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Invariants of third-order ordinary differential equations $y''' = f(x, y, y', y'')$ via point transformations

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Abstract

New systematic method to find the relative invariant differentiation operators is developed. We incorporate this new approach with Lie's infinitesimal method to study the general class $y''' = f(x, y, y', y'')$ under general point equivalence transformations in the generic case. As a result, all third-order differential invariants, relative and absolute invariant differentiation operators are determined for third-order ODEs $y''' = f(x, y, y', y'')$, which are not quadratic in the second-order derivative. These relative invariant differentiation operators are used to determine the fourth-order differential invariants and absolute invariant differentiation operators in a degenerate case of interest. As an application, invariant descriptions of all the canonical forms in the complex plane with four infinitesimal symmetries for third-order ODEs $y''' = f(x, y, y', y'')$, which are not quadratic in the second-order derivative, are provided.

Keywords: Lie's infinitesimal method, differential invariants, third-order ODEs, equivalence problem, point transformations, relative and absolute invariant differentiation operators.

1 Introduction

Differential invariants are of great importance in differential geometry [3], differential equations and variational structures [4, 5, 6, 7, 3, 8, 9, 10] as well as in general mathematical physics and various applications.

Lie [11, 12] was the first to show that that every invariant system of differential equations and variational problem [14] is expressible in terms of differential invariants. Lie [12] illustrated the use of differential invariants to integrate ODEs and then he completely classified all differential invariants for all possible finite-dimensional Lie groups of point transformations in the complex domain.

Tressé [15] and later Ovsiannikov [6], Olver [3] and Ibragimov [16] provided essential results on invariant differentiations and the existence of finite bases of differential invariants. We mention here that in Ibragimov [16, 18], a simple method for the construction of differential invariants for families of linear and nonlinear differential equations admitting infinite equivalence transformation groups was developed.

As a survey of known recent results, it is worth stating that Lie's infinitesimal method was applied to solve the equivalence problem for many significant classes of differential equations [19, 20, 21, 22, 23, 24, 25, 26, 27]. At the same time we mention that the Cartan's equivalence method [3, 28] has also been utilized for solution of equivalence problems for differential equations.

The group classification of scalar third-order ODEs have been investigated. Mahomed and Leach [30] obtained the canonical forms of scalar third-order equations that admit three-dimensional Lie algebras. Then Ibragimov and Mahomed [17] presented the complete group classification of such ODEs in the complex and real domains. The invariant description of such ODEs have only been performed for special cases as we discuss below. Our main purpose is to generalize this to scalar third-order ODEs which admit point symmetries and to provide an invariant description of the symmetry classes in the complex domain.

The linearization problem is a particular case of the equivalence problem which has attracted large attention since Lie's initial investigation of scalar second-order differential equations which are linearizable by point transformations [29].

Invariant linearization criteria for the third-order ODE class

$$y''' = a(x,y,y')y''^3 + b(x,y,y')y''^2 + c(x,y,y')y'' + d(x,y,y') \tag{1.1}$$

at most cubic in the second-order derivative has been studied in [31, 32] by Cartan's method and then in [23] by the direct approach. Lie [13] noted that the third-order ODE connected via contact transformations to the simplest linear third-order ODE is of form given in (1.1). Bagderina [2] presented the basis of differential invariants under the group of contact transformations for the class of ODEs at most cubic in the second-order derivative (1.1) by utilizing Lie's approach. The operators of invariant differentiations were also provided [2].

The equivalence problem for third-order ODEs which are at most quadratic in the second-order derivative, viz.

$$y''' = a(x,y,y')y''^2 + b(x,y,y')y'' + c(x,y,y') \tag{1.2}$$

with respect to the group of point equivalence transformations

$$\bar{x} = \phi(x,y), \bar{y} = \psi(x,y) \tag{1.3}$$

was investigated in Bagderina [1] via Lie's infinitesimal method.

The equivalence problem for scalar third-order ODEs under fiber preserving point transformations of dimension 4, 5, 6, and 7 was studied by Grebot [33]. The Cartan equivalence method was used.

Singularity analysis for scalar third-order were investigated as well. Here worth mentioning is that scalar third-order ODEs possessing the Painlevé property for polynomial in its lower order derivatives were investigated in [34, 35].

As an extension of the algebraic and geometric approaches, herein we invoke Lie's infinitesimal method to study differential invariants of third-order ODEs

$$y''' = f(x,y,y',y''), \tag{1.4}$$

which are not in general quadratic in the second-order derivative, under general point equivalence transformations (1.3).

The layout of this paper is as follows. In the next section, we present a concise description of Lie's infinitesimal method to find differential invariants and invariant differentiation operators of the class of ODEs (1.4) with respect to the general group of point equivalence transformations $\bar{x} = \phi(x,y), \bar{y} = \psi(x,y)$. Then in Section 3, we recover the infinitesimal point equivalence transformations. Then in Section 4, using the method described in Section 2, first, we find the third-order differential invariants, absolute and relative invariant differentiation operators of the class of ODEs (1.4), which are not quadratic in the second-order derivative, under the general group of point equivalence transformations. In Section 5, fourth-order differential invariants and absolute invariant differentiation operators are given for a degenerate case of interest. Section 6 provides illustrative examples of equations not quadratic in the second-order derivative taken from the works [17, 36]. This is motivated by studies of this general class for its symmetry group classification in [17] as well as interest in physics [36] for Einstein-Weyl geometry of hyper-CR type. Finally the conclusion is presented.

In the sequel, we use the notation $A = [a_1, a_2, ..., a_n]$ to express any differential operator $A = \sum_{j=1}^{n} a_j \frac{\partial}{\partial b_j}$. We denote y', y'' by p, q, respectively.

2 Lie's infinitesimal method

We briefly describe the Lie method which is invoked in the sequel to derive differential invariants by using point equivalence transformations.

Consider now the kth-order system of PDEs in n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$, viz.

$$E_\alpha(x, u, ..., u_{(k)}) = 0, \quad \alpha = 1, ..., m \ , \tag{2.5}$$

where $u_{(1)}, u_{(2)}, ..., u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, i.e., $u^\alpha_i = D_i(u^\alpha)$, $u^\alpha_{ij} = D_j D_i(u^\alpha)$,...,respectively, in which the operator of total differentiation with respect to x^i is

$$D_i = \tfrac{\partial}{\partial x^i} + u^\alpha_i \tfrac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \tfrac{\partial}{\partial u^\alpha_j} + ..., \quad i = 1, ..., n \ , \tag{2.6}$$

with summation implied for repeated indices.

The Lie-Bäcklund operator is

$$X = \xi^i \tfrac{\partial}{\partial x^i} + \eta^\alpha \tfrac{\partial}{\partial u^\alpha} \quad \xi^i, \eta^\alpha \in \mathcal{A} \ , \tag{2.7}$$

where $\mathcal{A}$ is the space of *differential functions.*

The operator (2.7) is an abbreviated form of the infinite formal sum

$$\begin{aligned} X^{(s)} &= \xi^i \tfrac{\partial}{\partial x^i} + \eta^\alpha \tfrac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1 i_2 \dots i_s} \tfrac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \\ &= \xi^i D_i + W^\alpha \tfrac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \tfrac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \end{aligned} \tag{2.8}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} &\zeta^\alpha_i = D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j) = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \\ &\zeta^\alpha_{i_1 \dots i_s} = D_{i_s}(\zeta^\alpha_{i_1 \dots i_{s-1}}) - u^\alpha_{j i_1 \dots i_{s-1}} D_{i_s}(\xi^j) = D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \dots i_s}, \quad s > 1, \end{aligned} \tag{2.9}$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{2.10}$$

Also it is well-known that *the point equivalence transformation of a class of PDEs (2.5)* is an invertible transformation of the independent and dependent variables of the form

$$\bar{x} = \phi(x, u), \bar{u} = \psi(x, u), \tag{2.11}$$

which maps every equation of the class into an equation of the same family, viz.

$$E_\alpha(\bar{x}, \bar{u}, \dots, \bar{u}_{(k)}) = 0, \quad \alpha = 1, \dots, m. \tag{2.12}$$

In order to describe Lie's infinitesimal method for deriving differential invariants using point equivalence transformations, we invoke the class of equations (1.4). It is well-known that the point equivalence transformation

$$\bar{x} = \phi(x, y), \bar{y} = \psi(x, y), \tag{2.13}$$

maps (1.4) into itself, i.e.

$$\bar{y}''' = \bar{f}(\bar{x}, \bar{y}, \bar{y}', \bar{y}''), \tag{2.14}$$

for arbitrary functions $\phi(x, y)$ and $\psi(x, y)$, in which $\bar{f}$ in general, can be different from the original function f. The set of all equivalence transformations forms a group denoted by $\mathcal{E}$.

The standard procedure for Lie's infinitesimal invariance criterion [6] is implemented in the following section to recover the continuous group of point equivalence transformations (2.13) for the class of third-order ODEs (1.4) with the corresponding infinitesimal point equivalence transformation operator

$$Y = \xi(x, y)D_x + W\partial_y + D_x(W)\partial_p + D_x^2(W)\partial_q + \mu(x, y, p, q, f)\partial_f, \tag{2.15}$$

where $\xi(x, y)$ and $\eta(x, y)$ are arbitrary functions obtained from

$$\bar{x} = x + \epsilon\xi(x, y) + O(\epsilon^2) = \phi(x, y), \tag{2.16}$$

$$\bar{y} = y + \epsilon\eta(x, y) + O(\epsilon^2) = \psi(x, y), \tag{2.17}$$

and

$$\mu = \dot{D}_x^3(W) + \xi(x, y)\dot{D}_x f, \tag{2.18}$$

with $W = \eta - \xi p$ and $\dot{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + f\frac{\partial}{\partial q}$.

Definition 2.1. *An invariant of a class of third-order ODEs (1.4)* is a function of the form

$$J = J(x, y, p, q, f), \tag{2.19}$$

which is invariant under the equivalence transformation (2.13).

Definition 2.2. *A differential invariant of order n of a class of third-order ODEs (1.4)* is a function of the form

$$J = J(x, y, p, q, f, f_{(1)}, f_{(2)}, ..., f_{(n)}), \tag{2.20}$$

which is invariant under the equivalence transformation (2.13) where $f_{(1)}, f_{(2)}, ..., f_{(n)}$ denote the collections of all first, second,..., nth-order partial derivatives.

Definition 2.3. *An invariant system of order n of a class of third-order ODEs (1.4)* is the system of the form $E_\alpha(x, y, p, q, f, f_{(1)}, f_{(2)}, ..., f_{(n)}) = 0, \ \alpha = 1, ..., m$ which satisfies the condition

$$Y^{(n)}E_\alpha(x, y, p, q, f, f_{(1)}, f_{(2)}, ..., f_{(n)}) = 0 \ (mod\ E_\alpha = 0, \ \alpha = 1, ..., m), \ \alpha = 1, ..., m. \tag{2.21}$$

An invariant system with $\alpha = 1$ is called an invariant equation.

Now, according to the theory of invariants of infinite transformation groups [6], the invariant criterion

$$YJ(x, y, p, q, f) = 0, \tag{2.22}$$

should be split by means of the functions $\xi(x, y)$ and $\eta(x, y)$ and their derivatives. This gives rise to a homogeneous linear system of PDEs whose solution gives the required invariants.

It should be noted that since the generator Y contains arbitrary functions $\xi(x, y)$ and $\eta(x, y)$, the corresponding identity (2.22) leads to m linear PDEs for J, where m is the number of the arbitrary functions and their derivatives that appear in Y. We point out that these m PDEs are not necessarily linearly independent.

In order to determine the differential invariants of order s, we need to calculate the prolongations of the operator Y using (2.8) by considering f as a dependent variable and the variables x, y, p, q as independent variables:

$$\begin{aligned} Y^{(s)} &= Y(x)\tilde{D}_x + Y(y)\tilde{D}_y + Y(p)\tilde{D}_p + Y(q)\tilde{D}_q + \tilde{W}\tfrac{\partial}{\partial f} + \sum_{s\geq 1} \tilde{D}_{i_1}...\tilde{D}_{i_s}(\tilde{W})\tfrac{\partial}{\partial f_{i_1 i_2 \cdots i_s}}, \\ & i_1, i_2, ..., i_s \in \{x, y, p, q\}, \end{aligned} \tag{2.23}$$

where

$$\tilde{D}_k = \partial_k + f_k\partial_f + f_{ki}\partial_{f_i} + f_{kij}\partial_{f_{ij}} + ..., \quad i,j,k \in \{x,y,p,q\}. \tag{2.24}$$

in which $\tilde{W}$ is the *Lie characteristic function*

$$\tilde{W} = \mu - Y(x)f_x - Y(y)f_y - Y(p)f_p - Y(q)f_q. \tag{2.25}$$

The differential invariants are determined by the equations

$$Y^{(s)}J(x,y,p,q,f,f_{(1)},f_{(2)},...,f_{(s)}) = 0. \tag{2.26}$$

It should be noted that since the generator $Y^{(s)}$ contains arbitrary functions $\xi(x,y)$ and $\eta(x,y)$, the corresponding identity (2.26) leads to m linear PDEs for J, where m is the number of the arbitrary functions and their derivatives that appear in $Y^{(s)}$.

For simplicity, from here on, we denote the derivative of $f(x,y,p,q)$ with respect to the independent variables x,y,p,q as f_1,f_2,f_3,f_4. The same notation will be used for higher-order derivatives.

Now, in order to find all the third order differential invariants of the third-order ODE (1.4), one can solve the invariant criterion (2.26) with $s=3$. However, for compactness of the derived differential invariants, one can replace any partial derivative with respect to x by the total derivative with respect to x. So, we need to solve the following invariant criterion

$$\begin{aligned}&Y^{(3)}J(x,y,p,q,f,f_2,f_3,f_4,f_{2,2},f_{2,3},f_{2,4},f_{3,3},f_{3,4},f_{4,4},f_{2,2,2},f_{2,2,3},f_{2,2,4},f_{2,3,3},f_{2,3,4},\\&f_{2,4,4},f_{3,3,3},f_{3,3,4},f_{3,4,4},f_{4,4,4},d_{1,1},d_{1,2},d_{1,3},d_{1,4},d_{1,5},d_{1,6},d_{1,7},d_{1,8},d_{1,9},d_{1,10},d_{2,1},\\&d_{2,2},d_{2,3},d_{2,4},d_{3,1}) = 0,\end{aligned} \tag{2.27}$$

by prolonging the infinitesimal operator $Y^{(3)}$ to the variables $d_{i,j}$ through the infinitesimals $Y^{(3)}d_{i,j}$, where

$$\begin{aligned}&d_{1,1} = \dot{D}_x f, d_{1,2} = \dot{D}_x f_2, d_{1,3} = \dot{D}_x f_3, d_{1,4} = \dot{D}_x f_4, d_{1,5} = \dot{D}_x f_{2,2},\\&d_{1,6} = \dot{D}_x f_{2,3}, d_{1,7} = \dot{D}_x f_{2,4}, d_{1,8} = \dot{D}_x f_{3,3}, d_{1,9} = \dot{D}_x f_{3,4}, d_{1,10} = \dot{D}_x f_{4,4},\\&d_{2,1} = \dot{D}_x^2 f, d_{2,2} = \dot{D}_x^2 f_2, d_{2,3} = \dot{D}_x^2 f_3, d_{2,4} = \dot{D}_x^2 f_4, d_{3,1} = \dot{D}_x^3 f.\end{aligned} \tag{2.28}$$

Definition 2.4. *An invariant differentiation operator of a class of third-order ODEs (1.4)* is a differential operator $\mathcal{D}$ which satisfies that if I is a differential invariant of ODE (1.4), then $\mathcal{D}I$ is its differential invariant too.

Theorem 2.5. *Let $\mathcal{D}$ be an invariant differentiation operator of a class of third-order ODEs (1.4). Then there is a relative invariant differentiation operator*

$$\Lambda = \mathcal{D} + r\ P + s\ Q, \quad : \mathcal{R}^{r,s} \to \mathcal{R}^{r,s}, \tag{2.29}$$

where $\mathcal{R}^{r,s}$ is the space of relative differential invariants of third-order ODEs (1.4) given by $\mathcal{R}^{r,s} = \{J(x, y, p, q, f, f_{(1)}, f_{(2)}, \ldots, f_{(n)}) : Y^{(n)}J = -(r\ D_x\xi + s\ W_y)J,\ n \in \mathbb{N}\}$ and P and Q satisfy the following system

$$Y^{(n)}P = \mathcal{D}(D_x\xi), \quad Y^{(n)}Q = \mathcal{D}(W_y). \tag{2.30}$$

Proof. Let $J \in \mathcal{R}^{r,s}$. Since the invariant differentiation operator $\mathcal{D}$ has the property $\mathcal{D}Y^{(n)} = Y^{(n)}\mathcal{D}$, then one can verify that $\Lambda J \in \mathcal{R}^{r,s}$ as follows

$$\begin{aligned} Y^{(n)}\Lambda J = &\ Y^{(n)}\mathcal{D}J + r\ Y^{(n)}(P\ J) + s\ Y^{(n)}(Q\ J), \\ = &\ \mathcal{D}Y^{(n)}J + r\ J\ Y^{(n)}P + r\ P\ Y^{(n)}J + s\ J\ Y^{(n)}Q + s\ Q\ Y^{(n)}J, \\ = &\ -\mathcal{D}((r\ D_x\xi + s\ W_y)J) + r\ J\ \mathcal{D}(D_x\xi) + s\ J\ \mathcal{D}(W_y) - (r\ D_x\xi + s\ W_y)(r\ P + s\ Q)\ J, \\ = &\ -(r\ D_x\xi + s\ W_y)\mathcal{D}\ J - (r\ D_x\xi + s\ W_y)(r\ P + s\ Q)\ J, \\ = &\ -(r\ D_x\xi + s\ W_y)\Lambda J. \end{aligned} \tag{2.31}$$

□

As it is shown in [6], the number of independent invariant differentiation operators $\mathcal{D}$ equals the number of independent variables x, y, p and q. The invariant differentiation operators $\mathcal{D}$ should take the form

$$\mathcal{D} = K\tilde{D}_x + L\tilde{D}_y + M\tilde{D}_p + N\tilde{D}_q, \tag{2.32}$$

with the coordinates K, L, M and N satisfying the non-homogeneous linear system

$$Y^{(3)}K = \mathcal{D}(Y(x)), \quad Y^{(3)}L = \mathcal{D}(Y(y)), \quad Y^{(3)}M = \mathcal{D}(Y(p)), \quad Y^{(3)}N = \mathcal{D}(Y(q)). \tag{2.33}$$

In this paper, we will construct the relative invariant differentiation operators Λ of the form

$$\Lambda = K\tilde{D}_x + L\tilde{D}_y + M\tilde{D}_p + N\tilde{D}_q + r\ P + s\ Q, \tag{2.34}$$

where K, L, M, N, P and Q can be calculated from (2.30) and (2.33) as functions of the following variables

$$x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, f_{2,4,4}, f_{3,3,3}, f_{3,3,4}, f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}. \tag{2.35}$$

The general solution of the system (2.30) and (2.33) gives both the differential invariants and the relative invariant differentiation operator of the form (2.34). We will use this relative invariant differentiation operator to find four relative and four absolute invariant differentiation operators in section 4.

This general solution can be found by prolonging the infinitesimal operator $Y^{(3)}$ to the variables K, L, M, N, P and Q through the infinitesimals $Y^{(3)}K, Y^{(3)}L, Y^{(3)}M, Y^{(3)}N, Y^{(3)}P$ and $Y^{(3)}Q$ respectively. Then solving the invariant criterion

$$Y^{(3)}J(x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, f_{2,4,4}, f_{3,3,3}, f_{3,3,4}, f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}, K, L, M, N, P, Q) = 0, \tag{2.36}$$

gives the implicit solution of the variables K, L, M, N, P and Q with the differential invariants.

In this paper, we are interested in finding the third-order differential invariants and differential operators of the general class $y''' = f(x, y, y', y'')$ under a group of point transformations (2.13). So, according to the theory of invariants of infinite transformation groups [6, 18], the invariant criterion (2.36) should be split by the functions $\xi(x, y)$ and $\eta(x, y)$

and their derivatives. This gives rise to a homogeneous linear system of partial differential equations (PDEs):

$$X_iJ = 0, \quad T_iJ = 0, \ i = 1 \dots 28, \tag{2.37}$$

where $X_i, i = 1 \dots 28$, are the differential operators corresponding to the coefficients of the following derivatives of $\eta(x, y)$ up to the sixth order in the invariant criterion

$$\eta, \eta_1, \eta_2, \eta_{1,1}, \eta_{1,2}, \eta_{2,2}, \eta_{1,1,1}, \eta_{1,1,2}, \eta_{1,2,2}, \eta_{2,2,2}, \eta_{1,1,1,1}, \eta_{1,1,1,2}, \eta_{1,1,2,2}, \eta_{1,2,2,2}, \eta_{2,2,2,2}, \eta_{1,1,1,1,1}, \eta_{1,1,1,1,2}, \eta_{1,1,1,2,2}, \eta_{1,1,2,2,2}, \eta_{1,2,2,2,2}, \eta_{2,2,2,2,2}, \eta_{1,1,1,1,1,1}, \eta_{1,1,1,1,1,2}, \eta_{1,1,1,1,2,2}, \eta_{1,1,1,2,2,2}, \eta_{1,1,2,2,2,2}, \eta_{1,2,2,2,2,2}, \eta_{2,2,2,2,2,2} \tag{2.38}$$

and $T_i, i = 1 \dots 28$, are the differential operators corresponding to the coefficients of the following derivatives of $\xi(x, y)$ up to the sixth order in the invariant criterion

$$\xi, \xi_1, \xi_2, \xi_{1,1}, \xi_{1,2}, \xi_{2,2}, \xi_{1,1,1}, \xi_{1,1,2}, \xi_{1,2,2}, \xi_{2,2,2}, \xi_{1,1,1,1}, \xi_{1,1,1,2}, \xi_{1,1,2,2}, \xi_{1,2,2,2}, \xi_{2,2,2,2}, \xi_{1,1,1,1,1}, \xi_{1,1,1,1,2}, \xi_{1,1,1,2,2}, \xi_{1,1,2,2,2}, \xi_{1,2,2,2,2}, \xi_{2,2,2,2,2}, \xi_{1,1,1,1,1,1}, \xi_{1,1,1,1,1,2}, \xi_{1,1,1,1,2,2}, \xi_{1,1,1,2,2,2}, \xi_{1,1,2,2,2,2}, \xi_{1,2,2,2,2,2}, \xi_{2,2,2,2,2,2}. \tag{2.39}$$

Functionally independent solutions of system (2.37) provide all independent differential invariants of $y''' = f(x, y, y', y'')$ up to the third order under point transformation, as well as an implicit solution of the variables K, L, M, N, P and Q which yield the relative invariant differentiation operator via (2.34). The solution of system (2.37) is found in many steps using Maple as follows:

First, let us consider the subsystem induced by the sixth derivatives of ξ and η

$$X_iJ = 0, \ T_iJ = 0, \ i = 22 \dots 28. \tag{2.40}$$

where the operators X_i and T_i, $i = 22 \dots 28$ are given in Appendix A in term of the variables $z_i, i = 1 \dots 45$ after relabeling the variables

$$x, y, p, q, f, f_2, f_3, f_4, f_{2,2}, f_{2,3}, f_{2,4}, f_{3,3}, f_{3,4}, f_{4,4}, f_{2,2,2}, f_{2,2,3}, f_{2,2,4}, f_{2,3,3}, f_{2,3,4}, f_{2,4,4}, f_{3,3,3}, f_{3,3,4}, f_{3,4,4}, f_{4,4,4}, d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, d_{1,5}, d_{1,6}, d_{1,7}, d_{1,8}, d_{1,9}, d_{1,10}, d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, d_{3,1}, K, L, M, N, P, Q \tag{2.41}$$

by the variables $z_i, i = 1 \dots 45$, respectively.

In 45-dimensional space of variables $z_i, i = 1 \dots 45$, the rank of the system (2.40) is 4, so it has 41 functionally independent solutions which are given as:

$$l_1 = z_1, l_2 = z_2, l_3 = z_3, l_4 = z_4, l_5 = z_5, l_6 = z_6, l_7 = z_7, l_8 = z_8, l_9 = z_9, l_{10} = z_{10}, l_{11} = z_{11}, l_{12} = z_{12},$$
$$l_{13} = z_{13}, l_{14} = z_{14}, l_{15} = z_{16}, l_{16} = z_{17}, l_{17} = z_{18}, l_{18} = z_{19}, l_{19} = z_{20}, l_{20} = z_{21}, l_{21} = z_{22}, l_{22} = z_{23},$$
$$l_{23} = z_{24}, l_{24} = z_{25}, l_{25} = z_{26}, l_{26} = z_{27}, l_{27} = z_{28}, l_{28} = z_{30}, l_{29} = z_{31}, l_{30} = z_{32}, l_{31} = z_{33}, l_{32} = z_{34},$$
$$l_{33} = z_{35}, l_{34} = z_{37}, l_{35} = z_{38}, l_{36} = z_{40}, l_{37} = z_{41}, l_{38} = z_{42}, l_{39} = z_{43}, l_{40} = z_{44}, l_{41} = z_{45}. \tag{2.42}$$

Second, let us consider the subsystem induced by the fifth derivatives of ξ and η

$$X_i J = 0,\ T_i J = 0,\ i = 16 \ldots 21. \tag{2.43}$$

where the inherited operators X_i and $T_i,\ i = 16 \ldots 21$ are given in Appendix B in term of the new variables $l_i, i = 1 \ldots 41$.

In 41-dimensional space of variables $l_i, i = 1 \ldots 41$, the rank of the system (2.43) is 6, so it has 35 functionally independent solutions which are given as:

$$m_1 = l_1, m_2 = l_2, m_3 = l_3, m_4 = l_4, m_5 = l_5, m_6 = l_6, m_7 = l_7, m_8 = l_8, m_9 = l_{10}, m_{10} = l_{11},$$
$$m_{11} = l_{12}, m_{12} = l_{13}, m_{13} = l_{14}, m_{14} = l_{16}, m_{15} = l_{17}, m_{16} = l_{18}, m_{17} = l_{19}, m_{18} = l_{20}, m_{19} = l_{21},$$
$$m_{20} = l_{22}, m_{21} = l_{23}, m_{22} = l_{24}, m_{23} = l_{26}, m_{24} = l_{27}, m_{25} = l_{29}, m_{26} = l_{30}, m_{27} = l_{31}, m_{28} = l_{32},$$
$$m_{29} = l_{35}, m_{30} = l_{36}, m_{31} = l_{37}, m_{32} = l_{38}, m_{33} = l_{39}, m_{34} = l_{40}, m_{35} = l_{41}. \tag{2.44}$$

Third, let us consider the subsystem induced by the fourth derivatives of ξ and η

$$X_i J = 0,\ T_i J = 0,\ i = 11 \ldots 15. \tag{2.45}$$

where the inherited operators X_i and $T_i,\ i = 11 \ldots 15$ are given in Appendix C in term of the new variables $m_i, i = 1 \ldots 35$.

In 35-dimensional space of variables $m_i, i = 1 \ldots 35$, the rank of the system (2.45) is 6 whenever $f_{4,4} \neq 0$, so it has 29 functionally independent solutions which are given as:

$$\begin{array}{l} n_1 = m_1, n_2 = m_2, n_3 = m_3, n_4 = m_4, n_5 = m_5, n_6 = m_7, n_7 = m_8, n_8 = m_{10}, n_9 = m_{11}, \\ n_{10} = m_{12}, n_{11} = m_{13}, n_{12} = m_{16}, n_{13} = m_{17}, n_{14} = m_{18}, n_{15} = m_{19}, n_{16} = m_{20}, n_{17} = m_{21}, \\ n_{18} = m_{24}, n_{19} = \frac{{m_{13}}^2 m_6 + (-3\,m_9 + m_{25}) m_{13} - 6\,m_{14} + 3\,m_{15}}{m_{13}}, n_{20} = \frac{(-6\,m_9 + m_{26}) m_{13} - 12\,m_{14} + 6\,m_{15}}{m_{13}}, \\ n_{21} = m_{27}, n_{22} = m_{28}, n_{23} = 6\,m_6 - 3\,m_{23} + m_{29}, n_{24} = m_{30}, n_{25} = m_{31}, n_{26} = m_{32}, \\ n_{27} = m_{33}, n_{28} = m_{34}, n_{29} = m_{35}. \end{array} \tag{2.46}$$

Finally, let us consider the subsystem induced by the zero, first, second and third derivatives of ξ and η

$$X_i J = 0,\ T_i J = 0,\ i = 1 \ldots 10. \tag{2.47}$$

where the inherited operators X_i and $T_i,\ i = 1 \ldots 10$ are given in Appendix D in term of the new variables $n_i, i = 1 \ldots 29$

One can see that the operators X_i and $T_i,\ i = 1 \ldots 10$ form a Lie algebra $\mathcal{L}_{20}$ with the nonzero commutators

$$\begin{array}{llll}
[X_2, X_3] = X_2, & [X_2, X_5] = 2\ X_4, & [X_2, X_6] = X_5, & [X_2, X_8] = 3\ X_7, \\
[X_2, X_9] = 2\ X_8, & [X_2, X_{10}] = X_9, & [X_2, T_2] = -X_2, & [X_2, T_3] = -X_3 + T_2, \\
[X_2, T_4] = -X_4, & [X_2, T_5] = -X_5 + 2\ T_4, & [X_2, T_6] = -X_6 + T_5, & [X_2, T_7] = -X_7, \\
[X_2, T_8] = -X_8 + 3\ T_7, & [X_2, T_9] = -X_9 + 2\ T_8, & [X_2, T_{10}] = -X_{10} + T_9, & [X_3, X_4] = -X_4, \\
[X_3, X_6] = X_6, & [X_3, X_7] = -X_7, & [X_3, X_9] = X_9, & [X_3, X_{10}] = 2\ X_{10}, \\
[X_3, T_3] = T_3, & [X_3, T_5] = T_5, & [X_3, T_6] = 2\ T_6, & [X_3, T_8] = T_8, \\
[X_3, T_9] = 2\ T_9, & [X_3, T_{10}] = 3\ T_{10}, & [X_4, X_5] = 3\ X_7, & [X_4, X_6] = X_8, \\
[X_4, T_2] = -2\ X_4, & [X_4, T_3] = -X_5 + T_4, & [X_4, T_4] = -3\ X_7, & [X_4, T_5] = -2\ X_8 + 3\ T_7, \\
[X_4, T_6] = -X_9 + T_8, & [X_5, X_6] = X_9, & [X_5, T_2] = -X_5, & [X_5, T_3] = -2\ X_6 + T_5, \\
[X_5, T_4] = -X_8, & [X_5, T_5] = -2\ X_9 + 2\ T_8, & [X_5, T_6] = -3\ X_{10} + 2\ T_9, & [X_6, T_3] = T_6, \\
[X_6, T_5] = T_9, & [X_6, T_6] = 3\ T_{10}, & [X_7, T_2] = -3\ X_7, & [X_7, T_3] = -X_8 + T_7, \\
[X_8, T_2] = -2\ X_8, & [X_8, T_3] = -2\ X_9 + T_8, & [X_9, T_2] = -X_9, & [X_9, T_3] = -3\ X_{10} + T_9, \\
[X_{10}, T_3] = T_{10}, & [T_2, T_3] = -T_3, & [T_2, T_4] = T_4, & [T_2, T_6] = -T_6, \\
[T_2, T_7] = 2\ T_7, & [T_2, T_8] = T_8, & [T_2, T_{10}] = -T_{10}, & [T_3, T_4] = T_5, \\
[T_3, T_5] = 2\ T_6, & [T_3, T_7] = T_8, & [T_3, T_8] = 2\ T_9, & [T_3, T_9] = 3\ T_{10}, \\
[T_4, T_5] = -T_8, & [T_4, T_6] = -T_9, & [T_5, T_6] = -3\ T_{10}. &
\end{array} \tag{2.48}$$

Moreover, the projection of the operators X_i and $T_i,\ i = 1 \ldots 10$ on the 5-dimensional space of variables $n_i, i = 1 \ldots 5$ are the generators of the original infinite Lie algebra spanned by the infinitesimal operators (2.15) before the prolongation to the third order.

In order to get the joint invariants of $\mathcal{L}_{20}$, one may start finding the relative invariants of

$\mathcal{L}_{20}$ by computing the joint invariants of the derived algebra $\mathcal{L}_{20}^{'}$.

The derived algebra $\mathcal{L}_{20}^{'}$ has the Levi Decomposition $\mathcal{L}_{20}^{'} = S \bigoplus R$ where the radical R is nilpotent with basis $< X_4, X_7, X_5, X_8, X_9, 2\ T_4 - X_5, 2T_8 - X_9, 3T_7 - X_8, T_5 - X_6, T_9 - X_{10}, X_{10}, X_6, T_{10}, T_6 >$ which defines an ascending chain of ideals, each of codimension one, and the the semi-simple part S is generated by $< X_2, X_3 - T_2, T_3 >$. The semi-simple part is $sl(2, \mathbb{R})$ because it has a two dimensional non-abelian subalgebra $< X_2, X_3 - T_2 >$.

We first find the joint invariants of the radical R using its ascending chain of ideals. Then finally we find the invariants of the semi-simple part S on the joint invariants of the radical.

In more details, if I is an ideal of codimension one in an ideal J, then J operates on the invariants of I. At every stage one is essentially dealing with the invariants of a single vector field and these can be determined using the method of characteristics [18]. There remains then the determination of the invariants of the semi-simple part on the invariants of the radical. To do this, we choose a Borel subalgebra $< X_2, X_3 - T_2 >$ and find its joint invariants in the invariants of the radical. This leaves the vector field T_3. Although T_3 is not inherited, nevertheless a functional multiple of the vector field T_3 is inherited. We then calculate the invariants of this inherited vector field to get the joint invariants of $\mathcal{L}_{20}^{'}$.

Using the procedure outline above, the joint invariants of the derived algebra $\mathcal{L}_{20}^{'}$ can be given for the case $f_{4,4,4} \neq 0$ after back substitution as an arbitrary function

$J\left(x, y, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8, \nu_9, \nu_{10}, \nu_{11}, \nu_{12}\right)$, where

$$\begin{aligned}
\nu_1 =&\ \tfrac{1}{3}\left(3\,\dot{D}_x f_4 - {f_4}^2 - 9\,f_3\right) f_{4,4} + 2\,f_{3,4} f_4 + 3\,f_{3,3} - 6\,f_{2,4},\\
\nu_2 =&\ {f_{4,4,4}}^{-\frac{3}{5}} \Delta_q \nu_1,\\
\nu_3 =&\ {f_{4,4,4}}^{-\frac{2}{5}} \Delta_p \nu_1,\\
\nu_4 =&\ {f_{4,4,4}}^{\frac{1}{5}} \Delta_x \nu_1,\\
\nu_5 =&\ {f_{4,4,4}}^{-\frac{1}{5}} \Delta_y \nu_1,\\
\nu_6 =&\ {f_{4,4,4}}^{\frac{3}{5}} \left(\tfrac{4}{9}\,{f_4}^3 + 2\,f_4 f_3 - 2\,f_4 \dot{D}_x f_4 + 6\,f_2 - 3\,\dot{D}_x f_3 + \dot{D}_x^2 f_4\right),
\end{aligned} \tag{2.49}$$

and

$$\begin{aligned}
\nu_7 =&\ {f_{4,4,4}}^{\frac{1}{5}} \left(L - p\ K\right),\\
\nu_8 =&\ \tfrac{1}{6} {f_{4,4,4}}^{-\frac{1}{5}} \left(6K + \ f_{4,4} \left(L - p\ K\right)\right),\\
\nu_9 =&\ \tfrac{1}{3} {f_{4,4,4}}^{\frac{2}{5}} \left(3(M - q\,K) + \ f_4 \left(L - p\ K\right)\right) + \tfrac{1}{6} {f_{4,4,4}}^{-\frac{3}{5}} ({f_{4,4}}^2 + 6\,f_{3,4,4}) \left(L - p\ K\right),\\
\nu_{10} =&\ \tfrac{1}{18} {f_{4,4,4}}^{\frac{3}{5}} \left(18\left(N - \ fK\right) + \left(3\,\dot{D}_x f_4 - {f_4}^2 - 9\,f_3\right)\left(L - p\ K\right)\right)\\
&+ \tfrac{1}{18} {f_{4,4,4}}^{-\frac{2}{5}} \left({f_{4,4}}^2 + 6\,f_{3,4,4}\right)\left(3\left(M - q\ K\right) + f_4\left(L - p\ K\right)\right)\\
&+ \tfrac{1}{72} {f_{4,4,4}}^{-\frac{7}{5}} \left({f_{4,4}}^2 + 6\,f_{3,4,4}\right)^2 \left(L - p\ K\right).\\
\nu_{11} =&\ \tfrac{1}{36}\left(\frac{\left((-4\,f_4 p - 6\,q) f_{4,4} - 6\,p\ \dot{D}_x f_{4,4} + 24\,f_4\right) f_{4,4,4} - \left(6\,f_{3,4,4} + {f_{4,4}}^2\right)(-6 + p\ f_{4,4})}{f_{4,4,4}}\right) K\\
&+ \tfrac{1}{36}\left(\frac{\left(4\,f_{4,4} f_4 + 6\,\dot{D}_x f_{4,4}\right) f_{4,4,4} + \left(6\,f_{3,4,4} + {f_{4,4}}^2\right) f_{4,4}}{f_{4,4,4}}\right) L + \tfrac{1}{6}\,f_{4,4} M + P\\
\nu_{12} =&\ \tfrac{1}{36}\left(\frac{\left((-4\,f_4 p + 6\,q) f_{4,4} + \left(12\,f_{3,4} - 18\,\dot{D}_x f_{4,4}\right) p + 12\,f_4\right) f_{4,4,4} - \left(6\,f_{3,4,4} + {f_{4,4}}^2\right)(-6 + p\ f_{4,4})}{f_{4,4,4}}\right) K\\
&+ \tfrac{1}{36}\left(\frac{\left(4\,f_{4,4} f_4 + 18\,\dot{D}_x f_{4,4} - 12\,f_{3,4}\right) f_{4,4,4} + \left(6\,f_{3,4,4} + {f_{4,4}}^2\right) f_{4,4}}{f_{4,4,4}}\right) L - \tfrac{1}{6}\,f_{4,4} M + Q
\end{aligned} \tag{2.50}$$

where $\Delta_q, \Delta_p, \Delta_x$ and Δ_y are the relative differential operators given by (4.59).

It should be noted here that $\nu_i,\ i = 1 \dots 6$ are relative invariants. Moreover, it is noted that the third-order relative invariants $\nu_i,\ i = 2 \dots 5$ vanish identically when the second-order relative invariant $\nu_1 = 0$.

Finally, since the quotient algebra $\mathcal{L}_{20}/\mathcal{L}_{20}^{'}$ is abelian generated by $< X_1, T_1, X_3 >$, these will operate as a commuting vector field on the joint invariants of $\mathcal{L}_{20}^{'}$ and their joint invariants will give the absolute invariants of $\mathcal{L}_{20}$.

The inheritance of the operators $< X_1, T_1, X_3 >$ in term of the new variables $x, y, \nu_i, i = 1 \ldots 12$ is

$$\begin{aligned} X_1 = &\ [0,1,0,0,0,0,0,0,0,0,0,0,0,0], \\ T_1 = &\ [1,0,0,0,0,0,0,0,0,0,0,0,0,0], \\ X_3 = &\ [0,0,-5\,\nu_1,-4\,\nu_2,-6\,\nu_3,-7\,\nu_4,-8\,\nu_5,-6\,\nu_6,3\,\nu_7,2\,\nu_8,\nu_9,-\nu_{10},0,0], \end{aligned} \tag{2.51}$$

The joint invariants of the operators (2.51) are the invariants of the operator

$$\begin{aligned} Z = &\ -5\,\nu_1\frac{\partial}{\partial\nu_1} - 4\,\nu_2\frac{\partial}{\partial\nu_2} - 6\,\nu_3\frac{\partial}{\partial\nu_3} - 7\,\nu_4\frac{\partial}{\partial\nu_4} - 8\,\nu_5\frac{\partial}{\partial\nu_5} - 6\,\nu_6\frac{\partial}{\partial\nu_6} + 3\,\nu_7\frac{\partial}{\partial\nu_7} + 2\,\nu_8\frac{\partial}{\partial\nu_8} \\ &+\nu_9\frac{\partial}{\partial\nu_9} - \nu_{10}\frac{\partial}{\partial\nu_{10}}. \end{aligned} \tag{2.52}$$

In section 4, the invariants of the operators (2.52) provide all absolute differential invariants and the invariant differentiation operators of $y''' = f(x, y, y', y'')$, with $f_{4,4,4} \neq 0$, up to the third-order under point transformations.

3 The infinitesimal point equivalence transformations

In order to find continuous group of equivalence transformations of the class (1.4) we consider the arbitrary function f that appears in our equation as a dependent variable and the variables $x, y, y' = p, y'' = q$ as independent variables and apply the Lie infinitesimal invariance criterion [6], that is we look for the infinitesimal ξ, η and μ of the equivalence operator Y:

$$Y = \xi(x,y)\partial_x + \eta(x,y)\partial_y + \mu(x,y,p,q,f)\partial_f, \tag{3.53}$$

such that its prolongation leaves the equation (1.4) invariant.

The prolongation of operator Y can be given using (2.8) as

$$Y = \xi(x,y)D_x + W\partial_y + D_x(W)\partial_p + D_x^2(W)\partial_q + D_x^3(W)\partial_{y'''} + \mu(x,y,p,q,f)\partial_f, \tag{3.54}$$

where

$$D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + y'''\frac{\partial}{\partial q} + y^{(4)}\frac{\partial}{\partial y'''} + ...$$

is the operator of total derivative and $W = \eta(x,y) - \xi(x,y)p$ is the characteristic of infinitesimal operator $X = \xi(x,y)\partial_x + \eta(x,y)\partial_y$.

So, the Lie infinitesimal invariance criterion gives $\mu = \dot{D}_x^3(W) + \xi(x,y)\dot{D}_x f$ for arbitrary functions $\xi(x,y)$ and $\eta(x,y)$ where $\dot{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p} + f\frac{\partial}{\partial q}$.

Thus, equation (1.4) admits an infinite continuous group of equivalence transformations generated by the Lie algebra $\mathcal{L}_{\mathcal{E}}$ spanned by the following infinitesimal operators

$$U = \xi(x,y)\frac{\partial}{\partial x} - pD_x(\xi)\partial_p - (2qD_x(\xi) + pD_x^2(\xi))\partial_q - (3fD_x(\xi) + 3qD_x^2(\xi) + p\dot{D}_x^3(\xi))\partial_f, \tag{3.55}$$

$$V = \eta(x,y)\partial_y + D_x(\eta)\partial_p + D_x^2(\eta)\partial_q + \dot{D}_x^3(\eta)\partial_f, \tag{3.56}$$

The infinitesimal point equivalence transformations (3.55)-(3.56) can be written in the finite form as in (2.16)-(2.17), respectively, where ϕ and ψ are arbitrary functions of the indicated variables.

4 Differential invariants, absolute and relative invariant differentiation operators of $y''' = f(x,y,y',y'')$ in the generic case

This section is devoted to the derivation of all third-order differential invariants of the general class $y''' = f(x,y,y',y'')$, with $f_{4,4,4} \neq 0$ and $\nu_1 \neq 0$, under point transformations (2.13). Moreover, the absolute and relative invariant differentiation operators are also constructed. Precisely, we obtain the following theorem.

Theorem 4.1. *Every third-order ODE $y''' = f(x, y, y', y'')$, with $f_{4,4,4} \neq 0$, belongs to one of two classes of equations. For the first class ($\nu_1 \neq 0$), there are five third-order differential invariants, under point transformations,*

$$\beta_1 = \frac{(\Delta_q \nu_1)^5}{I^3 {\nu_1}^4}, \quad \beta_2 = \frac{(\Delta_p \nu_1)^5}{I^2 {\nu_1}^6}, \quad \beta_3 = \frac{I(\Delta_x \nu_1)^5}{{\nu_1}^7} \quad \beta_4 = \frac{(\Delta_y \nu_1)^5}{I {\nu_1}^8}, \quad \beta_5 = \frac{I^3 K^5}{{\nu_1}^6}. \tag{4.57}$$

where ν_1, I, K are the basic relative invariants

$$\begin{aligned} \nu_1 &= \tfrac{1}{3}\left(3\,\dot{D}_x f_4 - {f_4}^2 - 9\,f_3\right) f_{4,4} + 2\,f_{3,4} f_4 + 3\,f_{3,3} - 6\,f_{2,4} \in \mathcal{R}^{1,1}, \\ I &= f_{4,4,4} \in \mathcal{R}^{-3,2}, \\ K &= \tfrac{4}{9}\,{f_4}^3 + 2\,f_4 f_3 - 2\,f_4 \dot{D}_x f_4 + 6\,f_2 - 3\,\dot{D}_x f_3 + \dot{D}_x^2 f_4 \in \mathcal{R}^{3,0}, \end{aligned} \tag{4.58}$$

the operators $\Delta_q, \Delta_p, \Delta_x, \Delta_y$ are the relative invariant differentiation operators

$$\begin{aligned} \Delta_q ={}& \tilde{D}_q \ : \mathcal{R}^{r,s} \to \mathcal{R}^{r-2,s+1}, \\ \Delta_p ={}& \tilde{D}_p - \left(\tfrac{{f_{4,4}}^2 + 6\,f_{3,4,4}}{6 f_{4,4,4}}\right) \tilde{D}_q - \tfrac{1}{6}\,(r-s)\,f_{4,4} \ : \mathcal{R}^{r,s} \to \mathcal{R}^{r-1,s+1}, \\ \Delta_x ={}& \dot{D}_x - \tfrac{1}{6}\,r\left(\tfrac{{f_{4,4}}^2 + 6\,f_{3,4,4}}{f_{4,4,4}} + 4 f_4\right) - \tfrac{1}{6}\,s\left(\tfrac{{f_{4,4}}^2 + 6\,f_{3,4,4}}{f_{4,4,4}} + 2 f_4\right) \ : \mathcal{R}^{r,s} \to \mathcal{R}^{r+1,s}, \\ \Delta_y ={}& \tilde{D}_y - \tfrac{1}{6} f_{4,4} \dot{D}_x - \tfrac{1}{3}\left(\tfrac{f_{4,4}^2 + 6 f_{3,4,4}}{2 f_{4,4,4}} + f_4\right) \tilde{D}_p + \tfrac{1}{18}\left(\tfrac{(f_{4,4}^2 + 6 f_{3,4,4})^2}{4 f_{4,4,4}^2} + 9\,f_3 + f_4^2 - 3\,\dot{D}_x f_4\right) \tilde{D}_q \\ &+ \tfrac{1}{36} r\left(2\,f_4 f_{4,4} + \tfrac{{f_{4,4}}^3 + 6 f_{3,4,4} f_{4,4}}{f_{4,4,4}} - 6 \dot{D}_x f_{4,4}\right) - \tfrac{1}{36} s\left(4\,f_4 f_{4,4} + \tfrac{{f_{4,4}}^3 + 6 f_{3,4,4} f_{4,4}}{f_{4,4,4}} - 12\,f_{3,4} + 18 \dot{D}_x f_{4,4}\right) \\ &: \mathcal{R}^{r,s} \to \mathcal{R}^{r,s+1}, \end{aligned} \tag{4.59}$$

and $\mathcal{R}^{r,s}$ is the space of relative invariants defined by

$$\mathcal{R}^{r,s} = \{J(x, y, p, q, f, f_{(1)}, f_{(2)}, \ldots, f_{(n)}) : Y^{(n)} J = -(r\ D_x \xi + s\ W_y) J,\ n \in \mathbb{N}\}. \tag{4.60}$$

Moreover, the invariant differentiation operators are

$$\begin{aligned} \mathcal{D}_1 &= {f_{4,4,4}}^{-\frac{3}{5}} {\nu_1}^{\frac{1}{5}} \tilde{D}_q, \\ \mathcal{D}_2 &= {f_{4,4,4}}^{-\frac{2}{5}} {\nu_1}^{-\frac{1}{5}} \left(\tilde{D}_p - \left(\tfrac{{f_{4,4}}^2 + 6\,f_{3,4,4}}{6 f_{4,4,4}}\right) \tilde{D}_q\right), \\ \mathcal{D}_3 &= {f_{4,4,4}}^{\frac{1}{5}} {\nu_1}^{-\frac{2}{5}} \dot{D}_x, \\ \mathcal{D}_4 &= {f_{4,4,4}}^{-\frac{1}{5}} {\nu_1}^{-\frac{3}{5}} \left(\tilde{D}_y - \tfrac{1}{6} f_{4,4} \dot{D}_x - \tfrac{1}{3}\left(\tfrac{f_{4,4}^2 + 6 f_{3,4,4}}{2 f_{4,4,4}} + f_4\right) \tilde{D}_p + \tfrac{1}{18}\left(\left(\tfrac{f_{4,4}^2 + 6 f_{3,4,4}}{2 f_{4,4,4}}\right)^2 + 9\,f_3 + f_4^2 - 3\,\dot{D}_x f_4\right) \tilde{D}_q\right). \end{aligned} \tag{4.61}$$

However, there is no third-order differential invariants for the second class ($\nu_1 = 0$).

Proof. The invariants of the operators (2.52) provide all absolute differential invariants and the invariant differentiation operators of $y''' = f(x, y, y', y'')$, with $f_{4,4,4} \neq 0$, up to the third-order under point transformations.

The invariants of the operators (2.52) can be given using characteristic method for two classes as follows:

(1) First class of equation ($\nu_1 \neq 0$)

$$\beta_1 = \frac{{\nu_2}^5}{{\nu_1}^4}, \quad \beta_2 = \frac{{\nu_3}^5}{{\nu_1}^6}, \quad \beta_3 = \frac{{\nu_4}^5}{{\nu_1}^7} \quad \beta_4 = \frac{{\nu_5}^5}{{\nu_1}^8}, \quad \beta_5 = \frac{{\nu_6}^5}{{\nu_1}^6}, \tag{4.62}$$

and

$$\gamma_1 = \nu_7 {\nu_1}^{\frac{3}{5}}, \ \gamma_2 = \nu_8 {\nu_1}^{\frac{2}{5}}, \ \gamma_3 = \nu_9 \nu_1^{\frac{1}{5}}, \ \gamma_4 = \nu_{10} \nu_1^{-\frac{1}{5}}, \ \gamma_5 = \nu_{11}, \ \gamma_6 = \nu_{12}. \tag{4.63}$$

(2) Second class of equation ($\nu_1 = 0$) does not have third-order differential invariants independent from the variables K, L, M, N, P and Q. This because of vanishing the third-order relative invariants $\nu_i, \ i = 2 \ldots 5$ identically when the second-order relative invariant $\nu_1 = 0$.

Regarding the invariant differentiation operators, since $\gamma_i, \ i = \ldots 6$ are the only invariants depending on the variables K, L, M, N, P and Q, then the general solution of (2.33) can be given implicitly as

$$\gamma_1 = F_1, \ \gamma_2 = F_2, \ \gamma_3 = F_3, \ \gamma_4 = F_4, \ \gamma_5 = F_5, \ \gamma_6 = F_6, \tag{4.64}$$

where $F_i, \ i = 1 \ldots 6$ are arbitrary functions of differential invariants $\beta_i, \ i = 1 \ldots 5$.

Solving system (4.64) gives the variables K, L, M, N, P and Q in terms of six arbitrary functions $F_i, \ i = 1 \ldots 6$ which provide four invariant differentiation operators $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and $\mathcal{D}_4$ via (2.32) and four relative invariant differentiation operators $\Delta_q, \Delta_p, \Delta_x, \Delta_y$ via (2.34). $\square$

5 Differential invariants and absolute invariant differentiation operators of $y''' = f(x, y, y', y'')$ in a degenerate case

In order to give an invariant description of all the canonical forms in the complex plane with four infinitesimal symmetries [17] for third-order ODEs $y''' = f(x, y, y', y'')$ which are not quadratic in the second-order derivative, one needs to study the degenerate class $y''' = f(x, y, y', y'')$, $f_{4,4,4} \neq 0$, $\nu_1 = 0$ and $\Delta_q I \neq 0$, which appears in the application section.

In reality, one can use the relative invariant differentiation operators (4.59) to study the degenerate classes of the class $y''' = f(x, y, y', y'')$ with $f_{4,4,4} \neq 0$. The next theorem is devoted to the derivation of fourth-order differential invariants and the invariant differentiation operators of the degenerate class $y''' = f(x, y, y', y'')$, $f_{4,4,4} \neq 0$, $\nu_1 = 0$ and $\Delta_q I \neq 0$, under point transformations (2.13).

Theorem 5.1. *The class of third-order ODEs $y''' = f(x, y, y', y'')$, with $f_{4,4,4} \neq 0$, $\nu_1 = 0$ and $\Delta_q I \neq 0$, has the following eight fourth-order differential invariants, under point transformations,*

$$\begin{array}{llll} \beta_1 = {J_1}^4 {J_2}^{-3} \Delta_p I, & \beta_2 = {J_1}^2 {J_2}^{-2} \Delta_x I, & \beta_3 = {J_1}^3 {J_2}^{-3} \Delta_y I, & \beta_4 = {J_1}^{-3} K, \\ \beta_5 = {J_1}^{-1} {J_2}^{-1} \Delta_q K, & \beta_6 = {J_1}^{-2} {J_2}^{-1} \Delta_p K, & \beta_7 = {J_1}^{-4} \Delta_x K, & \beta_8 = {J_1}^{-3} {J_2}^{-1} \Delta_y K. \end{array} \tag{5.65}$$

Moreover, the invariant differentiation operators are

$$\begin{aligned} \mathcal{D}_1 &= {J_1}^2 {J_2}^{-1} \tilde{D}_q, \\ \mathcal{D}_2 &= {J_1}^1 {J_2}^{-1} \left(\tilde{D}_p - \left(\frac{{f_{4,4}}^2 + 6\, f_{3,4,4}}{6 f_{4,4,4}} \right) \tilde{D}_q \right), \\ \mathcal{D}_3 &= {J_1}^{-1} \dot{D}_x, \\ \mathcal{D}_4 &= {J_2}^{-1} \left(\tilde{D}_y - \tfrac{1}{6} f_{4,4} \dot{D}_x - \tfrac{1}{3} \left(\frac{f_{4,4}^2 + 6 f_{3,4,4}}{2 f_{4,4,4}} + f_4 \right) \tilde{D}_p + \tfrac{1}{18} \left(\frac{(f_{4,4}^2 + 6 f_{3,4,4})^2}{4 f_{4,4,4}^2} + 9\, f_3 + f_4^2 - 3\, \dot{D}_x f_4 \right) \tilde{D}_q \right), \end{aligned} \tag{5.66}$$

where $J_1 = I^3(\Delta_q I)^{-2}$, $J_2 = I^5(\Delta_q I)^{-3}$ *and* $\Delta_q, \Delta_p, \Delta_x, \Delta_y$ *are the relative invariant differentiation operators given by (4.59).*

Proof. The class of third-order ODE $y''' = f(x, y, y', y'')$, with $f_{4,4,4} \neq 0$, $\nu_1 = 0$ and $\Delta_q I \neq 0$, has the basic relative differential invariants

$$I \in \mathcal{R}^{-3,2}, \quad \Delta_q I \in \mathcal{R}^{-5,3}. \tag{5.67}$$

Since the space of relative differential invariants $\mathcal{R}^{r,s}$ has the properties

$$\mathcal{R}^{r,s}\mathcal{R}^{\bar{r},\bar{s}} \subset \mathcal{R}^{r+\bar{r},s+\bar{s}} \quad \text{and} \quad (\mathcal{R}^{r,s})^{\alpha} \subset \mathcal{R}^{\alpha r,\alpha s}, \tag{5.68}$$

then we have the following two relative differential invariants

$$J_1 = I^3(\Delta_q I)^{-2} \in \mathcal{R}^{1,0}, \quad J_2 = I^5(\Delta_q I)^{-3} \in \mathcal{R}^{0,1}. \tag{5.69}$$

Moreover, since we have the following fourth-order relative differential invariants

$$\begin{aligned} &\Delta_p I \in \mathcal{R}^{-4,3}, \quad \Delta_x I \in \mathcal{R}^{-2,2}, \quad \Delta_y I \in \mathcal{R}^{-3,3}, \quad K \in \mathcal{R}^{3,0}, \\ &\Delta_q K \in \mathcal{R}^{1,1}, \quad \Delta_p K \in \mathcal{R}^{2,1}, \quad \Delta_x K \in \mathcal{R}^{4,0}, \quad \Delta_y K \in \mathcal{R}^{3,1}. \end{aligned} \tag{5.70}$$

then the fourth-order absolute differential invariants (5.65) can be obtained by multiplying a proper choice of the powers of J_1 and J_2 with the fourth-order relative differential invariants (5.70).

Similarly, the invariant differentiation operators (5.66) can be found using the relative invariant differentiation operators (4.59) as follows.

$$\begin{aligned} &\mathcal{D}_1 = {J_1}^2 {J_2}^{-1} \Delta_q|_{r=0,s=0}, \quad \mathcal{D}_2 = {J_1}^1 {J_2}^{-1} \Delta_p|_{r=0,s=0}, \\ &\mathcal{D}_3 = {J_1}^{-1} \Delta_x|_{r=0,s=0}, \qquad \mathcal{D}_4 = {J_2}^{-1} \Delta_y|_{r=0,s=0}. \end{aligned} \tag{5.71}$$

□

6 Application

In this section, invariant descriptions of all the canonical forms in the complex plane with four infinitesimal symmetries [17] for third-order ODEs $y''' = f(x, y, y', y'')$ which are not quadratic in the second-order derivative, ($I \neq 0$), are provided. Two more examples of equations not quadratic in the second-order derivative with three Lie symmetries taken from the works [17, 36] are also discussed.

Example 6.1. Consider the canonical form of third order ODE in the complex plane with four infinitesimal symmetries [17]

$$y''' = C\exp(-y''),\ C \neq 0. \tag{6.72}$$

It is an equation of the first class ($I \neq 0$ and $\nu_1 \neq 0$). The five third-order differential invariants, by Theorem 4.1, are

$$\beta_1 = 162,\ \beta_2 = -\frac{3}{64},\ \beta_3 = \frac{3125}{108},\ \beta_4 = -\frac{15386239549}{573308928}, \beta_5 = \frac{16}{81}. \tag{6.73}$$

Example 6.2. Consider the canonical form of third order ODE in the complex plane with four infinitesimal symmetries [17]

$$y''' = Cy''^{\left(\frac{b-2}{b-1}\right)},\ b \neq 0, 1, 2,\ C \neq 0. \tag{6.74}$$

For $b \neq -1$, it is an equation of the first class ($I \neq 0$ and $\nu_1 \neq 0$). The five third-order differential invariants, by Theorem 4.1, are

$$\begin{aligned}&\beta_1 = -\frac{1}{3}\frac{(b+2)^5(b+1)}{b^3(b-2)},\ \beta_2 = \frac{1}{2592}\frac{(b+2)^5(b-2)}{b^7(b+1)},\ \beta_3 = -\frac{1}{27}\frac{(3\,b+2)^5(2\,b-1)^5}{b^4(b+1)^2(b-2)^3},\\ &\beta_4 = -\frac{1}{71663616}\frac{\left(4\,b^4+68\,b^3+41\,b^2-4\right)^5}{b^{11}(b+1)^3(b-2)^2}\ \beta_5 = \frac{32}{81}\frac{b^3(2\,b-1)^5}{(b+1)(b-2)^4}.\end{aligned} \tag{6.75}$$

For $b = -1$, we have the equation

$$y''' = Cy''^{\frac{3}{2}},\ C \neq 0. \tag{6.76}$$

It is an equation of the second class ($I \neq 0, \nu_1 = 0$ and $\Delta_q I \neq 0$). The only non-zero fourth-order differential invariants, by Theorem 5.1, are

$$\beta_1 = 1,\ \beta_2 = -\tfrac{3}{2},\ \beta_3 = -\tfrac{9}{8}. \tag{6.77}$$

Example 6.3. Consider the canonical form of third order ODE in the complex plane with four infinitesimal symmetries [17]

$$y''' = 3\frac{y''^2}{y'} + C\frac{(2xy'' + y')^{\frac{3}{2}}}{x^2\sqrt{y'}},\ C \neq 0. \tag{6.78}$$

It is an equation of the second class ($I \neq 0, \nu_1 = 0$ and $\Delta_q I \neq 0$). The only non-zero fourth-order differential invariants, by Theorem 5.1, are

$$\beta_1 = 1,\ \beta_2 = -\tfrac{3}{2},\ \beta_3 = -\tfrac{3}{8}\,\tfrac{(3\,C^2+4)}{C^2}. \tag{6.79}$$

Example 6.4. Consider the ODE

$$y''' = \ln y''. \tag{6.80}$$

belonging to the canonical form of third order ODE in the complex plane with three infinitesimal symmetries [17]. It is an equation of the first class ($I \neq 0$ and $\nu_1 \neq 0$). All the third-order differential invariants, by Theorem 4.1, are not identically constant. However, since the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5)}{\partial(x,y,y',y'')}$ has rank one, then all of them are functions of one variable s. One can see that the invariant variable s can be written in term of invariant differentiation of the differential invariant β_1 as

$$s = \mathcal{D}_1\beta_1(\mathcal{D}_3\beta_1)^2(\mathcal{D}_2\beta_1)^{-3}. \tag{6.81}$$

Moreover, the third-order differential invariants, by Theorem 4.1, can be given on the invariant surface $s = 0$ as follows:

$$\beta_1 = -\tfrac{1}{24},\ \beta_2 = \tfrac{1}{331776},\ \beta_3 = \tfrac{8192}{81},\ \beta_4 = -\tfrac{1}{146767085568},\ \beta_5 = -\tfrac{16807}{432}. \tag{6.82}$$

This characterization can be applied as explained below.

By means of the point transformation

$$\bar{x} = -y,\ \bar{y} = x, \tag{6.83}$$

equation (6.80) transforms to

$$\bar{y}''' = \frac{3\,\bar{y}''^2}{\bar{y}'} + \bar{y}'^4 \ln\left(\frac{\bar{y}''}{\bar{y}'^3}\right). \tag{6.84}$$

It can be seen that the Jacobian matrix $\frac{\partial(\bar{\beta}_1,\bar{\beta}_2,\bar{\beta}_3,\bar{\beta}_4,\bar{\beta}_5)}{\partial(\bar{x},\bar{y},\bar{y}',\bar{y}'')}$ has rank one and the differential invariants, $\bar{\beta}_i,\ i = 1\dots 5$, for the transformed equation (6.84) match the differential invariants, $\beta_i,\ i = 1\dots 5$, for the original equation (6.80) after replacing s by $\bar{s}$ where $\bar{s}$ can be evaluated by

$$\bar{s} = \bar{\mathcal{D}}_1\bar{\beta}_1(\bar{\mathcal{D}}_3\bar{\beta}_1)^2(\bar{\mathcal{D}}_2\bar{\beta}_1)^{-3}. \tag{6.85}$$

Furthermore, the third-order differential invariants, by Theorem 4.1, on the surface $\bar{s} = 0$ are

$$\bar{\beta}_1 = -\tfrac{1}{24},\ \bar{\beta}_2 = \tfrac{1}{331776},\ \bar{\beta}_3 = \tfrac{8192}{81},\ \bar{\beta}_4 = -\tfrac{1}{146767085568},\ \bar{\beta}_5 = -\tfrac{16807}{432}. \tag{6.86}$$

Remark 6.5. The transformation (6.83) transforms the variable s to the variable $\bar{s}$. Moreover, the variable s is a second-order joint invariant of the three Lie symmetries of the equation (6.80) and similarly, the transformed variable $\bar{s}$ is a second-order joint invariant of the three Lie symmetries of the transformed equation (6.84)

Example 6.6. Consider the ODE

$$y''' = 24\,\frac{y''^3}{\left(-3+\sqrt{9-2\,y'y''}\right)^3} + 12\,\frac{y'y''^4}{\left(-3+\sqrt{9-2\,y'y''}\right)^4} \tag{6.87}$$

that defines Einstein-Weyl geometry which is not of hyper-CR type and is of recent interest in physics [36]. It is an equation of the second class ($I \neq 0, \nu_1 = 0$ and $\Delta_q I \neq 0$). The only non-zero fourth-order differential invariants, by Theorem 5.1, are

$$\beta_1 = f_1(s),\ \beta_2 = f_2(s),\ \beta_3 = f_3(s). \tag{6.88}$$

which are functions of the second-order joint invariant $s = y'y''$ of the three Lie point symmetries of (6.87). One can verify that the values of these invariants when the parameter s equals zero are

$$\beta_1 = 0,\ \beta_2 = 0,\ \beta_3 = 0. \tag{6.89}$$

Remark 6.7. Given a third-order ODE $y''' = f(x, y, y', y'')$ belonging to the second class ($I \neq 0, \nu_1 = 0$ and $\Delta_q I \neq 0$) and having three Lie symmetries with the second-order joint invariant t. If the only non-zero fourth-order differential invariants, by Theorem 5.1, are

$$\bar{\beta}_1 = g_1(t),\ \bar{\beta}_2 = g_2(t),\ \bar{\beta}_3 = g_3(t), \tag{6.90}$$

then it will be equivalent to the ODE (6.87) if the algebraic system

$$f_1(s) = g_1(t),\ f_2(s) = g_2(t),\ f_3(s) = g_3(t), \tag{6.91}$$

is consistent. In particular,

$$\bar{\beta}_1 = 0,\ \bar{\beta}_2 = 0,\ \bar{\beta}_3 = 0, \tag{6.92}$$

for the parameter $t = t_0$ which is the solution of the system (6.91) when $s = 0$.

7 Conclusion

Lie's infinitesimal method is utilized to study the differential invariants of the general class of ODEs $y''' = f(x, y, y', y'')$ which is not quadratic in the second-order derivative under an arbitrary point equivalence transformation. All third-order differential invariants and the invariant differentiation operators in the generic case ($I \neq 0$ and $\nu_1 \neq 0$) are determined which are given in Theorem 4.1.

As an application, the symmetry algebra of the third-order ODE $y''' = f(x, y, y', y'')$ where both of the two relative invariants I and ν_1 are nonzero is characterized, by direct calculation of Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3,\beta_4)}{\partial(x,y,y',y'')}$ for the canonical forms in Appendix E, as follows:

1) The symmetry algebra is 4-dimensional iff the rank of the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3,\beta_4)}{\partial(x,y,y',y'')}$ is zero (the differential invariants $\beta_1, \beta_2, \beta_3$ and β_4 are constant).

2) The symmetry algebra is 3-dimensional iff the rank of the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3,\beta_4)}{\partial(x,y,y',y'')}$ is one.

3) The symmetry algebra is 2-dimensional iff the rank of the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3,\beta_4)}{\partial(x,y,y',y'')}$ is two.

4) The symmetry algebra is 1-dimensional iff the rank of the Jacobian matrix $\frac{\partial(\beta_1,\beta_2,\beta_3,\beta_4)}{\partial(x,y,y',y'')}$ is three.

All fourth-order differential invariants and the invariant differentiation operators in the degenerate case $I \neq 0,\ \nu_1 = 0$ and $\Delta_q I \neq 0$ are determined in Theorems 5.1.

As an application, we provide invariant descriptions of all the canonical forms in the complex plane with four infinitesimal symmetries [17] for third-order ODEs $y''' = f(x, y, y', y'')$ which are not quadratic in the second-order derivative as constant invariants.

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Appendix A: The differential operators of the homogeneous linear system of PDEs (2.40)

$$X_{22} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]$$
$$X_{23} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 6\,z_3, 0, 0, 0, 0, 0, 0]$$
$$X_{24} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 5\,z_3, 0, 0, 15\,{z_3}^2, 0, 0, 0, 0, 0, 0]$$
$$X_{25} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4\,z_3, 0, 0, 0, 0, 0, 0, 10\,{z_3}^2, 0, 0, 20\,{z_3}^3, 0, 0, 0, 0, 0, 0]$$
$$X_{26} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3\,z_3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6\,{z_3}^2, 0, 0, 0, 0, 0, 0, 10\,{z_3}^3, 0, 0, 15\,{z_3}^4, 0, 0, 0, 0, 0, 0]$$
$$X_{27} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3\,{z_3}^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4\,{z_3}^3, 0, 0, 0, 0, 0, 0, 5\,{z_3}^4, 0, 0, 6\,{z_3}^5, 0, 0, 0, 0, 0, 0]$$
$$X_{28} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, {z_3}^3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, {z_3}^4, 0, 0, 0, 0, 0, 0, {z_3}^5, 0, 0, {z_3}^6, 0, 0, 0, 0, 0, 0]$$
$$T_{22} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, 0, 0, 0, 0, 0]$$
$$T_{23} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, 0, -6\,{z_3}^2, 0, 0, 0, 0, 0, 0]$$
$$T_{24} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, 0, 0, 0, 0, 0, -5\,{z_3}^2, 0, 0, -15\,{z_3}^3, 0, 0, 0, 0, 0, 0]$$
$$T_{25} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -z_3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4\,{z_3}^2, 0, 0, 0, 0, 0, 0, -10\,{z_3}^3, 0, 0, -20\,{z_3}^4, 0, 0, 0, 0, 0, 0]$$
$$T_{26} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3\,{z_3}^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -6\,{z_3}^3, 0, 0, 0, 0, 0, 0, -10\,{z_3}^4, 0, 0, -15\,{z_3}^5, 0, 0, 0, 0, 0, 0]$$
$$T_{27} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3\,{z_3}^3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4\,{z_3}^4, 0, 0, 0, 0, 0, 0, -5\,{z_3}^5, 0, 0, -6\,{z_3}^6, 0, 0, 0, 0, 0, 0]$$
$$T_{28} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -{z_3}^4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -{z_3}^5, 0, 0, 0, 0, 0, 0, -{z_3}^6, 0, 0, -{z_3}^7, 0, 0, 0, 0, 0, 0]$$

Appendix B: The differential operators of the homogeneous linear system of PDEs (2.43)

$$X_{16} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]$$
$$X_{17} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 5\,l_3, 3, 0, 0, 0, 0, 0, 0, 0]$$
$$X_{18} = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4\,l_3, 0, 0, 3, 0, 0, 0, 0, 10\,{l_3}^2, 12\,l_3, 0, 0, 0, 0, 0, 0, 0]$$
$$X_{19} = [0, 0, 0, 0, 0, 0, 0, 0, 3\,l_3, 0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6\,{l_3}^2, 0, 0, 9\,l_3, 0, 0, 0, 0, 10\,{l_3}^3, 18\,{l_3}^2, 0, 0, 0, 0, 0, 0, 0]$$
$$X_{20} = [0, 0, 0, 0, 0, 0, 0, 0, 3\,{l_3}^2, 0, 0, 0, 0, 0, 6\,l_3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4\,{l_3}^3, 0, 0, 9\,{l_3}^2, 0, 0, 0, 0, 5\,{l_3}^4, 12\,{l_3}^3, 0, 0, 0, 0, 0, 0, 0]$$
$$X_{21} = [0, 0, 0, 0, 0, 0, 0, 0, {l_3}^3, 0, 0, 0, 0, 0, 3\,{l_3}^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, {l_3}^4, 0, 0, 3\,{l_3}^3, 0, 0, 0, 0, {l_3}^5, 3\,{l_3}^4, 0, 0, 0, 0, 0, 0, 0]$$
$$T_{16} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -l_3, -1, 0, 0, 0, 0, 0, 0, 0]$$
$$T_{17} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -l_3, 0, 0, -1, 0, 0, 0, 0, -5\,{l_3}^2, -8\,l_3, 0, 0, 0, 0, 0, 0, 0]$$
$$T_{18} = [0, 0, 0, 0, 0, 0, 0, 0, -l_3, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4\,{l_3}^2, 0, 0, -7\,l_3, 0, 0, 0, 0, -10\,{l_3}^3, -22\,{l_3}^2, 0, 0, 0, 0, 0, 0, 0]$$
$$T_{19} = [0, 0, 0, 0, 0, 0, 0, 0, -3\,{l_3}^2, 0, 0, 0, 0, 0, -6\,l_3, 0, 0, 0, 0, 0, 0, 0, 0, 0, -6\,{l_3}^3, 0, 0, -15\,{l_3}^2, 0, 0, 0, 0, -10\,{l_3}^4, -28\,{l_3}^3, 0, 0, 0, 0, 0, 0, 0]$$
$$T_{20} = [0, 0, 0, 0, 0, 0, 0, 0, -3\,{l_3}^3, 0, 0, 0, 0, 0, -9\,{l_3}^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4\,{l_3}^4, 0, 0, -13\,{l_3}^3, 0, 0, 0, 0, -5\,{l_3}^5, -17\,{l_3}^4, 0, 0, 0, 0, 0, 0, 0]$$
$$T_{21} = [0, 0, 0, 0, 0, 0, 0, 0, -{l_3}^4, 0, 0, 0, 0, 0, -4\,{l_3}^3, 0, 0, 0, 0, 0, 0, 0, 0, 0, -{l_3}^5, 0, 0, -4\,{l_3}^4, 0, 0, 0, 0, -{l_3}^6, -4\,{l_3}^5, 0, 0, 0, 0, 0, 0, 0]$$

Appendix C: The differential operators of the homogeneous linear system of PDEs (2.45)

$X_{11} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

$X_{12} = [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4\,m_3, 3, 0, -m_{13}, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0]$

$X_{13} = [0, 0, 0, 0, 0, 3\,m_3, 0, 0, 3, 0, 0, 0, 0, -m_{13}, 0, 0, 0, 0, 0, 0, 0, 6\,{m_3}^2, 9\,m_3, 0, 3 - 3\,m_3m_{13}, 6, 0, 0, 9\,m_3, 0, 0, 0, 0, 0, 0]$

$X_{14} = [0, 0, 0, 0, 0, 3\,{m_3}^2, 0, 0, 6\,m_3, 0, 0, 0, 0, 3 - 2\,m_3m_{13}, 6, 0, 0, 0, 0, 0, 0, 4\,{m_3}^3, 9\,{m_3}^2, 0, 6\,m_3 - 3\,{m_3}^2m_{13}, 12\,m_3, 0, 0, 9\,{m_3}^2, 0, 0, 0, 0, 0, 0]$

$X_{15} = [0, 0, 0, 0, 0, {m_3}^3, 0, 0, 3\,{m_3}^2, 0, 0, 0, 0, -{m_3}^2m_{13} + 3\,m_3, 6\,m_3, 0, 0, 0, 0, 0, 0, {m_3}^4, 3\,{m_3}^3, 0, -{m_3}^3m_{13} + 3\,{m_3}^2, 6\,{m_3}^2, 0, 0, 3\,{m_3}^3, 0, 0, 0, 0, 0, 0]$

$T_{11} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -m_3, -1, 0, 0, 0, 0, 0, -3, 0, 0, 0, 0, 0, 0]$

$T_{12} = [0, 0, 0, 0, 0, -m_3, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4\,{m_3}^2, -7\,m_3, 0, m_3m_{13} - 3, -6, 0, 0, -15\,m_3, 0, 0, 0, 0, 0, 0]$

$T_{13} = [0, 0, 0, 0, 0, -3\,{m_3}^2, 0, 0, -6\,m_3, 0, 0, 0, 0, m_3m_{13} - 3, -6, 0, 0, 0, 0, 0, 0, -6\,{m_3}^3, -15\,{m_3}^2, 0, -12\,m_3 + 3\,{m_3}^2m_{13}, -24\,m_3, 0, 0, -27\,{m_3}^2,$
$0, 0, 0, 0, 0, 0]$

$T_{14} = [0, 0, 0, 0, 0, -3\,{m_3}^3, 0, 0, -9\,{m_3}^2, 0, 0, 0, 0, 2\,{m_3}^2m_{13} - 9\,m_3, -18\,m_3, 0, 0, 0, 0, 0, 0, -4\,{m_3}^4, -13\,{m_3}^3, 0, -15\,{m_3}^2 + 3\,{m_3}^3m_{13}, -30\,{m_3}^2,$
$0, 0, -21\,{m_3}^3, 0, 0, 0, 0, 0, 0]$

$T_{15} = [0, 0, 0, 0, 0, -{m_3}^4, 0, 0, -4\,{m_3}^3, 0, 0, 0, 0, {m_3}^3m_{13} - 6\,{m_3}^2, -12\,{m_3}^2, 0, 0, 0, 0, 0, 0, -{m_3}^5, -4\,{m_3}^4, 0, -6\,{m_3}^3 + {m_3}^4m_{13}, -12\,{m_3}^3,$
$0, 0, -6\,{m_3}^4, 0, 0, 0, 0, 0, 0]$

Appendix D: The differential operators of the homogeneous linear system of PDEs (2.47)

$X_1 = [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$

$X_2 = [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, n_{24}, 0, 0, 0, 0]$

$X_3 = [0, 0, n_3, n_4, n_5, 0, 0, -n_8, -n_9, -n_{10}, -n_{11}, -2\,n_{12}, -2\,n_{13}, -2\,n_{14}, -2\,n_{15}, -2\,n_{16}, -2\,n_{17}, 0, -n_{19}, -n_{20}, -n_{21}, -n_{22}, 0, 0, n_{25}, n_{26}, n_{27}, 0, 0]$

$X_4 = [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, n_{24}, 0, 0, 0]$

$X_5 = [0, 0, 0, 2\,n_3, 3\,n_4, -2\,n_7, 3, -n_{10}, -4\,n_{10}, -2\,n_{11}, 0, -2\,n_{13} - n_{15}, -n_{16}, -6\,n_{15}, -4\,n_{16}, -2\,n_{17}, 0, 0, -n_6n_{11} + 5\,n_8 + 3\,n_9 - 3\,\frac{n_{14}}{n_{11}} - n_{21},$
$12\,n_8 + 5\,n_9 - 6\,\frac{n_{14}}{n_{11}} - 4\,n_{21}, -n_{10} - 2\,n_{22}, -n_{11}, -6\,n_6 + 6\,n_{18}, 0, 0, n_{25} + n_{24}n_3, 2\,n_{26} + n_{24}n_4, 0, n_{24}]$

$X_6 = [0, 0, 0, {n_3}^2, 3\,n_3n_4, -2\,n_7n_3 + 3\,n_4, 3\,n_3, -n_4n_{11} - n_3n_{10}, -4\,n_3n_{10} - 2\,n_7, 3 - 2\,n_3n_{11}, 0, -2\,n_3n_{13} - n_3n_{15} - n_{10} - n_4n_{16}, -n_3n_{16}$
$-n_{11} - n_4n_{17}, -6\,n_3n_{15} - 6\,n_{10}, -4\,n_3n_{16} - 2\,n_{11}, -2\,n_3n_{17}, 0, 3\,n_4, -n_{11}n_7n_4 - n_{11}n_6n_3 + 5\,n_3n_8 + 3\,n_3n_9 + 2\,n_4n_{10} + 12\,\frac{n_4n_{13}}{n_{11}}$
$-3\,\frac{n_3n_{14}}{n_{11}} - 3\,\frac{n_4n_{15}}{n_{11}} - 3\,\frac{n_9}{n_{11}} - n_3n_{21} - n_4n_{22}, 12\,n_3n_8 + 5\,n_3n_9 + 2\,n_4n_{10} + 24\,\frac{n_4n_{13}}{n_{11}} - 6\,\frac{n_3n_{14}}{n_{11}} - 6\,\frac{n_4n_{15}}{n_{11}} - 6\,\frac{n_9}{n_{11}} - 4\,n_3n_{21} - 2\,n_{18},$
$-n_3n_{10} - 2\,n_3n_{22} - 2\,n_4n_{11}, -n_3n_{11}, -6\,n_6n_3 + 6\,n_3n_{18}, 0, 0, n_{25}n_3, n_{25}n_4 + 2\,n_{26}n_3, 0, n_{25}]$

$X_7 = [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, n_{24}, 0, 0]$

$X_8 = [0, 0, 0, 0, 3\,n_3, 3, 0, -n_{11}, 0, 0, 0, -n_{16}, -n_{17}, 0, 0, 0, 0, 3, -n_7n_{11} + 2\,n_{10} + 12\,\frac{n_{13}}{n_{11}} - 3\,\frac{n_{15}}{n_{11}} - n_{22}, 2\,n_{10} + 24\,\frac{n_{13}}{n_{11}} - 6\,\frac{n_{15}}{n_{11}}, -2\,n_{11}, 0, 0, 0, 0, 0,$
$n_{25} + 2\,n_{24}n_3, 0, 0]$

$X_9 = [0, 0, 0, 0, 3\,{n_3}^2, 6\,n_3, 0, 3 - 2\,n_3n_{11}, 6, 0, 0, -2\,n_{11} - 2\,n_3n_{16}, -2\,n_3n_{17}, 0, 0, 0, 0, 6\,n_3, -2\,n_{11}n_7n_3 + 6\,n_7 + 4\,n_3n_{10} - 6\,\frac{n_{10}}{n_{11}} + 24\,\frac{n_3n_{13}}{n_{11}}$
$-6\,\frac{n_3n_{15}}{n_{11}} - 2\,n_3n_{22}, 10\,n_7 + 4\,n_3n_{10} - 12\,\frac{n_{10}}{n_{11}} + 48\,\frac{n_3n_{13}}{n_{11}} - 12\,\frac{n_3n_{15}}{n_{11}}, 3 - 4\,n_3n_{11}, 0, 0, 0, 0, 0, n_{24}{n_3}^2 + 2\,n_{25}n_3, 0, 0]$

$X_{10} = [0, 0, 0, 0, {n_3}^3, 3\,{n_3}^2, 0, -{n_3}^2n_{11} + 3\,n_3, 6\,n_3, 0, 0, 3 - 2\,n_3n_{11} - {n_3}^2n_{16}, -{n_3}^2n_{17}, 6, 0, 0, 0, 3\,{n_3}^2, -n_{11}n_7{n_3}^2 + 2\,{n_3}^2n_{10} + 6\,n_7n_3$
$+12\,\frac{{n_3}^2n_{13}}{n_{11}} - 6\,\frac{n_3n_{10}}{n_{11}} - 3\,\frac{{n_3}^2n_{15}}{n_{11}} - 6\,\frac{n_7}{n_{11}} - {n_3}^2n_{22}, 2\,{n_3}^2n_{10} + 10\,n_7n_3 + 24\,\frac{{n_3}^2n_{13}}{n_{11}} - 12\,\frac{n_3n_{10}}{n_{11}} - 6\,\frac{{n_3}^2n_{15}}{n_{11}} - 12\,\frac{n_7}{n_{11}}, -2\,{n_3}^2n_{11}$
$+3\,n_3, 0, 0, 0, 0, 0, n_{25}{n_3}^2, 0, 0]$

$$
\begin{aligned}
T_1 = \quad & [1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] \\
T_2 = \quad & [0,0,-n_3,-2\,n_4,-3\,n_5,-2\,n_6,-n_7,-n_8,-n_9,0,n_{11},0,n_{13},0,n_{15},2\,n_{16},3\,n_{17},-2\,n_{18},-2\,n_{19},-2\,n_{20},-n_{21},0,-3\,n_{23},n_{24},0, \\
& -n_{26},-2\,n_{27},0,0] \\
T_3 = \quad & [0,0,-{n_3}^2,-3\,n_3n_4,-3\,{n_4}^2-4\,n_5n_3,-4\,n_5+3\,n_7n_4-2\,n_6n_3,-n_7n_3-6\,n_4,-n_{18}+n_4n_{10}+n_5n_{11},-6\,n_6+6\,n_4n_{10},-n_7+n_3n_{10} \\
& +3\,n_4n_{11},-6+2\,n_3n_{11},n_4n_{15}+3\,n_4n_{13}-n_{21}-n_8+2\,n_3n_{12}+n_5n_{16},-n_{22}+n_4n_{16}+3\,n_3n_{13}+n_5n_{17},-6\,n_9+2\,n_3n_{14}+9\,n_4n_{15}, \\
& 3\,n_3n_{15}+6\,n_4n_{16},2\,n_{11}+4\,n_3n_{16}+3\,n_4n_{17},5\,n_3n_{17},-6\,n_5-n_7n_4-2\,n_3n_{18},-3\,\frac{n_{20}}{n_{11}}+12\,\frac{n_{19}}{n_{11}}+3\,\frac{n_5n_{15}}{n_{11}}+3\,\frac{n_4n_{14}}{n_{11}}+6\,\frac{n_4n_{12}}{n_{11}} \\
& +n_{11}n_6n_4-12\,\frac{n_5n_{13}}{n_{11}}+n_{11}n_5n_7-2\,n_5n_{10}+n_4n_{21}-3\,n_4n_9-9\,n_4n_8+n_5n_{22}-n_3n_{19}-n_{23},-6\,\frac{n_{20}}{n_{11}}+24\,\frac{n_{19}}{n_{11}}+6\,\frac{n_5n_{15}}{n_{11}}-n_3n_{20} \\
& +6\,\frac{n_4n_{14}}{n_{11}}+12\,\frac{n_4n_{12}}{n_{11}}-24\,\frac{n_5n_{13}}{n_{11}}+6\,n_4n_{21}-6\,n_4n_9-18\,n_4n_8,3\,n_4n_{22}+n_4n_{10}-n_{18}+3\,n_5n_{11},2\,n_4n_{11}+n_3n_{22},-4\,n_5n_7+12\,n_6n_4 \\
& -12\,n_4n_{18}-3\,n_3n_{23},n_{25},0,-2\,n_{26}n_3,-3\,n_{26}n_4-3\,n_{27}n_3,n_{26},-n_{26}] \\
T_4 = \quad & [0,0,0,-n_3,-3\,n_4,n_7,-3,0,2\,n_{10},n_{11},0,n_{13},0,3\,n_{15},2\,n_{16},n_{17},0,-n_7,-4\,n_8+6\,\frac{n_{12}}{n_{11}},-6\,n_8+12\,\frac{n_{12}}{n_{11}}+2\,n_{21}-n_9,n_{22},n_{11},6\,n_6-6\,n_{18}, \\
& 0,0,-n_{24}n_3,-n_{26}-2\,n_{24}n_4,n_{24},0] \\
T_5 = \quad & [0,0,0,-2\,{n_3}^2,-9\,n_3n_4,-9\,n_4+4\,n_7n_3,-9\,n_3,2\,n_4n_{11}+n_3n_{10}-n_7,8\,n_3n_{10}+4\,n_7,-9+4\,n_3n_{11},0,n_3n_{15}+4\,n_3n_{13}+2\,n_4n_{16},n_{11} \\
& +n_3n_{16}+2\,n_4n_{17},12\,n_{10}+12\,n_3n_{15},4\,n_{11}+8\,n_3n_{16},4\,n_3n_{17},0,-15\,n_4-2\,n_7n_3,n_{11}n_6n_3+2\,n_{11}n_7n_4-13\,n_3n_8-3\,n_3n_9+6\,n_6 \\
& -4\,n_4n_{10}+24\,\frac{n_8}{n_{11}}-24\,\frac{n_4n_{13}}{n_{11}}+12\,\frac{n_3n_{12}}{n_{11}}+3\,\frac{n_3n_{14}}{n_{11}}+6\,\frac{n_4n_{15}}{n_{11}}-3\,\frac{n_9}{n_{11}}+n_3n_{21}+2\,n_4n_{22}-2\,n_{18},-24\,n_3n_8-7\,n_3n_9+6\,n_6+2\,n_4n_{10} \\
& +48\,\frac{n_8}{n_{11}}-48\,\frac{n_4n_{13}}{n_{11}}+24\,\frac{n_3n_{12}}{n_{11}}+6\,\frac{n_3n_{14}}{n_{11}}+12\,\frac{n_4n_{15}}{n_{11}}-6\,\frac{n_9}{n_{11}}+8\,n_3n_{21}+4\,n_{18},n_3n_{10}+4\,n_3n_{22}+7\,n_4n_{11}-n_7,3\,n_3n_{11}-6,18\,n_6n_3 \\
& -12\,n_7n_4-18\,n_3n_{18},0,0,-n_{24}{n_3}^2-n_{25}n_3,-3\,n_3n_{24}n_4-4\,n_{26}n_3-2\,n_{25}n_4,n_{25}+n_{24}n_3,-n_{24}n_3] \\
T_6 = \quad & [0,0,0,-{n_3}^3,-6\,{n_3}^2n_4,-12\,n_3n_4+3\,n_7{n_3}^2,-6\,{n_3}^2,{n_3}^2n_{10}-n_7n_3+3\,n_3n_4n_{11}-6\,n_4,-12\,n_4+6\,{n_3}^2n_{10}+6\,n_7n_3,-12\,n_3+3\,{n_3}^2n_{11}, \\
& 0,{n_3}^2n_{15}+3\,{n_3}^2n_{13}+3\,n_3n_4n_{16}+n_3n_{10}-n_7+3\,n_4n_{11},-6+{n_3}^2n_{16}+3\,n_3n_4n_{17}+2\,n_3n_{11},6\,n_7+18\,n_3n_{10}+9\,{n_3}^2n_{15},-12+6\,n_3n_{11} \\
& +6\,{n_3}^2n_{16},3\,{n_3}^2n_{17},0,-18\,n_3n_4-n_7{n_3}^2,n_{11}n_6{n_3}^2-6\,n_3n_4n_{10}+3\,\frac{{n_3}^2n_{14}}{n_{11}}+6\,\frac{{n_3}^2n_{12}}{n_{11}}+3\,n_3n_4n_{22}+12\,\frac{n_4n_{10}}{n_{11}}+24\,\frac{n_3n_8}{n_{11}} \\
& +3\,n_{11}n_3n_7n_4-36\,\frac{n_3n_4n_{13}}{n_{11}}+9\,\frac{n_3n_4n_{15}}{n_{11}}+{n_3}^2n_{21}-3\,{n_3}^2n_9-9\,{n_3}^2n_8+6\,\frac{n_{18}}{n_{11}}-18\,\frac{n_6}{n_{11}}-10\,n_7n_4+6\,n_6n_3-2\,n_3n_{18},-6\,{n_3}^2n_9 \\
& -18\,{n_3}^2n_8+6\,n_6n_3-12\,n_7n_4+12\,\frac{{n_3}^2n_{12}}{n_{11}}-72\,\frac{n_3n_4n_{13}}{n_{11}}+48\,\frac{n_3n_8}{n_{11}}+12\,\frac{n_{18}}{n_{11}}+24\,\frac{n_4n_{10}}{n_{11}}+6\,\frac{{n_3}^2n_{14}}{n_{11}}+18\,\frac{n_3n_4n_{15}}{n_{11}}-36\,\frac{n_6}{n_{11}}+6\,{n_3}^2n_{21} \\
& +6\,n_3n_{18},3\,{n_3}^2n_{22}+{n_3}^2n_{10}+9\,n_3n_4n_{11}-n_7n_3-12\,n_4,-6\,n_3+2\,{n_3}^2n_{11},12\,n_6{n_3}^2-12\,n_3n_7n_4-12\,{n_3}^2n_{18},0,0,-n_{25}{n_3}^2, \\
& -3\,n_{25}n_3n_4-3\,n_{26}{n_3}^2,n_{25}n_3,-n_{25}n_3] \\
T_7 = \quad & [0,0,0,0,-n_3,-1,0,0,0,0,0,0,0,0,0,0,0,-3,0,2\,n_{10},n_{11},0,-4\,n_7,0,0,0,-n_{24}n_3,0,0] \\
T_8 = \quad & [0,0,0,0,-3\,{n_3}^2,-6\,n_3,0,n_3n_{11}-3,-6,0,0,n_{11}+n_3n_{16},n_3n_{17},0,0,0,0,-12\,n_3,n_{11}n_7n_3-4\,n_7-2\,n_3n_{10}-12\,\frac{n_3n_{13}}{n_{11}}+6\,\frac{n_{10}}{n_{11}} \\
& +3\,\frac{n_3n_{15}}{n_{11}}+n_3n_{22},-2\,n_7+4\,n_3n_{10}-24\,\frac{n_3n_{13}}{n_{11}}+12\,\frac{n_{10}}{n_{11}}+6\,\frac{n_3n_{15}}{n_{11}},-9+5\,n_3n_{11},0,-12\,n_7n_3,0,0,0,-2\,n_{24}{n_3}^2-n_{25}n_3,0,0] \\
T_9 = \quad & [0,0,0,0,-3\,{n_3}^3,-9\,{n_3}^2,0,2\,{n_3}^2n_{11}-9\,n_3,-18\,n_3,0,0,-9+2\,{n_3}^2n_{16}+4\,n_3n_{11},2\,{n_3}^2n_{17},-18,0,0,0,-15\,{n_3}^2,2\,n_{11}n_7{n_3}^2-4\,{n_3}^2n_{10} \\
& -14\,n_7n_3+18\,\frac{n_7}{n_{11}}-24\,\frac{{n_3}^2n_{13}}{n_{11}}+18\,\frac{n_3n_{10}}{n_{11}}+6\,\frac{{n_3}^2n_{15}}{n_{11}}+2\,{n_3}^2n_{22},2\,{n_3}^2n_{10}-14\,n_7n_3+36\,\frac{n_7}{n_{11}}-48\,\frac{{n_3}^2n_{13}}{n_{11}}+36\,\frac{n_3n_{10}}{n_{11}}+12\,\frac{{n_3}^2n_{15}}{n_{11}}, \\
& 7\,{n_3}^2n_{11}-21\,n_3,0,-12\,n_7{n_3}^2,0,0,0,-n_{24}{n_3}^3-2\,n_{25}{n_3}^2,0,0] \\
T_{10} = \quad & [0,0,0,0,-{n_3}^4,-4\,{n_3}^3,0,{n_3}^3n_{11}-6\,{n_3}^2,-12\,{n_3}^2,0,0,3\,{n_3}^2n_{11}+{n_3}^3n_{16}-12\,n_3,{n_3}^3n_{17},-24\,n_3,0,0,0,-6\,{n_3}^3,{n_3}^3n_{11}n_7-2\,{n_3}^3n_{10} \\
& -10\,n_7{n_3}^2-12\,\frac{{n_3}^3n_{13}}{n_{11}}+12\,\frac{{n_3}^2n_{10}}{n_{11}}+24\,\frac{n_7n_3}{n_{11}}+3\,\frac{{n_3}^3n_{15}}{n_{11}}+{n_3}^3n_{22},-12\,n_7{n_3}^2-24\,\frac{{n_3}^3n_{13}}{n_{11}}+24\,\frac{{n_3}^2n_{10}}{n_{11}}+48\,\frac{n_7n_3}{n_{11}}+6\,\frac{{n_3}^3n_{15}}{n_{11}}, \\
& 3\,{n_3}^3n_{11}-12\,{n_3}^2,0,-4\,n_7{n_3}^3,0,0,0,-n_{25}{n_3}^3,0,0]
\end{aligned}
$$

Appendix E: Group classification of third-order ODEs in the complex domain

Algebra	Representative Equations
L_I	$y''' = f(y, y', y'')$
$L_{2:1}^{I}$	$y''' = f(y', y'')$
$L_{2:1}^{II}$	$y''' = f(x, y'')$
$L_{2:2}^{I}$	$y''' = y''^2 f(y', xy'')$
$L_{2:2}^{II}$	$y''' = y' f(x, y''/y')$
$L_{3:2}$	$y''' = f(y'')$
$L_{3:3}^{I}$	$y''' = y''^2 f(y'' \exp y')$
$L_{3:3}^{II}$	$y''' = y'' f(\exp x / y'')$
$L_{3:4}^{I}$	$y''' = y''^{3/2} f(y''\ y'^{-2})$
$L_{3:4}^{II}$	$y''' = y''^2 f(xy'')$
$L_{3:5}^{I}$	$y''' = y''^2 f(y')$
$L_{3:6}^{I}$	$y''' = y''^{\frac{a-3}{a-2}} f(y''^{1-a}\ y'^{a-2}),\ a \neq 0, 1, 2: \quad y''' = y'^{-1} f(y''), a = 2$
$L_{3:6}^{II}$	$y''' = y''^{\frac{2-3a}{1-2a}} f(x^{1-2a}\ y''^{1-a}),\ a \neq 0, 1, \frac{1}{2}: \quad y''' = x^{-1} f(y''), a = \frac{1}{2}$
$L_{3:8}^{I}$	$y''' = 3\frac{y''^2}{y'} + \frac{y'^4}{x^2} f\left(\frac{2xy'' + y'}{y'^3}\right)$
$L_{3:8}^{III}$	$y''' = 3\frac{y' y''^2}{y'^2 - 1} + \frac{(y'^2 - 1)^2}{x^2} f\left(\frac{xy'' - y' + y'^3}{(1 - y'^2)^{3/2}}\right)$
$L_{3:8}^{IV}$	$y''' = \frac{3}{2}\frac{y''^2}{y'} + y' f(x)$

$L_{4:1}$	$y''' = \frac{h'''(x)}{h''(x)} y'', \quad h''' \neq 0$
$L_{4:2}$	$y''' = C\frac{y''^2}{y'}, \quad C \neq 3, \frac{3}{2}$
$L_{4:3}$	$y''' = Cy''^{\left(\frac{b-2}{b-1}\right)}, \ b \neq 0, 1, 2$
$L^I_{4:5}$	$y''' = \frac{3}{2}\frac{y''^2}{y'} + Cy'$
$L^{II}_{4:5}$	$y''' = 3\frac{y''^2}{y'} + C\frac{(2xy'' + y')^{3/2}}{x^2\sqrt{y'}}$
$L_{4:6}$	$y''' = C \exp(-y'')$
L_5	$y''' = ky' + ly, l \neq 0$
$L_{6:1}$	$y''' = \frac{3}{2}\frac{y''^2}{y'}$
$L_{6:2}$	$y''' = \frac{3y'y''^2}{1 + y'^2}$
L_7	$y''' = 0$

In the above table, f is an arbitrary function of its argument(s) and C is an arbitrary non-zero constant.

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