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On the joint distribution of sample mean and variance in some multivariate distributions

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Abstract: A simpler proof of the joint moment generating function (MGF) of sample mean and variance in the multivariate normal case is presented, as a prelude to independence of sample mean and variance. Thanks to the joint MGF of sample mean and variance, a result is given for the MGF of a singular Wishart distribution. The Joint distribution of sample mean and variance is exhibited for a class of elliptically contoured distributions and the result is concluded by illustrations for the univariate Students t-distribution.

Key Words: Correlation; Elliptically contoured distribution; Multivariate normal; Independence; Sample mean; Sample variance.

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1 Introduction

Sample mean and variance based on normal population are independent. A number of proofs has been in the literature. For proofs based on Helmerts transformation, we refer to Rao (1973, 182) and that for depending on moment generating functions, we refer to Hogg and Craig (1978, 172) and Rohatgi (1984, 523). In case of multivariate normal distribution, the proof of statistical independence between sample mean and variance based on moment generating function is also known. We present a simple proof along the line of Laradji and Joarder (2014). It is known

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that the sample mean and variance are independent if and only if the sample comes from normal distribution (Zinger, 1951, Kagan, Linnik and Rao, 1973). It would be desirable to know if the sample mean and variance in other distributions are uncorrelated. Of particular interest would be to check the uncorrelation of distributions that shares similar properties as the normal distribution, say, for example, symmetry.

Consider a random sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, $(n = 2, 3, \ldots)$ from a p-dimensional cumulative distribution function (cdf) $\mathcal{F}$ with p-dimensional probability density function f. We define the sample mean vector $\bar{\boldsymbol{X}}$ and variance-covariance (variance for short) matrix $\boldsymbol{S}$ respectively by

$$\begin{aligned} \bar{\boldsymbol{X}} &= \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{X}_j = (\bar{X}_1, \ldots, \bar{X}_p)^T, \quad \bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}, \\ &\text{and} \\ \boldsymbol{S}_{\boldsymbol{x}} &= \frac{1}{n-1}\sum_{j=1}^{n} (\boldsymbol{X}_j - \bar{\boldsymbol{X}})(\boldsymbol{X}_j - \bar{\boldsymbol{X}})^T, \ n \geq 2. \end{aligned}$$

In Section 3, we provide a simple proof of joint moment generating function of sample mean and variance for broad spectrum of readers, where an effort is put to find the moment generating function of a singular Wishart distribution. Section 4 is allocated for the joint density of sample mean and variance in elliptically contoured distributions. Finally in Section 5, we provide some illustrations.

2 Preliminaries & Notation

We denote the space of all positive definite (pd) matrices of order p by $\mathcal{S}(p)$.

In the following we present a new and very simple proof of the independence of sample mean vector and variance matrix for independently and identically distributed (iid) normal random variables, extending the result of Laradji and Joarder (2014) to the multivariate case.

Theorem 2.1. *Let the random variables $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ $(n \geq 2)$ be iid according to $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma} \in \mathcal{S}(p)$. Then the joint mgf of the sample mean and variance satisfies*

$$\mathcal{M}_{\bar{\boldsymbol{X}}, \boldsymbol{S}_{\boldsymbol{x}}}(\boldsymbol{t}, \boldsymbol{T}) = \mathcal{M}_{\bar{\boldsymbol{X}}}(\boldsymbol{t}) \mathcal{M}_{\boldsymbol{S}_{\boldsymbol{x}}}(\boldsymbol{T}),$$

where $\boldsymbol{t} = (t_1, \ldots, t_p)^T \in \mathbb{R}^p$ and $\boldsymbol{T} = (T_{ij}) \in \mathcal{S}(p)$, $i, j = 1, \ldots, p$. Then $\bar{\boldsymbol{X}}$ and $\boldsymbol{S}_{\boldsymbol{x}}$ are independent.

Proof: Without loss of generality, we assume $\boldsymbol{\mu} = \boldsymbol{0}$, $\boldsymbol{\Sigma} = \boldsymbol{I}_p$. Then the joint mgf of $\bar{\boldsymbol{X}}$ and $\boldsymbol{S}_{\boldsymbol{x}}$ is given by

$$\begin{aligned} \mathcal{M}_{\bar{\boldsymbol{X}}, \boldsymbol{S}_{\boldsymbol{x}}}(\boldsymbol{t}, \boldsymbol{T}) &= \frac{1}{(2\pi)^{\frac{np}{2}}} \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} \exp\left(\boldsymbol{t}^T \bar{\boldsymbol{x}} + \operatorname{tr} \boldsymbol{T} \boldsymbol{s}_{\boldsymbol{x}}\right) \exp\left(-\frac{1}{2}\sum_{i=1}^{n} \boldsymbol{x}_i^T \boldsymbol{x}_i\right) \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_n \\ &= \frac{1}{(2\pi)^{\frac{np}{2}}} \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} \operatorname{etr}\left(\boldsymbol{T} \boldsymbol{s}_{\boldsymbol{x}}\right) \exp\left(-\frac{1}{2} Q\right) \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_n \end{aligned}$$

where $\operatorname{etr}(.) = \exp(\operatorname{tr}(.))$, $\bar{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i$, $(n-1)\boldsymbol{s}_{\boldsymbol{x}} = \sum_{i=1}^{n} (\boldsymbol{x}_i - \bar{\boldsymbol{x}})(\boldsymbol{x}_i - \bar{\boldsymbol{x}})^T$, and

$$Q = \sum_{i=1}^{n} \boldsymbol{x}_i^T \boldsymbol{x}_i - 2\boldsymbol{t}_1^T \bar{\boldsymbol{x}} = \sum_{i=1}^{n} \left(\boldsymbol{x}_i - \frac{\boldsymbol{t}_1}{n}\right)^T \left(\boldsymbol{x}_i - \frac{\boldsymbol{t}_1}{n}\right) - \boldsymbol{t}_1^T \boldsymbol{t}_1$$

Then, we obtain

$$\begin{aligned}\mathcal{M}_{\bar{\boldsymbol{X}},\boldsymbol{S_x}}(\boldsymbol{t},\boldsymbol{T}) &= \frac{1}{(2\pi)^{\frac{np}{2}}}\exp\left(\frac{\boldsymbol{t}_1^T\boldsymbol{t}_1}{2n}\right)\\ &\times\int_{\mathbb{R}^p}\cdots\int_{\mathbb{R}^p}\operatorname{etr}(\boldsymbol{T}\boldsymbol{s_x})\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(\boldsymbol{x}_i-\frac{\boldsymbol{t}_1}{n}\right)^T\left(\boldsymbol{x}_i-\frac{\boldsymbol{t}_1}{n}\right)\right)\mathrm{d}\boldsymbol{x}_1\ldots\mathrm{d}\boldsymbol{x}_n\end{aligned}$$

Now, make the transformation $\boldsymbol{u}_i=\boldsymbol{x}_i-\frac{1}{n}\boldsymbol{t}_1$, $(i=1,\ldots,n)$. Then we have $\boldsymbol{s_x}=\boldsymbol{s_u}$, and hence

$$\mathcal{M}_{\bar{\boldsymbol{X}},\boldsymbol{S_x}}(\boldsymbol{t},\boldsymbol{T})=\exp\left(\frac{\boldsymbol{t}_1^T\boldsymbol{t}_1}{2n}\right)I(\boldsymbol{T};\boldsymbol{u}), \tag{1}$$

where

$$I(\boldsymbol{T};\boldsymbol{u})=\int_{\mathbb{R}^p}\cdots\int_{\mathbb{R}^p}\frac{1}{(2\pi)^{\frac{np}{2}}}\operatorname{etr}(\boldsymbol{T}\boldsymbol{s_u})\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\boldsymbol{u}_i^T\boldsymbol{u}_i\right)\mathrm{d}\boldsymbol{u}_1\ldots\mathrm{d}\boldsymbol{u}_n.$$

Obviously, $I(\boldsymbol{T};\boldsymbol{u})=\mathcal{M}_{\boldsymbol{S_u}}(\boldsymbol{T})$ where $(n-1)\boldsymbol{S_u}=\sum_{i=1}^{n}(\boldsymbol{U}_i-\bar{\boldsymbol{U}})(\boldsymbol{U}_i-\bar{\boldsymbol{U}})^T$ and $\boldsymbol{U}_i\sim\mathcal{N}_p(\boldsymbol{0},\boldsymbol{I}_p)$, $(i=1,\ldots,n)$. Thus, $I(\boldsymbol{T};\boldsymbol{u})=I(\boldsymbol{T};\boldsymbol{x})=\mathcal{M}_{\boldsymbol{S_x}}(\boldsymbol{T})$ for the random sample $\boldsymbol{X}_i\sim\mathcal{N}_p(\boldsymbol{0},\boldsymbol{I}_p)$, $(i=1,\ldots,n)$. Inasmuch as $\mathcal{M}_{\bar{\boldsymbol{X}}}(\boldsymbol{t}_1)=\exp\left(\frac{\boldsymbol{t}_1^T\boldsymbol{t}_1}{2n}\right)$, in this case; it follows from (1), that $\mathcal{M}_{\bar{\boldsymbol{X}},\boldsymbol{S_x}}(\boldsymbol{t},\boldsymbol{T})=\mathcal{M}_{\bar{\boldsymbol{X}}}(\boldsymbol{t})\mathcal{M}_{\boldsymbol{S_x}}(\boldsymbol{T})$. By uniqueness property of the mgf, this proves that the sample mean and variance are independent. ■

3 Joint Moment Generating Function of Sample Mean and Variance in Multivariate Normal Distribution

In this section, we assume that we have a iid sample from a p-dimensional (dim) normal ($\mathcal{N}_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$) distribution. We present an inductive proof of the joint moment generating function of sample mean and variance. We will require the following theorem which is an extension to the result of Hogg and Craig (1978, 84-85).

Theorem 3.1. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be p-dim random vector and $p\times p$-dim random matrix respectively, that have the joint pdf $f_{\boldsymbol{X},\boldsymbol{Y}}(.,.)$ and the marginal pdfs $f_{\boldsymbol{X}}(.)$ and $f_{\boldsymbol{Y}}(.)$ respectively. Furthermore, let $\mathcal{M}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{t},\boldsymbol{T})$ be the mgf of the distribution. Then $\boldsymbol{X}$ and $\boldsymbol{Y}$ are stochastically independent iff*

$$\mathcal{M}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{t},\boldsymbol{T})=\mathcal{M}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{t},\boldsymbol{0})\mathcal{M}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{0},\boldsymbol{T})$$

Proof: Without loss of generality, suppose both random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are continuous. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be independent. Then

$$\begin{aligned}\mathcal{M}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{t},\boldsymbol{T}) &= E\left[\exp\left(\boldsymbol{t}^T\boldsymbol{X}+\operatorname{tr}\boldsymbol{T}\boldsymbol{Y}\right)\right]\\ &= \mathcal{M}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{t},\boldsymbol{0})\mathcal{M}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{0},\boldsymbol{T}).\end{aligned}$$

Conversely, if $\mathcal{M}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{t},\boldsymbol{T})=\mathcal{M}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{t},\boldsymbol{0})\mathcal{M}_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{0},\boldsymbol{T})$, then

$$\int_{\boldsymbol{y}\in\mathcal{S}(p)}\int_{\boldsymbol{x}\in\mathbb{R}^p}e^{\boldsymbol{t}_1^T\boldsymbol{x}+\operatorname{tr}\boldsymbol{T}\boldsymbol{y}}f_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} = \left[\int_{\mathbb{R}^p}e^{\boldsymbol{t}^T\boldsymbol{x}}f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right]\left[\int_{\mathcal{S}(p)}e^{\operatorname{tr}\boldsymbol{T}\boldsymbol{y}}f_{\boldsymbol{Y}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right]$$

$$= \int_{\mathcal{S}(p)} \int_{\mathbb{R}^p} e^{\boldsymbol{t}^T \boldsymbol{x} + \operatorname{tr} \boldsymbol{T}\boldsymbol{y}} f_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{Y}}(\boldsymbol{y}) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}.$$

By the uniqueness of mgf we must have $f_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{x},\boldsymbol{y}) = f_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{Y}}(\boldsymbol{y})$. It follows that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent.

The following result is an attempt to find the mgf of the singular Wishart distribution.

Theorem 3.2. *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n, \boldsymbol{X}_{n+1}$ be iid observations of size $n+1 \geq 3$ from $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma} \in \mathcal{S}(p)$. Further, let $\boldsymbol{U} = \frac{n(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}})(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}})^T}{(n+1) \operatorname{tr} \boldsymbol{\Sigma}}$. Then*

$$\mathcal{M}_{\boldsymbol{U}} \left(\frac{\operatorname{tr} \boldsymbol{\Sigma}}{n} \boldsymbol{t}_2 \right) = |\boldsymbol{I}_p - \frac{2}{n} \boldsymbol{U} \boldsymbol{\Sigma}|^{-\frac{1}{2}}$$

Proof: For our purpose and notational convenience, let $\bar{\boldsymbol{X}}_{n+1}$ and $\boldsymbol{S}_{n+1}$ denote the sample mean and variance based on $n+1$ observations. Then we have the following relations

$$\begin{aligned} \bar{\boldsymbol{X}}_{n+1} &= \frac{n}{n+1} \bar{\boldsymbol{X}} + \frac{1}{n+1} \boldsymbol{X}_{n+1}, \\ \text{and} \qquad \boldsymbol{S}_{n+1} &= \frac{n-1}{n} \boldsymbol{S}_{\boldsymbol{x}} + \frac{1}{n+1} (\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}})(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}})^T. \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathcal{M}_{\bar{\boldsymbol{X}}_{n+1}, \boldsymbol{S}_{n+1}}(\boldsymbol{t}, \boldsymbol{T}) &= E\left[\exp\left(\boldsymbol{t}^T \bar{\boldsymbol{X}}_{n+1} + \operatorname{tr} \boldsymbol{T} \boldsymbol{S}_{n+1}\right)\right] \\ &= E\left\{ \boldsymbol{t}^T \left(\frac{n}{n+1} \bar{\boldsymbol{X}} + \frac{1}{n+1} \boldsymbol{X}_{n+1} \right) \right. \\ &\quad \left. + \operatorname{tr} \boldsymbol{T} \left(\left(1 - \frac{1}{n}\right) \boldsymbol{S}_{\boldsymbol{x}} + \frac{1}{n+1} (\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}})(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}})^T \right) \right\} \\ &= \mathcal{M}_{\bar{\boldsymbol{X}}, \boldsymbol{S}_{\boldsymbol{x}}} \left(\frac{n}{n+1} \boldsymbol{t}, \frac{n-1}{n} \boldsymbol{T} \right) \mathcal{M}_{\boldsymbol{X}_{n+1}} \left(\frac{1}{n+1} \boldsymbol{t} \right) \mathcal{M}_{\boldsymbol{U}} \left(\frac{\operatorname{tr} \boldsymbol{\Sigma}}{n} \boldsymbol{T} \right). \end{aligned} \tag{1}$$

According to Theorem 2.1, and using the facts that $\boldsymbol{S}_{\boldsymbol{x}} \sim W_p(n-1, \boldsymbol{\Sigma}/(n-1))$,

$$\begin{aligned} \mathcal{M}_{\bar{\boldsymbol{X}}}(\boldsymbol{t}) &= \exp\left(\boldsymbol{t}^T \boldsymbol{\mu} + \frac{1}{2n} \boldsymbol{t}^T \boldsymbol{\Sigma} \boldsymbol{t} \right) \\ \mathcal{M}_{\boldsymbol{S}_{\boldsymbol{x}}}(\boldsymbol{T}) &= |\boldsymbol{I}_p - \frac{2}{n-1} \boldsymbol{U} \boldsymbol{\Sigma}|^{-\frac{1}{2}(n-1)}, \end{aligned}$$

where $\boldsymbol{U} = \boldsymbol{U}^T$, $u_{jj} = t_{jj}$ and $u_{jk} = u_{kj} = \frac{1}{2} t_{jk}$ $(j < k)$, we have

$$\mathcal{M}_{\bar{\boldsymbol{X}}, \boldsymbol{S}_{\boldsymbol{x}}}(\boldsymbol{t}, \boldsymbol{T}) = \exp\left(\boldsymbol{t}^T \boldsymbol{\mu} + \frac{1}{2n} \boldsymbol{t}^T \boldsymbol{\Sigma} \boldsymbol{t} \right) |\boldsymbol{I}_p - \frac{2}{n-1} \boldsymbol{U} \boldsymbol{\Sigma}|^{-\frac{1}{2}(n-1)}.$$

Thus,

$$\begin{aligned} \mathcal{M}_{\bar{\boldsymbol{X}}, \boldsymbol{S}_{\boldsymbol{x}}} \left(\frac{n}{n+1} \boldsymbol{t}, \frac{n-1}{n} \boldsymbol{T} \right) &= \exp\left(\frac{n}{n+1} \boldsymbol{t}^T \boldsymbol{\mu} + \frac{n}{2(n+1)^2} \boldsymbol{t}^T \boldsymbol{\Sigma} \boldsymbol{t} \right) |\boldsymbol{I}_p - \frac{2}{n} \boldsymbol{U} \boldsymbol{\Sigma}|^{-\frac{1}{2}(n-1)} \\ \mathcal{M}_{\bar{\boldsymbol{X}}_{n+1}, \boldsymbol{S}_{n+1}}(\boldsymbol{t}, \boldsymbol{T}) &= \exp\left(\boldsymbol{t}^T \boldsymbol{\mu} + \frac{1}{2(n+1)} \boldsymbol{t}^T \boldsymbol{\Sigma} \boldsymbol{t} \right) |\boldsymbol{I}_p - \frac{2}{n} \boldsymbol{U} \boldsymbol{\Sigma}|^{-\frac{1}{2}n}. \end{aligned} \tag{2}$$

Substituting (2) in (1), and after some algebra, yields the results.

4 Joint Distribution of Sample Mean and Variance for a Class of Elliptically Contoured Distributions (ECD)

Firstly, we give a general definition of ECD and secondly, the weighting representation of ECDs due to Chu (1973) will be presented. This representation suggests a broader class than that of variance mixture of normal distributions, since the weights are not always positive, however in the latter class we involve positive weights; see Arashi et al. (2013) for more details.

Definition 4.1. *It is said that the random vector $\boldsymbol{X}$ has a p-dim elliptically contoured distribution with location $\boldsymbol{\mu} \in \mathbb{R}^p$, scale $\boldsymbol{\Sigma} \in \mathcal{S}(p)$ and density generator $g : \mathbb{R}^+ \to \mathbb{R}^+$, if it has the following density function*

$$f(\boldsymbol{x}) = |\boldsymbol{\Sigma}|^{-\frac{1}{2}} g\left\{(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$

In this case, we designate $\boldsymbol{X} \sim \mathcal{E}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$.

In the following, the mixture (weighting) representation of the density function of ECDs, due to Chu (1973) is provided.

Lemma 4.1. *Let $\boldsymbol{X} \sim \mathcal{E}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$, $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma} \in \mathcal{S}(p)$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$. Then the density function of $\boldsymbol{X}$ has the following weighting representation*

$$f(\boldsymbol{x}) = \int w(t) h_{\mathcal{N}_p(\boldsymbol{\mu}, t^{-1}\boldsymbol{\Sigma})}(\boldsymbol{x}) \mathrm{d}t \tag{1}$$

where $w(t)$ is the weighting function given by

$$w(t) = (2\pi)^{\frac{p}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}} t^{-\frac{p}{2}} \mathcal{L}^{-1}[f(s)], \quad s = \frac{1}{2}\left(\boldsymbol{x}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{x}\right) \tag{2}$$

where $\mathcal{L}^{-1}[f(s)]$ denotes the inverse Laplace transform of $f(s)$.

Not necessarily that all the distributions in the ECD class as in Definition 4.1 satisfy the weighting representation. However, most well-known distributions have this representation feature. See Arashi et al. (2013) and references therein for details and applications.

Theorem 4.1. *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be iid observations of size $n \geq 2$ from $\mathcal{E}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$, where $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma} \in \mathcal{S}(p)$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$. Then the joint density of $(\bar{\boldsymbol{X}}, \boldsymbol{S_x})$ is given by*

$$f_{\bar{\boldsymbol{X}}, \boldsymbol{S_x}}(\bar{\boldsymbol{x}}, \boldsymbol{s_x}) = \frac{(n-1)^{\frac{p(n-1)}{2}} |\boldsymbol{\Sigma}|^{-\frac{n}{2}}}{\pi^{\frac{p}{2}} 2^{\frac{np}{2}} \Gamma_p\left(\frac{n-1}{2}\right)} |\boldsymbol{s_x}|^{\frac{n-p-2}{2}} \int t^{\frac{np}{2}} \operatorname{etr}\left(-\frac{1}{2} t \boldsymbol{H}\right) w(t) \mathrm{d}t$$

where

$$\boldsymbol{H} = \boldsymbol{\Sigma}^{-1}\left[n(\bar{\boldsymbol{x}}-\boldsymbol{\mu})(\bar{\boldsymbol{x}}-\boldsymbol{\mu})^T + (n-1)\boldsymbol{s_x}\right]$$

Proof: Suppose that primarily we have a random sample from $\mathcal{N}_p(\boldsymbol{\mu}, t^{-1}\boldsymbol{\Sigma})$. Since under normality $\bar{\boldsymbol{X}}$ and $\boldsymbol{S_x}$ are independent (see Theorem 2.1), the joint density of $(\bar{\boldsymbol{X}}, \boldsymbol{S_x})$, under normality, is given by

$$\begin{aligned} f^t_{\bar{\boldsymbol{X}},\boldsymbol{S_x}}(\bar{\boldsymbol{x}}, \boldsymbol{s_x}) &= f^t_{\bar{\boldsymbol{X}}}(\bar{\boldsymbol{x}}) f^t_{\boldsymbol{S_x}}(\boldsymbol{s_x}) \\ &= \frac{(n-1)^{\frac{p(n-1)}{2}} t^{\frac{np}{2}} |\boldsymbol{\Sigma}|^{-\frac{n}{2}}}{\pi^{\frac{p}{2}} 2^{\frac{np}{2}} \Gamma_p\left(\frac{n-1}{2}\right)} |\boldsymbol{s_x}|^{\frac{n-p-2}{2}} \\ &\quad \times \operatorname{etr}\left\{-\frac{t}{2}\boldsymbol{\Sigma}^{-1}\left[n(\bar{\boldsymbol{x}}-\boldsymbol{\mu})(\bar{\boldsymbol{x}}-\boldsymbol{\mu})^T + (n-1)\boldsymbol{s_x}\right]\right\} \end{aligned}$$

Thus, applying Lemma 4.1, the joint density of $(\bar{\boldsymbol{X}}, \boldsymbol{S_x})$ is achieved as

$$f_{\bar{\boldsymbol{X}},\boldsymbol{S_x}}(\bar{\boldsymbol{x}}, \boldsymbol{s_x}) = \int w(t) f^t_{\bar{\boldsymbol{X}},\boldsymbol{S_x}}(\bar{\boldsymbol{x}}, \boldsymbol{s_x}) \mathrm{d}t$$

which with some algebra, completes the proof.

4.1 Joint Distribution of Sample Mean and Variance for the t-Distribution

Definition 4.2. *It is said that the random vector $\boldsymbol{X}$ has p-dimensional t-distribution with location $\boldsymbol{\mu} \in \mathbb{R}^p$, scale $\boldsymbol{\Sigma} \in \mathcal{S}(p)$ and degrees of freedom $\gamma_o \geq 1$, if it has the following density function*

$$f(\boldsymbol{x}) = \frac{\Gamma\left(\frac{\gamma_o+p}{2}\right)|\boldsymbol{\Sigma}|^{-\frac{1}{2}}}{(\pi\gamma_o)^{\frac{p}{2}}\Gamma\left(\frac{\gamma_o}{2}\right)}\left\{1 + \frac{1}{\gamma_o}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}^{-\frac{1}{2}(\gamma_o+p)}$$

In this case, we designate $\boldsymbol{X} \sim M_t^{(p)}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \gamma_o)$.

Since the t-distribution is a member of elliptical class, according to Lemma 4.1, the density function of $\boldsymbol{X} \sim M_t^{(p)}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \gamma_o)$ has a weighting representation where $w(t)$ is the inverse gamma distribution with

$$\left\{\begin{array}{c} w(t) = \frac{1}{\Gamma\left(\frac{\gamma_o}{2}\right)}\left(\frac{\gamma_o t}{2}\right)^{\frac{\gamma_o}{2}} e^{-\frac{\gamma_o t}{2}} t^{-1}, \quad t \in \mathbb{R}^+ \\ \text{and} \\ \kappa^{(h)} = E(t^{-h}) = \int \left(\frac{1}{h}\right)^k w(t)\mathrm{d}t = \left(\frac{\gamma_o}{2}\right)^h \left(\frac{\Gamma\left(\frac{\gamma_o}{2}-h\right)}{\Gamma\left(\frac{\gamma_o}{2}\right)}\right) \end{array}\right. \tag{3}$$

In the following we give the joint density of $(\bar{\boldsymbol{X}}, \boldsymbol{S_x})$ for a t-distributed random sample.

Theorem 4.2. *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be iid observations of size $n \geq 2$ from $M_t^{(p)}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \gamma_o)$, where $\gamma_o \geq 1$, $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma} \in \mathcal{S}(p)$. Then the joint density of $(\bar{\boldsymbol{X}}, \boldsymbol{S_x})$ is given by*

$$\begin{aligned} f_{\bar{\boldsymbol{X}},\boldsymbol{S_x}}(\bar{\boldsymbol{x}}, \boldsymbol{s_x}) &= \frac{\Gamma\left(\frac{np+\gamma_o}{2}\right)}{\gamma_o^{\frac{np}{2}}\Gamma\left(\frac{\gamma_o}{2}\right)} \frac{(n-1)^{\frac{p(n-1)}{2}}|\boldsymbol{\Sigma}|^{-\frac{n}{2}}}{\pi^{\frac{p}{2}}\Gamma_p\left(\frac{n-1}{2}\right)} \\ &\quad \times |\boldsymbol{s_x}|^{\frac{n-p}{2}-1}\left\{1 + \frac{1}{\gamma_o}\left(\operatorname{tr}\boldsymbol{\Sigma}^{-1}\left[n(\bar{\boldsymbol{x}}-\boldsymbol{\mu})(\bar{\boldsymbol{x}}-\boldsymbol{\mu})^T + (n-1)\boldsymbol{s_x}\right]\right)\right\}^{-\frac{1}{2}(np+\gamma_o)} \end{aligned}$$

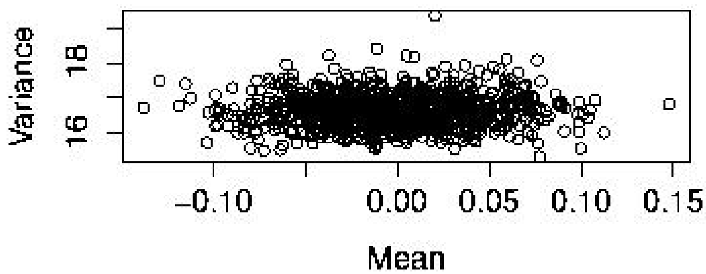

Figure 1: Mean vs Variance for $n = 10000$, $\mu = 0$, $\sigma^2 = 10$, $\gamma_o = 5$

Proof: Making use of the result of Theorem 4.1 and Eq. (3), we have

$$\begin{aligned} f_{\bar{\boldsymbol{X}},\boldsymbol{S_x}}(\bar{\boldsymbol{x}}, \boldsymbol{s_x}) &= \frac{(n-1)^{\frac{p(n-1)}{2}}|\boldsymbol{\Sigma}|^{-\frac{n}{2}}}{\pi^{\frac{p}{2}}2^{\frac{np}{2}}\Gamma_p\left(\frac{n-1}{2}\right)}|\boldsymbol{s_x}|^{\frac{n-p-2}{2}}\int t^{\frac{np}{2}}\operatorname{etr}\left(-\frac{1}{2}t\boldsymbol{H}\right)w(t)\mathrm{d}t \\ &= \frac{1}{\Gamma\left(\frac{\gamma_o}{2}\right)}\left(\frac{\gamma_o}{2}\right)^{\frac{\gamma_o}{2}}\frac{(n-1)^{\frac{p(n-1)}{2}}|\boldsymbol{\Sigma}|^{-\frac{n}{2}}}{\pi^{\frac{p}{2}}2^{\frac{np}{2}}\Gamma_p\left(\frac{n-1}{2}\right)}|\boldsymbol{s_x}|^{\frac{n-p-2}{2}}\left(2[\operatorname{tr}\boldsymbol{H}+\gamma_o]^{-1}\right)^{\frac{np+\gamma_o}{2}}\Gamma\left(\frac{np+\gamma_o}{2}\right), \end{aligned}$$

where $\boldsymbol{H} = \boldsymbol{\Sigma}^{-1}\left[n(\bar{\boldsymbol{x}}-\boldsymbol{\mu})(\bar{\boldsymbol{x}}-\boldsymbol{\mu})^T + (n-1)\boldsymbol{s_x}\right]$. After simplifications, the result follows.

Corollary 4.2.1. *For the special case $p = 1$ and $\gamma_o = n-1$, the joint density of $(\bar{X}, S_x^2)$, according to the random sample $X_1, \ldots, X_n$ from $t(\mu, \sigma^2, \gamma_o)$, where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, and $n \geq 2$, after some simplification, is given by*

$$\begin{aligned} f_{\bar{X},S_x^2}(\bar{x}, s_x^2) &= \left(\frac{\gamma_o}{2}\right)^{\gamma_o}\frac{\sqrt{n}\Gamma\left(\gamma_o+\frac{1}{2}\right)}{\sqrt{2\pi}\sigma^n\Gamma^2\left(\frac{n-1}{2}\right)} \\ &\quad\times\left(s_x^2\right)^{\frac{\gamma_o}{2}-1}\left[\frac{1}{2\sigma^2}\left(n(\bar{x}-\mu)^2+(n-1)s_x^2+\gamma_o\sigma^2\right)\right]^{-\left(\gamma_o+\frac{1}{2}\right)}, \quad \gamma_o = n-1 \geq 1. \end{aligned}$$

5 Illustrations

In this section, we draw samples from a univariate Student's t-distribution, calculate mean, variance and correlation for different degrees of freedom, to show the uncorrelation structures.

From Figures 1-2, it can be seen that the sample mean and variance are much scattered as the degrees of freedom gets larger. In Table 1, the sample correlation coefficient r between sample mean $\bar{x}$ and sample variance s_x^2 of a sample of size $n = 10000$ observations from $t(0, \sigma^2, \gamma_o)$ is tabulated using a Monte Carlo simulation with 1000 replications. Smaller values of the sample correlation coefficient between the sample mean and variance is indicative of their uncorrelation.

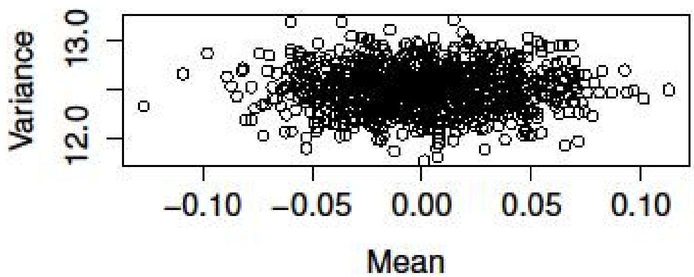

Figure 2: Mean vs Variance for $n = 10000$, $\mu = 0$, $\sigma^2 = 10$, $\gamma_o = 10$

σ^2	γ_o	r	r^2
1	5	0.0101	0.0001
	10	-0.0201	0.0004
	15	0.0051	0.0000
	100	-0.0350	0.0012
	1000	0.0311	0.0009
10	5	-0.0111	0.0001
	10	0.0217	0.0004
	15	-0.0154	0.0002
	100	0.0156	0.0002
	1000	-0.0359	0.0012
100	5	0.0113	0.0001
	10	-0.0240	0.0005
	100	0.0211	0.0004
	1000	0.0112	0.0001

Table 1: Sample correlation coefficient between sample mean and sample variance for different values σ^2 and γ_o.

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