\partial z}.$$

Note that $[X, Y] = 0$. Now, because the fields are commuting, we can compute the basic invariants of any one of them, say X ; then Y will operate on the invariants of X .

The invariants for X are given by the standard method of Cauchy characteristics as follows [Ib1, p. 67]: We want to solve

$$\frac{dx}{1} = \frac{dy}{0} = \frac{-zdz}{x}.$$

The basic invariants of X are $x^2 + z^2 =: \xi$, $y =: \eta$. As Y commutes with X , it operates on invariants of X . Now $Y(\xi) = -2\eta$, $Y(\eta) = 1$. Thus on the invariants of X the field induced by Y is

$$-2\eta \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

The corresponding characteristic system is

$$\frac{d\xi}{-2\eta} = \frac{d\eta}{1},$$

so we get the basic invariant – which must be a joint invariant – as $\xi + \eta^2 = x^2 + y^2 + z^2$.

Examples given below shows what happens if we just work with finitely many vector fields.

2.2. Example: The rotations in \mathbb{R}^n with metric signature (p, q) , where $p + q = n$. The group $\text{SO}(p, q)$ operates transitively on every nonzero level set of the function $\sum_{i=1}^p x_i^2 - \sum_{i=1}^q x_{i+p}^2$, and it operates transitively on the nonzero vectors in the zero level set of this function. Therefore, it is clear that there is only one basic joint invariant. Let us recover this by Lie algebra calculations in a manner that is applicable in general.

The Lie group $\text{SO}(p, q)$ is generated by ordinary rotations in the (x_1, x_2) -plane, the (x_2, x_3) -plane, \dots , the (x_{p-1}, x_p) -plane, the (x_{p+1}, x_{p+2}) -plane, \dots , the (x_{p+q-1}, x_{p+q}) -plane, and hyperbolic rotations in the (x_p, x_{p+1}) -plane. The fundamental vector fields generated by these rotations in the coordinate planes are

$$x_{i+1} \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_{i+1}}, i \in \{1, \dots, p+q-1\} \setminus \{p\} \quad \text{and} \quad x_{p+1} \frac{\partial}{\partial x_p} + x_p \frac{\partial}{\partial x_{p+1}}.$$

The reduced row echelon form is the augmented $(n-1) \times (n-1)$ identity matrix, augmented by column vector

$$\frac{x_1}{x_n}, \dots, \frac{x_p}{x_n}, -\frac{x_{p+1}}{x_n}, \dots, -\frac{x_{p+q-1}}{x_n}.$$

Thus we get the corresponding vector fields

$$\frac{\partial}{\partial x_i} + \frac{x_i}{x_n} \frac{\partial}{\partial x_n}, i \leq p \quad \text{and} \quad \frac{\partial}{\partial x_j} - \frac{x_j}{x_n} \frac{\partial}{\partial x_n}, p < j \leq n-1.$$

Since for independent variables x, y, z ,

$$\left[\frac{\partial}{\partial x} + \frac{x}{z} \frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \frac{y\epsilon}{z} \frac{\partial}{\partial z} \right] = \left[\frac{x}{z} \frac{\partial}{\partial z}, \frac{y\epsilon}{z} \frac{\partial}{\partial z} \right] = 0,$$

where $\epsilon = \pm 1$, we conclude that these vector fields commute and each such field operates on the invariants of the remaining. By calculations as in Example 1 we see that the basic joint invariant is $\sum_{i=1}^p x_i^2 - \sum_{i=1}^q x_{i+p}^2$.

2.3. Proof of Theorem 1. We will use the notation in the statement of Theorem 1. Take a point $p \in U$, and let $\mathcal{L}(p)$ be the linear span of $X(p)$ with $X \in \mathcal{L}$. Let $r(p)$ be the dimension of $\mathcal{L}(p)$, and let $r = \max \{r(p)\}_{p \in U}$. Choose a point p with $r(p) = r$.

By renaming the basis for \mathcal{L} , we may assume that $X_1(p), \dots, X_r(p)$ is a basis for $\mathcal{L}(p)$. Therefore, the determinant $X_1(p) \wedge \dots \wedge X_r(p) \in \wedge^r T_p U$ is nonzero. Hence $X_1(q) \wedge \dots \wedge X_r(q) \in \wedge^r T_q U$ is nonzero for all q in a neighborhood of p . In particular, $r(q) = r(p) = r$ at all such points q .

Replacing U by this open neighborhood of p , we may suppose that $r(q) = r$ for all points $q \in U$. This implies that $X_{r+k}(q)$ is a linear combination of $X_1(q), \dots, X_r(q)$ with coefficients that depend differentiably on $q \in U$. Moreover, for any $X, Y \in \mathcal{L}$, as $[X, Y](q)$ is a linear combination of $X_1(q), \dots, X_d(q)$ with scalar coefficients, we see that for $1 \leq i, j \leq r$, the Lie bracket $[X_i, X_j](q)$ is a linear combination of $X_1(q), \dots, X_r(q)$ with coefficients that depend differentiably on q . Also, for $1 \leq i, j \leq r$ and any differentiable function f ,

$$[X_i, fX_j]$$

is a linear combination of X_1, \dots, X_r with coefficients that are differentiable functions. If

$$X_j = \sum_{k=1}^n a_{jk} \frac{\partial}{\partial x_k}, \quad 1 \leq j \leq r,$$

we put these operators in reduced row echelon form with coefficients as differentiable functions. Therefore, taking possibly a smaller open subset of U , we obtain a family of vector fields which span $\mathcal{L}(q)$, $q \in U$, and is closed under Lie brackets with differentiable functions as coefficients. Also, the local invariants for this family are the same as for X_1, \dots, X_r .

After changing indices, we may suppose that

$$X_j = \frac{\partial}{\partial x_j} + \sum_{k=r+1}^n b_{jk} \frac{\partial}{\partial x_k}, \quad 1 \leq j \leq r.$$

We want to show that $[X_i, X_j] = 0$ for all $i, j \leq r$. A straightforward computation shows that

$$[X_i, X_j] \equiv 0 \text{ modulo } \frac{\partial}{\partial x_{r+1}}, \dots, \frac{\partial}{\partial x_n},$$

meaning $[X_i, X_j] = \sum_{\ell=r+1}^n \phi_{i,j}^\ell \frac{\partial}{\partial x_\ell}$, where $\phi_{i,j}^\ell$ are smooth functions. On the other hand, $[X_i, X_j]$ is a linear combination of X_1, \dots, X_r with functions as coefficients. From these we conclude that $[X_i, X_j] = 0$. This completes the proof of the theorem.

2.4. Proof of Theorem 2. We use the notation of Theorem 2. Since Y_1, \dots, Y_r are in row reduced echelon form, and $[Y_i, Y_j] \neq 0$, it follows that $[Y_i, Y_j]$ is not in the linear span of Y_1, \dots, Y_r with smooth functions as coefficients. Therefore, when we go back to Step 1 and construct the row reduced echelon form of $Y_1, \dots, Y_r, [Y_i, Y_j]$, there are $r+1$ vector fields in it. Consequently, each time we come back and complete Step 1, the number of vector fields goes up by one. This immediately implies that the process stops after at most n iterations. The final statement of the theorem is obvious.

Remark 3. Theorem 1 and Theorem 2 are valid in algebraic category for any field – working with the Zariski topology. For the field \mathbb{R} , one has the standard refinement that r commuting fields of rank r are, in suitable coordinates $\frac{\partial}{\partial x_i}$ (Frobenius' theorem). The reason is that any nonzero vector field X in suitable local coordinates is $\frac{\partial}{\partial x_1}$ and any vector field that commutes with X operates on the invariants of X .

Let us illustrate Theorem 2 by two examples.

Taking the example in [O11, p. 64], consider the following three vector fields on \mathbb{R}^3 :

$$V_+ := 2y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}, \quad V_0 := -2x \frac{\partial}{\partial x} + 2z \frac{\partial}{\partial z}, \quad V_- := -x \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial z}.$$

Although they are closed under Lie bracket, we do not need this fact to compute the joint invariants.

The row reduced echelon form of the matrix of coefficients is

$$\begin{pmatrix} 1 & 0 & -z/x \\ 0 & 1 & 2y/x \end{pmatrix}$$

Let

$$X := \frac{\partial}{\partial x} - \frac{z}{x} \frac{\partial}{\partial z} \quad \text{and} \quad Y := \frac{\partial}{\partial y} + \frac{2y}{x} \frac{\partial}{\partial z}.$$

Sine $[X, Y] = 0$, we stop at this stage. The invariants of X are

$$\xi = xz \quad \text{and} \quad \eta = y.$$

Since X commutes with Y , the action of Y preserves the invariants of X . We have

$$Y(\xi) = 2\eta \quad \text{and} \quad Y(\eta) = 1.$$

So Y on invariants of X is

$$Y = 2\eta \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

Its invariants are given by $\frac{d\xi}{2\eta} = \frac{d\eta}{1}$. So the basic invariant is

$$\xi - \eta^2 = xz - y^2.$$

The next example is from [Ib1].

Take the following two vector fields on \mathbb{R}^4 with coordinates (x, y, z, w) :

$$X_1 := (0, z, -y, 0), X_2 := (1, w, 0, y).$$

Its row reduced echelon form Y_1, Y_2 is not closed under Lie bracket. We have

$$[Y_1, Y_2] = X_3 := (0, 0, -w/z, -1).$$

The row reduced echelon form for X_1, X_2, X_3 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y/w \\ 0 & 0 & 1 & z/w \end{pmatrix}$$

which give commutative vector fields. Consequently, the joint invariant is $y^2 + z^2 - w^2$.

3. MORE EXAMPLES

An efficient way to get invariants of a solvable algebra L is to first determine the joint invariants of the commutator algebra — which is always nilpotent and thus one can use the central series for systematic reductions — and then find the joint invariants of the full algebra as they are the same as those of L/L' on the invariants of L' .

Also for semi-direct products $L \rtimes V$ one can first find the joint invariants of V , and then the invariants of L on the invariants of V to find the joint invariants of the full algebra.

Before giving examples, let us record the formulas for the fundamental vector fields as differential operators in the adjoint and coadjoint representations of Lie groups.

Let \mathcal{L} be a finite dimensional Lie algebra, and let X_1, \dots, X_d be a basis of \mathcal{L} . Let $\omega_1, \dots, \omega_d$ be the dual basis of \mathcal{L}^* .

For $X \in \mathcal{L}$, the fundamental vector fields $X_{\mathcal{L}}$ and $X_{\mathcal{L}^*}$ corresponding to X in the adjoint and coadjoint representations are given as differential operators by the formulas:

$$X_{\mathcal{L}} = \sum_{1 \leq i, j \leq d} x_i \omega_j([X, X_i]) \frac{\partial}{\partial x_j} \quad \text{and} \quad X_{\mathcal{L}^*} = - \sum_{1 \leq i, j \leq d} x_i \omega_i([X, X_j]) \frac{\partial}{\partial x_j}.$$

Several examples of invariants of solvable algebras are computed in [Nd, Pe1]. Also invariants of real low dimensional algebras and some general classical algebras are calculated in several papers, for example [PSWZ1, PSWZ2, Pe2, PN]. We now give some examples of fundamental invariants of certain solvable Lie algebras and Lie algebras of low rank.

3.1. Examples from [BPP]. For the convenience of the reader, we will refer to the online version of the paper [BPP] — available at <http://arxiv.org/pdf/math-ph/0602046.pdf>.

3.1.1. *Example 1.* We will use the variable x, y, z, w for the variable $\{e_i\}_{i=1}^4$ in Example 1 of [BPP].

After writing the matrix of the operators in the coadjoint representation, Maple directly gives two joint invariant, one of which is in integral form. Working with the reduced row echelon form we easily get one invariant

$$I_1 = (x^2 + y^2) \exp(-2b \cdot \tan^{-1}(y/x)).$$

A second invariant can be obtained by using elementary implications like

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{\lambda a + \mu c}{\lambda b + \mu d}.$$

This gives a second independent invariant

$$I_2 = \frac{w^{2b}}{(x^2 + y^2)^a};$$

this corrects a misprint in this example from [BPP].

3.1.2. *Example 2.* We will use the variable s, w, x, y, z for the variable $\{e_i\}_{i=1}^5$ in Example 2 of [BPP].

After writing down the matrix of coefficients of the operators in the coadjoint representation corresponding to the given basis and using the operators corresponding to the row reduced form, we find that there is only basic joint invariant

$$\frac{w - s \cdot \ln s}{s}.$$

Maple gives this directly — without any row reductions.

3.1.3. *Example 3.* We will use the variable s, w, x, y, z for the variable $\{e_i\}_{i=1}^5$ in Example 3 of [BPP].

Using the same procedure as in Example 2, Maple gives directly the invariant

$$\frac{xw + zs}{s}.$$

3.1.4. *Example 4.* We will use the variable r, s, w, x, y, z for the variable $\{e_i\}_{i=1}^6$ in Example 4 of [BPP].

Maple cannot find directly joint invariants from the matrix of operators for the coadjoint representation. However, when one works with the row reduced echelon form, the situation simplifies dramatically. One gets two basic invariants

$$I_1 = r^{-2b}(x^2 + w^2) \exp(-2a \cdot \tan^{-1}(w/x)) \quad \text{and} \quad I_2 = \frac{s}{r} - \frac{1}{2a} \ln \frac{x^2 + w^2}{r^{2b}}.$$

3.2. Invariants of $\mathfrak{sl}(3, \mathbb{R})$ in its adjoint and coadjoint representations. The non-zero commutation relations are

$$\begin{aligned} [e_1, e_2] &= e_2, [e_1, e_3] = 2e_3, [e_1, e_4] = -e_4, [e_1, e_6] = e_6, [e_1, e_7] = -2e_7, [e_1, e_8] = -e_8, [e_2, e_4] = e_1 - e_5, \\ [e_2, e_5] &= e_2, [e_2, e_6] = e_3, [e_2, e_7] = -e_8, [e_3, e_4] = -e_6, [e_3, e_5] = -e_3, [e_3, e_7] = e_1, [e_3, e_8] = e_2, [e_4, e_5] = -e_4, \\ [e_4, e_8] &= -e_7, [e_5, e_6] = 2e_6, [e_5, e_7] = -e_7, [e_5, e_8] = -2e_8, [e_6, e_7] = e_4, [e_6, e_8] = e_5 \end{aligned}$$

Writing the operators $\sum_{i=1}^8 x_i X_i$ as $[x_1, x_2, \dots, x_8]$, the coadjoint representation of the basis of $\mathfrak{sl}(3, \mathbb{R})$ is

$$\begin{aligned}
X_1 &= [0, -x_2, -2x_3, x_4, 0, -x_6, 2x_7, x_8] \\
X_2 &= [x_2, 0, 0, x_5 - x_1, -x_2, -x_3, x_8, 0] \\
X_3 &= [2x_3, 0, 0, x_6, x_3, 0, -x_1, -x_2] \\
X_4 &= [-x_4, -x_5 + x_1, -x_6, 0, x_4, 0, 0, x_7] \\
X_5 &= [0, x_2, -x_3, -x_4, 0, -2x_6, x_7, 2x_8] \\
X_6 &= [x_6, x_3, 0, 0, 2x_6, 0, -x_4, -x_5] \\
X_7 &= [-2x_7, -x_8, x_1, 0, -x_7, x_4, 0, 0] \\
X_8 &= [-x_8, 0, x_2, -x_7, -2x_8, x_5, 0, 0]
\end{aligned}$$

The reduced echelon form

$$\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 & \frac{2x_6^2 x_8 - x_3 x_7 x_6 - x_2 x_4 x_6 - x_3 x_4 x_5 + 2x_1 x_4 x_3 + 2x_1 x_5 x_6 - 2x_1^2 x_6}{3(-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2)} & \frac{2x_3 x_8 x_6 - x_3^2 x_7 - x_5 x_6 x_2 + x_3 x_5^2 - x_1 x_3 x_5 - x_2 x_4 x_3 + 2x_1 x_2 x_6}{3(x_3 x_5 x_6 - x_6 x_3 x_1 + x_4 x_3^2 - x_6^2 x_2)} \\
0 & 1 & 0 & 0 & 0 & 0 & -\frac{x_7 x_6^2 + x_4 x_6 x_1 - x_4^2 x_3}{-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2} & -\frac{-x_3 x_7 x_6 - x_3 x_4 x_5 + x_2 x_4 x_6}{-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2} \\
0 & 0 & 1 & 0 & 0 & 0 & -\frac{x_7 x_6 x_1 - x_7 x_4 x_3 - x_7 x_5 x_6 + x_6 x_4 x_8}{-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2} & -\frac{x_7 x_6 x_2 - x_4 x_3 x_8}{-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2} \\
0 & 0 & 0 & 1 & 0 & 0 & -\frac{x_1 x_2 x_6 + x_3 x_8 x_6 - x_2 x_4 x_3}{-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2} & -\frac{-x_3^2 x_8 - x_2 x_5 x_3 + x_2^2 x_6}{-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2} \\
0 & 0 & 0 & 0 & 1 & 0 & \frac{-x_2 x_4 x_6 + 2x_3 x_4 x_5 - x_1 x_4 x_3 - x_6^2 x_8 + 2x_3 x_7 x_6 - x_1 x_5 x_6 + x_1^2 x_6}{3(-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2)} & \frac{-2x_5 x_6 x_2 + 2x_3 x_5^2 - 2x_1 x_3 x_5 + x_3 x_8 x_6 - 2x_3^2 x_7 + x_2 x_4 x_3 + x_1 x_2 x_6}{3(-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2)} \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{x_7 x_6 x_2 - x_4 x_3 x_8}{-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2} & -\frac{x_8 x_6 x_2 - x_8 x_5 x_3 + x_8 x_3 x_1 - x_3 x_7 x_2}{-x_3 x_5 x_6 + x_6 x_3 x_1 - x_4 x_3^2 + x_6^2 x_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right]$$

leads to commuting operators, and implies that there are two joint invariants which can be found using Maple as

$$\begin{aligned}
I_1 &= x_5^2 + x_1^2 - x_1 x_5 + 3x_7 x_3 + 3x_8 x_6 + 3x_2 x_4 \\
I_2 &= 2x_1^3 - 3x_5 x_1^2 + 9x_2 x_4 x_1 - 3x_1 x_5^2 - 18x_1 x_8 x_6 + 9x_7 x_3 x_1 + 2x_5^3 + 9x_5 x_8 x_6 - 18x_7 x_5 x_3 + 9x_5 x_2 x_4 + 27x_7 x_6 x_2 + 27x_4 x_3 x_8
\end{aligned}$$

The adjoint representation of the basis of $\mathfrak{sl}(3, \mathbb{R})$ is

$$\begin{aligned}
X_1 &= [0, x_2, 2x_3, -x_4, 0, x_6, -2x_7, -x_8] \\
X_2 &= [x_4, x_5 - x_1, x_6, 0, -x_4, 0, 0, -x_7] \\
X_3 &= [x_7, x_8, -x_5 - 2x_1, 0, 0, -x_4, 0, 0] \\
X_4 &= [-x_2, 0, 0, -x_5 + x_1, x_2, x_3, -x_8, 0] \\
X_5 &= [0, -x_2, x_3, x_4, 0, 2x_6, -x_7, -2x_8] \\
X_6 &= [0, 0, -x_2, x_7, x_8, -2x_5 - x_1, 0, 0] \\
X_7 &= [-x_3, 0, 0, -x_6, 0, 0, x_5 + 2x_1, x_2] \\
X_8 &= [0, -x_3, 0, 0, -x_6, 0, x_4, 2x_5 + x_1]
\end{aligned}$$

The reduced echelon form

$$\left[\begin{array}{cccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & -\frac{-2x_3x_1x_4-x_3x_5x_4+x_4x_2x_6-x_6^2x_8-x_6x_5^2-x_6x_5x_1+2x_6x_1^2}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} & & & & -\frac{2x_2x_6x_1-x_3x_4x_2+x_6x_8x_3+x_6x_5x_2}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} \\ 0 & 1 & 0 & 0 & 0 & 0 & -\frac{x_4x_5x_6+2x_4x_6x_1+x_6^2x_7-x_4^2x_3}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} & & & & \frac{x_3x_1x_4+x_7x_3x_6+2x_3x_5x_4-x_4x_2x_6}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{x_6x_7x_1+x_6x_4x_8-x_7x_4x_3-x_6x_7x_5}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} & & & & -\frac{x_2x_6x_7-x_4x_8x_3}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{2x_2x_6x_1-x_3x_4x_2+x_6x_8x_3+x_6x_5x_2}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} & & & & \frac{x_2x_3x_1+x_3^2x_8+2x_3x_5x_2-x_2^2x_6}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{x_3x_1x_4+x_7x_3x_6+2x_3x_5x_4-x_4x_2x_6}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} & & & & -\frac{x_3x_1^2+x_2x_6x_1+x_3x_5x_1+x_3^2x_7-2x_3x_5^2-x_3x_4x_2+2x_6x_5x_2}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{x_2x_6x_7-x_4x_8x_3}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} & & & & -\frac{x_3x_8x_1+x_2x_8x_6-x_2x_7x_3-x_3x_8x_5}{x_3x_1x_6+x_6^2x_2-x_3^2x_4-x_5x_3x_6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 0 \end{array} \right]$$

leads to commuting operators, and implies that there are two joint invariants which can be found using Maple as

$$\begin{aligned} I_1 &= x_5^2 + x_1x_5 + x_1^2 + x_7x_3 + x_8x_6 + x_4x_2 \\ I_2 &= -x_1^2x_5 - x_1x_6x_8 + x_1x_4x_2 - x_1x_5^2 - x_3x_7x_5 + x_4x_8x_3 + x_2x_6x_7 + x_4x_5x_2 \end{aligned}$$

3.3. Invariants of forms of $\mathfrak{so}(4)$ in their adjoint and coadjoint representations. The basic invariants for real forms of $\mathfrak{so}(4)$ in suitable coordinates obtained as in 3.2 are

$\mathfrak{so}(4)$:

$$\begin{aligned} x_4^2 + x_3^2 + 2x_4x_3 + (x_1 + x_6)^2 + (x_5 - x_2)^2 \\ x_4^2 + x_3^2 - 2x_4x_3 + (x_1 - x_6)^2 + (x_5 + x_2)^2 \end{aligned}$$

$\mathfrak{so}(2, 2)$:

$$\begin{aligned} x_4^2 + x_3^2 - 2x_4x_3 + (x_5 + x_2)^2 - (x_1 - x_6)^2 \\ x_4^2 + x_3^2 + 2x_4x_3 + (x_2 - x_5)^2 - (x_1 + x_6)^2 \end{aligned}$$

$\mathfrak{so}(1, 3)$:

$$\begin{aligned} -x_4^2 + x_3^2 - 2Ix_4x_3 - (x_1 + Ix_6)^2 + (x_5 + Ix_2)^2 \\ -x_4^2 + x_3^2 + 2Ix_4x_3 - (x_1 - Ix_6)^2 + (x_5 - Ix_2)^2 \end{aligned}$$

The real invariants are obtained from taking the real and imaginary parts of either of the above two invariants.

3.4. Concluding remarks. The commuting vector fields which give the invariants of the exceptional groups can also be computed because explicit structure constants, which are programmable, are available – as indicated e.g in [Az1], [Az2, p. 9]; see also [Va].

The exceptional groups are also of interest to theoretical physicists [Ca], [Ra]. In certain cases, the joint invariants in the fundamental representations of certain exceptional groups can also be obtained algorithmically.

For example, one can realize D_4 is the Levi complement of a maximal parabolic subgroups of D_5 as in [ABS], use a choice of structure constants which are integers and use triality to obtain G_2 as a subgroup of D_5 with its maximal torus as a subgroup of a maximal torus of D_5 . Then the root vector corresponding to the simple root of D_5 which is not a simple root of D_4 would be a high weight vector for G_2 and it translates under G_2 would give the seven dimensional fundamental representation of G_2 .

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