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Abstract

We consider a stochastic differential equation $\partial_t X = f(X_t, t) + \eta$, where η is a mixed noise. The Itô formula, for both cases Poisson and mixed noise, will be given. The truncated moments of mixed noise will be calculated. The generalized Feynman graphs and rules will be introduced, and a graphical representations of the truncated moments of mixed noise will be provided.

Key words: SDE, Itô formula, Feynman graphs and rules, mixed noise, truncated moments.
MSC (2010): 60H10, 60H35, 05C62.

1 Introduction and overview

We consider the stochastic differential equation (SDE):

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} X_t(x) = f(X_t, t) + \eta(t), \ \ (t, x) \in]0, \infty[\times \mathbb{R}^d \\ X_0(x) = x_0, \ \ x_0 \in \mathbb{R}^d. \end{array} \right. \tag{1}$$

where $\eta = \eta(t)$ is a stochastic noise, $X_t, \ f(x,t) \in C^2(\mathbb{R}^d \times [0, \infty[)$.
It is known that for Riemann integrals one use the fundamental theorem of calculus, which establishes a connection between integration and differentiation, however for stochastic integration we do not have such results but we have an Itô integral version of the chain rule called Itô formula.
In the case when $f(x,t) \in C^2(\mathbb{R}^d \times [0, \infty[)$, and η a Gaussian noise the Itô formula is known, see, e.g, [11].

The Itô formula is one of the most powerful tools of the stochastic analysis due to its vast range of applications, for example in Mathematical Finance, see. e.g [11]. However the applications of this formula to other fields like graph theory seems to be not well studied. It is therefore the aim of this work to provide a link between graph theory and Itô calculus.
In this work the Itô formula for pure jump noise is obtained for a given polynomial f, but it can be extended further by interpreting $\sum_{l=1}^{q} \frac{\eta^l}{l!} \left(\frac{\partial}{\partial X}\right)^l f$, as a pseudo differential operator in X. Moreover, we will show how the Poisson noise can be scaled to obtain a Gaussian noise,

and hence we get back to the classical Itô formula. Finding graphical representations to the truncated moments of mixed noise seems to be of a great importance since it will simplify the stochastic integrations and formulas therein, also it will establishes a connection between graph theory and Itô calculus.
The graphs introduced, during this work, are called Feynman graphs see, e.g [7, 15], and a numerical value[1] will be assigned to each graph.
Our graphical model is not restricted to SDE of type (1), in fact one can generalize equation (1) and make it more complicated, e.g by introduction of non linear terms, e.g for force F of gradient type, $F = \nabla V$ we obtain a non-linear SDE and one can ask the same questions as before, again expansion into graphs is possible. Generalizing further we pass from SDE's to the stochastic partial differential equations (SPDE's), see.e.g [8, 15].
Before we go over to describe the contents of the present paper, let us mention that, to the best of our knowledge, a graphical representations of the noise as done in this work, have not been considered before.
Let us now describe the single sections of the article:
In the next section we will start by some useful notations and representations of the Poisson noise, we consider the case of a polynomial function f, then we prove the Itô formula for pure jump noise.
Section 3 will be devoted to Gaussian and mixed noise, we will show how one can scale the Poisson noise to obtain Gaussian noise.
Section 4 is concerned with the proofs of the main results on the graphical representations of the truncated moments of mixed noise, for this aim we will introduce the Feynman graph and Feynman rules.

2 Stochastic equations driven by Poisson noise

In this section we start by introducing some useful notations:
Let $S(\mathbb{R}^d)$ be the Schwartz Space of all rapidly decreasing functions on $\mathbb{R}^d$ endowed with the Schwartz topology, its topological dual is the Space of tempered distribution noted by $S'(\mathbb{R}^d)$. We denote by $\langle ., . \rangle$ the dual pairing between $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$.
Let $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ be the Lévy characteristic, see [14], represented by

$$\psi(t) = i\, at - \frac{\sigma^2\, t^2}{2} + z \int_{\mathbb{R}^d \setminus \{0\}} (e^{ist} - 1)\, dr(s),\ \forall\, t \in \mathbb{R}. \tag{2}$$

Here $a \in \mathbb{R}^d$, $\sigma^2 \geq 0$, $z \geq 0$ and r a probability measure on $\mathbb{R}^d \setminus \{0\}$ such that its Fourier transform is entirely analytic.
Now we will consider a representation of the Poisson noise in terms of corresponding Poisson distribution:
Let $\Lambda_n \subset\subset \mathbb{R}^d$ be a monotone sequence of compact sets s.t, $\Lambda_n \uparrow \mathbb{R}^d$ as $n \longrightarrow \infty$ and $\Lambda_0 = \emptyset$.

[1]The procedure of given each graph a numerical value is called "perturbation theory".

For $n \in \mathbb{N}$, Let $L_n = \Lambda_n \backslash \Lambda_{n-1}$ and we denote the Lebesgue volume of L_n by $\mid L_n \mid$. Let

$$\eta_n^p = \sum_{j=1}^{N_T^z} S_j^n \delta_{T_j^n}, \quad T_j^n \sim \frac{dx}{T}|_{[0,T]}, \quad T > 0, ^2 \tag{3}$$

where δ_x is the Dirac measure of mass one in x, $\{S_j^n\}_{j \in \mathbb{N}}$ is a family of real valued and independent random variables with law given by r and N_T^z is a poisson random variable with intensity $z \mid L_n \mid$, i.e,

$$P(N_T^z = k) = e^{-z| L_n|} \frac{(z \mid L_n \mid)^k}{k!} \;;\; k \in \mathbb{N}_0 \tag{4}$$

The characteristic functional of the noise η_n^p for any function $h \in S(\mathbb{R}^d)$ such that $supph \subseteq L_n$ is given by:

$$\begin{aligned}
\langle e^{i\, \eta_n^p(h)} \rangle &= \langle e^{i \sum_{j=1}^{N_T^z} S_j^n\, h(X_j^n)} \rangle \\
&= e^{-z\,| L_n|} \sum_{l=0}^{\infty} \frac{(z \mid L_n \mid)^l}{l!} \Big(\int_{L_n} \int_{\mathbb{R}^d \setminus 0} e^{i\, s\, h(x)}\, dr(s)\, \frac{dx}{\mid L_n \mid} \Big)^l \\
&= \exp\{ z \int_{L_n} \int_{\mathbb{R}^d \setminus 0} (\, e^{i\, s\, h(x)} - 1) dr(s)\, dx\} = C_{\eta_n^p}(h), \forall\, h \in S(\mathbb{R}^d).
\end{aligned} \tag{5}$$

For a fixed random parameter w, let $T_1(w)$, $T_2(w)$, $\cdots$, $T_{N_T^z}(w)$ be the random times at which the jumps occur.

To avoid notations complications we will restrict to one dimension,(generalizing to $d > 1$ will be straightforward), we consider also polynomial transformations, i.e, we let $f(X,t) = \sum_{q=0}^{N} C_q(t)\, X^q,\ N \in \mathbb{N}$.

Lemma 2.1. *Let X and S be two random variables then:*

$$(X + S)^q - X^q = \sum_{l=1}^{q} \frac{S^l}{l!} \Big(\frac{\partial}{\partial X}\Big)^l X^q, \quad q \in \mathbb{N} \tag{6}$$

Proof. We have

$$\begin{aligned}
(X + S)^q - X^q &= \sum_{l=0}^{q} \frac{q!}{(q-l)!\, l!} X^{q-l}\, S^l - X^q \\
&= \sum_{l=1}^{q} \frac{q!}{(q-l)!\, l!} X^{q-l}\, S^l \\
&= \sum_{l=1}^{q} \frac{q!}{(q-l)!\, l!} \Big(\frac{\partial}{\partial X}\Big)^l X^q \times \frac{(q-l)!}{q!} S^l \\
&= \sum_{l=1}^{q} \frac{S^l}{l!} \Big(\frac{\partial}{\partial X}\Big)^l X^q
\end{aligned}$$

[2]The index p here, is just a notation and we mean poisson noise.

■

Since $\frac{df}{dt}(X,t) = \dot{f}(X,t) = \sum_{q=0}^{N} \left(C_q(t) \dot{X^q}\right)$, it suffices then to determine $\left(C_q \dot{X^q}\right)$ for $t \notin \{T_1, \cdots, T_{N_T^z}\}$.

Proposition 2.2. *Let $T_1, \cdots, T_{N_T^z}$ be a discrete random times and $\breve{X}$ be a derivative without stochastic terms then:*

$$\left(C_q \dot{X^q}\right) = \dot{C}_q X^q + q X^{q-1} \breve{X} + \sum_{l=1}^{q} \frac{\eta^l}{l!} \left(\frac{\partial}{\partial X}\right)^l C_q X^q \tag{7}$$

Proof. Let $X(T_j^-) = \lim_{t \uparrow T_j} X(t)$ and $S_j = X(T_j) - X(T_j^-)$, we have

$$\begin{aligned} \left(C_q \dot{X^q}\right) &= \dot{C}_q X^q + q X^{q-1} \dot{X} \\ &= \dot{C}_q X^q + q X^{q-1} \breve{X} + \sum_{j=1}^{N_T^z} \delta_{T_j}(t) C_q \left[\left(X(T_j^-) + S_j\right)^q - X^q(T_j^-)\right], \end{aligned}$$

using lemma (2.1) and the fact that $\eta^l = \sum_{j=1}^{N_T^z} S_j^l \delta_{T_j}$, which is again a poisson random field, we obtain

$$\begin{aligned} \left(C_q \dot{X^q}\right) &= \dot{C}_q X^q + q X^{q-1} \breve{X} + \sum_{j=1}^{N_T^z} \delta_{T_j}(t) \sum_{l=1}^{q} \frac{S_j^l}{l!} \left(\frac{\partial}{\partial X}\right)^l X^q \\ &= \dot{C}_q X^q + q X^{q-1} \breve{X} + \sum_{l=1}^{q} \frac{\eta^l}{l!} \left(\frac{\partial}{\partial X}\right)^l C_q X^q. \end{aligned} \tag{8}$$

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By lemma (2.1) and proposition (2.2), the following results holds:

Theorem 2.3. *The Itô formula for pure jump noise is given by:*

$$\dot{f}(X,t) = \frac{\partial f}{\partial t}(X,t) + \frac{\partial f}{\partial X}(X,t) \dot{X} + \sum_{l=2}^{q} \frac{\eta^l}{l!} \left(\frac{\partial}{\partial X}\right)^l f(X,t) \tag{9}$$

Proof. From proposition (2.2) we have:

$$\begin{aligned} \dot{f}(X,t) &= \sum_{q=0}^{N} \dot{C}_q X^q + \sum_{q=0}^{N} q X^{q-1} \breve{X} + \sum_{l=1}^{q} \frac{\eta^l}{l!} \left(\frac{\partial}{\partial X}\right)^l \sum_{q=0}^{N} C_q X^q \\ &= \frac{\partial f}{\partial t}(X,t) + \frac{\partial f}{\partial X}(X,t) \breve{X} + \sum_{l=1}^{q} \frac{\eta^l}{l!} \left(\frac{\partial}{\partial X}\right)^l f(X,t) \\ &= \frac{\partial f}{\partial t}(X,t) + \frac{\partial f}{\partial X}(X,t) \dot{X} + \sum_{l=2}^{q} \frac{\eta^l}{l!} \left(\frac{\partial}{\partial X}\right)^l f(X,t) \end{aligned} \tag{10}$$

■

3 How to obtain a Gaussian noise from Poisson process

We consider the generalizing functional of a function f for a Poisson noise η^p:

$$C_{\eta^p}(f) = \exp\Big(z \int_0^T \int_{\mathbb{R}^d\setminus\{0\}} (e^{i\,s\,f(t)} - 1)\,dr(s)\,dt\Big), \quad z > 0, \tag{11}$$

where $\int_{\mathbb{R}^d\setminus\{0\}} s\,dr(s) = 0$, and let $c_2 = \int_{\mathbb{R}^d\setminus\{0\}} s^2\,dr(s)$.

Lemma 3.1. *Let C_{η^p} be the generalizing function of a function f given by equation (11) and η^g the Gaussian noise, then:*

$$\lim_{z\longrightarrow\infty} C_{\eta^p}(f) = C_{\eta^g}(f) = \begin{cases} \exp\Big(-\frac{c_2}{2}\int_0^T f^2(t)\,dt\Big) & ,if\ n=2 \\ 0 & ,\ otherweise \end{cases} \tag{12}$$

Proof. Consider the transformation $s \longrightarrow \frac{s}{\sqrt{z}}$ and let $z \longrightarrow \infty$, then:

$$\begin{aligned} \lim_{z\longrightarrow\infty} C_{\eta^p}(f) &= \lim_{z\longrightarrow\infty} \exp\Big(z \int_0^T \int_{\mathbb{R}\setminus\{0\}} (e^{i\,\frac{s}{\sqrt{z}}\,f(t)} - 1)\,dr(s)\,dt\Big) \\ &= \lim_{z\longrightarrow\infty} \exp\Big(z \int_0^T \int_{\mathbb{R}\setminus\{0\}} e^{i\,\frac{s}{\sqrt{z}}\,f(t)}\,dr(s)\,dt\Big) \\ &= \lim_{z\longrightarrow\infty} \exp\Big(z \int_0^T \int_{\mathbb{R}\setminus\{0\}} \sum_{n=0}^{\infty} \frac{i^n}{n!}\frac{s^n}{z^{\frac{n}{2}}} f^n(t)\,dr(s)\,dt\Big) \\ &= \exp\Big(\int_0^T \int_{\mathbb{R}\setminus\{0\}} -\frac{1}{2} s^2 f^2(t)\,dr(s)\,dt\Big) \\ &= \exp\Big(-\frac{c_2}{2}\int_0^T f^2(t)\,dt\Big) \\ &= C_{\eta^g} \end{aligned} \tag{13}$$

■

Lemma (3.1) shows that one can scale the Poisson noise to obtain a Gaussian noise with intensity c_2, under this scale it is therefore important to think about the Itô formula in case of higher order $\eta^{q\prime}s$.

Lemma 3.2. *Let η^p be the jump noise given by equation (3), then the following holds:*

$$\lim_{z\longrightarrow\infty} C_{(\eta^p)^q}(f) = \begin{cases} \exp\Big(i\frac{c_2}{2}\int_0^T f(t)\,dt\Big) & ,if\ q=2,\ n=1 \\ 1 & ,if\ q>\ 2,\ n\neq 1 \end{cases} \tag{14}$$

Proof.

$$\begin{aligned} \lim_{z\longrightarrow\infty} C_{(\eta^p)^q}(f) &= \lim_{z\longrightarrow\infty} \exp\Big(z \int_0^T \int_{\mathbb{R}\setminus\{0\}} \sum_{n=0}^{\infty} \frac{i^n}{n!}\frac{s^{qn}}{z^{\frac{qn}{2}}} f^n(t)\,dr(s)\,dt\Big) \\ &= \begin{cases} \exp\Big(i\frac{c_2}{2}\int_0^T f(t)\,dt\Big) & ,if\ q=2,\ n=1 \\ 1 & ,if\ q>\ 2,\ n\neq 1 \end{cases} \end{aligned} \tag{15}$$

empty vertex	full vertex	leg of type 1	leg of type 2
○	●		

Table 1: Different types of vertices and legs.

■

Applying lemmas (3.1) and (3.2) to the Itô formula obtained by theorem (2.3), one thus get the following result:

Theorem 3.3. *Let η be a Poisson noise given by its generalizing functional (11) then:*

$$\lim_{z \longrightarrow \infty} \dot{f}(X,t) = \frac{\partial f}{\partial t}(X,t) + \frac{\partial f}{\partial X}(X,t) + \frac{c_2}{2}\left(\frac{\partial}{\partial X}\right)^2 f(X,t) \quad (16)$$

Proof.

$$\begin{aligned} \lim_{z \longrightarrow \infty} \dot{f}(X,t) &= \lim_{z \longrightarrow \infty} \left[\frac{\partial f}{\partial t}(X,t) + \frac{\partial f}{\partial X}(X,t)\,\dot{X} + \sum_{l=1}^{q} \frac{\eta^l}{l!}\left(\frac{\partial}{\partial X}\right)^l f(X,t)\right] \\ &= \frac{\partial f}{\partial t}(X,t) + \frac{\partial f}{\partial X}(X,t) + \frac{c_2}{2}\left(\frac{\partial}{\partial X}\right)^2 f(X,t) \quad (17) \end{aligned}$$

■

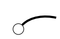
If we assume that the random variable X depend on z, i.e, $X = X_z$, and in the Itô formula, η^p will be replaced by η^g, a pure Gaussian noise, we obtain the classical Itô formula with $c_2 = \sigma^2$.

4 A graphical representation of the noise

In this section we will develop a graphical representation for the mixed noise, for this aim we need first to introduce the Feynman graph and Feynman rules and to calculate the truncated moments of mixed noise.
Let $G_r(t,\, x)$ be the Green function which satisfies :

$$\begin{cases} \frac{\partial G_r(t,x)}{\partial t} = G_r(t,\, x) + \delta(x) \quad ,(t,\, x) \in]0,\infty[\times \mathbb{R}^d \\ G_r(t,\, x) = 0, \quad t < 0 \end{cases} \quad (18)$$

Here $\delta(x)$ is the Dirac distribution.
In the following we will assign to each numerical expression a graphical symbol:

- A "propagator" of type $G_r(t-s)$ will be symbolized by 
- A "propagator" of type $\delta(t-s)$ will be symbolized by 
- A contribution $\eta(s)$ will be symbolized by a full vertex ●
- For the power of X, i.e, X^p we will consider p legs of type 1 connected to the same full vertex.

The different types of vertices and legs are resumed in table 1.

Definition 4.1. *A Feynman graph is a graph with two types of vertices called empty vertex and full vertex, by definition full vertices are distinguishable and have distinguishable legs whereas empty vertices are non-distinguishable and have non distinguishable legs. Edges are non-directed and connect full and empty vertices, but never connect two full or two empty vertices. The legs of type 1 connect the full vertices whereas the legs of type 2 connect the empty vertices.*

We denote the family of all Feynman graph G with n vertices by $\mathcal{F}(n)$.

Definition 4.2. Let $x_1, ..., x_k \in \mathbb{R}^d$, I a partition of the set $\{1, ..., n+p\}$, $I \in \mathcal{P}(n+p)$, $I = \{I_1, ..., I_k\}$ the truncated moments functions $\langle \eta(x_1) \cdots \eta(x_{n+p}) \rangle^T$ are recursively defined by:

$$\langle \prod_{i=1}^{n+p} \eta(z_i) \rangle = \sum_{\substack{I \in \mathcal{P}(n+p) \\ I=\{I_1,...,I_k\}}} \prod_{l=1}^{k} \langle I_l \rangle^T \tag{19}$$

where $\langle I_l \rangle^T = \langle \prod_{j \in I_l} \eta(x_j) \rangle^T$.

Proposition 4.3. *Let η be a mixed noise, then the ordinary moments $\langle X^p(t) \eta(s_1) \cdots \eta(s_n) \rangle$ are given by:*

$$\langle X^p(t)\eta(s_1)\cdots\eta(s_n)\rangle = \int_{\mathbb{R}_+} G_r(t-t_1)\cdots \int_{\mathbb{R}_+} G_r(t-t_p) \sum_{\substack{I \in \mathcal{P}(n+p) \\ I=\{I_1,...,I_k\}}} \prod_{l=1}^{k} \langle \prod_{j \in I_l} \eta(t_j) \rangle^T ds_1\cdots ds_n \, dt_1\cdots dt_p \, . \tag{20}$$

Proof. Let $I = \{I_1, \cdots, I_k\} \in \mathcal{P}(n+p)$, then

$$\begin{aligned} \langle X^p(t)\eta(s_1)\cdots\eta(s_n)\rangle &= \langle X(t)\cdots X(t)\eta(s_1)\cdots\eta(s_n)\rangle \\ &= \int_{\mathbb{R}_+} G_r(t-t_1)\cdots \int_{\mathbb{R}_+} G_r(t-t_p) \, \langle \eta(t_1)\cdots\eta(t_p)\eta(s_1)\cdots\eta(s_n)\rangle \\ &\qquad\qquad ds_1\cdots ds_n \, dt_1\cdots dt_p \\ &= \int_{\mathbb{R}_+} G_r(t-t_1)\cdots \int_{\mathbb{R}_+} G_r(t-t_p) \sum_{\substack{I \in \mathcal{P}(n+p) \\ I=\{I_1,...,I_k\}}} \prod_{l=1}^{k} \langle \prod_{j \in I_l} \eta(t_j) \rangle^T ds_1\cdots ds_n \, dt_1\cdots dt_p \end{aligned}$$

Where $s_1 = t_{p+1}, \cdots, s_l = t_{p+l}$.

The following result simplify the calculus of the truncated moments $\prod_{l=1}^{k} \langle \prod_{j \in I_l} \eta(t_j) \rangle^T$:

Theorem 4.4. *The truncated moment functions of the (mixed) noise η are given by the following formula*

$$\langle \eta(x_1)\cdots\eta(x_n)\rangle^T = c_n \int_{\mathbb{R}^d} \delta(x-x_1)\cdots\delta(x-x_n)dx \, . \tag{21}$$

where

$$\begin{aligned} c_n &= (-i)^n \frac{d^n \psi(t)}{dt^n} \mid t = 0 \\ &= \delta_{n,1} a + \delta_{n,2} \sigma^2 + z \int_{\mathbb{R}^d \setminus \{0\}} s^n dr(s) \end{aligned} \tag{22}$$

$\delta_{n,n'}$ *being the Kronecker symbol.*

Proof. For the proof we refer to [2, 8].

$\blacksquare$

Definition 4.5. The following algorithm denoted by $B(G)$ and known as Feynman rule assign a numerical value to each Feynman graph G:

- Assign a value $s \in \mathbb{R}+$ to each empty vertex.
- For each leg of type 1 multiply with $G_r(t-s)$ and for each j-th leg of type 2 multiply with $\delta(s-s_j)$.
- For each empty vertex with n legs multiply with c_n.
- Integrate, with respect to Lebesgue measure, over all s.

Theorem 4.6. *Let $I=\{I_1,\cdots,I_k\}\in\mathcal{P}(n+p)$, then the ordinary moments of order p are given by:*

$$\langle X^p(t)\eta(s_1)\cdots\eta(s_n)\rangle=\int_{\mathbb{R}_+}G_r(t-t_1)\cdots\int_{\mathbb{R}_+}G_r(t-t_p)\sum_{\substack{I\in\mathcal{P}(n+p)\\ I=\{I_1,\ldots,I_k\}}}\prod_{l=1}^{k}c_{\sharp I_l}\int_{\mathbb{R}_+}\prod_{j\in I_l}\delta(t_j-t)\,ds_1\cdots ds_n\,dt_1\cdots dt_p \tag{23}$$

Proof. By proposition (4.3) and theorem (4.4), we have:

$$\begin{aligned}\langle X^p(t)\eta(s_1)\cdots\eta(s_n)\rangle &= \int_{\mathbb{R}_+}G_r(t-t_1)\cdots\int_{\mathbb{R}_+}G_r(t-t_p)\sum_{\substack{I\in\mathcal{P}(n+p)\\ I=\{I_1,\ldots,I_k\}}}\prod_{l=1}^{k}\langle\prod_{j\in I_l}\eta(t_j)\rangle^T\,ds_1\cdots ds_n\,dt_1\cdots dt_p\\ &= \int_{\mathbb{R}_+}G_r(t-t_1)\cdots\int_{\mathbb{R}_+}G_r(t-t_p)\sum_{\substack{I\in\mathcal{P}(n+p)\\ I=\{I_1,\ldots,I_k\}}}\prod_{l=1}^{k}\int_{\mathbb{R}_+}\prod_{j\in I_l}c_{\sharp I_l}\,\delta(t_j-t)\,ds_1\cdots dt_p\\ &= \int_{\mathbb{R}_+}G_r(t-t_1)\cdots\int_{\mathbb{R}_+}G_r(t-t_p)\sum_{\substack{I\in\mathcal{P}(n+p)\\ I=\{I_1,\ldots,I_k\}}}\prod_{l=1}^{k}c_{\sharp I_l}\int_{\mathbb{R}_+}\prod_{j\in I_l}\delta(t_j-t)\,ds_1\cdots dt_p\end{aligned}$$

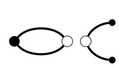
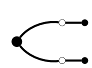
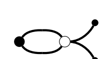
$\blacksquare$.

The following example provide a graphical representation of the moments of the noise in the case when $n=2$:

Example. The ordinary moments of order 2 are given by:

$$\begin{aligned}\langle X^2(t)\eta(s_1)\eta(s_2)\rangle &= c_2^2\int_0^\infty G_r^2(t-s)\,\delta(s_1-s_2)\,ds+2c_2^2\int_0^\infty G_r(t-s)\,G_r(t-\bar{s})\delta(s_1-s_2)\,dsd\bar{s}\\ &+ c_4\int_0^\infty G_r^2(t-s)\,\delta(s_1-s_2)\,ds,\end{aligned} \tag{24}$$

following definition (4.5) the corresponding graphical representation will be:

$\langle X^2(t)\eta(s_1)\eta(s_2)\rangle =$  $+\,2$  $+$  (25)

To derive now the graphical representation of the truncated moments, we need first mixed truncated expectations of $\eta^{q}{'s}$ and $\eta's$:

Proposition 4.7. *The truncated moment of the (mixed) noise is given by:*

$$\langle \eta^q(s_1) \cdots \eta^q(s_j)\eta(s_{j+1}) \cdots \eta(s_n)\rangle^T = c_{jq+n-j} \int_{\mathbb{R}_+} \prod_{l=1}^{n} \delta(s_l - s)\, ds \tag{26}$$

Proof. By application of the Fourier transform we have:

$$\begin{aligned} \mathbb{E}[e^{i\,(\eta(f)+\eta^q(g))}] &= e^{-t\,z} \sum_{n=0}^{\infty} \frac{(zT)^n}{n!} \Big(\int_0^T \int_{\mathbb{R}\setminus\{0\}} e^{i\,(s\,f(t)+s^q\,g(t))}\, dr(s)\, dt \Big)^n \\ &= \exp\Big(z \int_0^T \int_{\mathbb{R}\setminus\{0\}} (e^{i\,(s\,f(t)+s^q\,g(t))} - 1)\, dr(s)\, dt \Big) \\ &= C_{\eta(f)+\eta^q(g)} \end{aligned} \tag{27}$$

Hence

$$\langle \eta^q(s_1) \cdots \eta^q(s_j)\eta(s_{j+1}) \cdots \eta(s_n)\rangle^T = c_{jq+n-j} \int_{\mathbb{R}_+} \prod_{l=1}^{n} \delta(s_l - s)\, ds \tag{28}$$

■

By definition (4.5) and proposition (4.7) we derive the following result:

Theorem 4.8. *The graphical representation of the truncated moments of the mixed noise is given by a sum over all Feynman graphs that are evaluated according to the rule fixed by definition (4.5), i.e:*

$$\langle \eta^q(s_1) \cdots \eta^q(s_j)\eta(s_{j+1}) \cdots \eta(s_n)\rangle^T = \sum_{G \in \mathcal{F}(n)} B(G, \eta). \tag{29}$$

Proof. Taking expectations with a collection of $\eta' s$, we see from the form of the truncated functions calculated above that the $q-$legs, $1 \leq q \leq p$, can only be connected with the same empty vertex and the $q-$lines is also evaluated with a dirac distribution δ, however to each empty vertex with $q_1 + \cdots + q_n + l$ legs we should multiply with $C_{q_1 + \cdots q_n + l}$.

■

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